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A ring R (associative with identity) is called *right* ℵ-QF 3 if it has a faithful right ideal which is a direct sum of a family of injective envelopes of pairwise non-isomorphic simple right R-modules. A right QF 3 ring is just a right ℵ-QF 3 ring where the above family is finite. The aim of the present work is to give a structure theorem for semiprime ℵ-QF 3 rings. It is proved, among others, that the following conditions are equivalent for a given ring R: (a) R is a semiprime right ℵ-QF 3 ring, (b) there is a ring Q, which is a direct product of right full linear rings, such that $\operatorname{Soc} Q \subset R \subset Q$, (c) R is right nonsingular and every non-singular right R-module is cogenerated by simple and projective modules.

A ring R is called a *right* QF 3 ring if there is a minimal faithful module U_R, in the sense that every faithful right R-module contains a direct summand which is isomorphic to U; one proves that if there exists such a module U, then it is unique up to an isomorphism. It was proved by Colby and Rutter [**5**, Theorem 1] that R is right QF 3 if and only if it contains a faithful right ideal of the form $E(S_1) \oplus \cdots \oplus E(S_n)$, where each $E(S_i)$ is the injective envelope of a simple module S_i, and the S_i's are pairwise non-isomorphic. Following Kawada [**10**], we say that R is a *right* ℵ-QF 3 ring if there is a family $(e_\lambda)_{\lambda \in \Lambda}$ of pairwise orthogonal and pairwise non isomorphic (in the sense that $e_\lambda R \neq e_\mu R$ whenever $\lambda \neq \mu$) idempotents of R such that: (a) each $e_\lambda R$ is the injective envelope of a minimal right ideal, (b) the right ideal $W_R = \sum_{\lambda \in \Lambda} e_\lambda R$ is faithful; here ℵ stands for the cardinality of the set Λ. It is clear from Colby and Rutter's result that a right QF 3 ring is nothing other than a right ℵ-QF 3 ring where ℵ is a finite cardinal. By a ℵ-QF 3 ring we shall mean a ring which is both right and left ℵ-QF 3; similarly for QF 3 rings.

In [**4**] we studied those right ℵ-QF 3 rings which have zero right singular ideal. Our purpose in the present paper is to characterize the semiprime right ℵ-QF 3 rings. Our main result is that the following conditions are equivalent for a given ring R: (a) R is a semiprime right ℵ-QF 3 ring, (b) R is a semiprime ring with essential socle and every simple projective right R-module is injective. (c) R is right nonsingular and every nonsingular right R-module is cogenerated by simple projective modules, (d) R is (isomorphic to) a subring of a direct product $\prod_{\lambda \in \Lambda} Q_\lambda$ of right full linear rings and $\oplus_{\lambda \in \Lambda} \operatorname{Soc} Q_\lambda \subset R$. As a consequence we obtain that R is a semiprime ℵ-QF 3 ring if and only if it satisfies one (and hence all) of the following conditions: (a) R is a subring of the direct product of a family $(Q_\lambda)_{\lambda\in\Lambda}$ of simple artinian rings and contains the direct sum $\bigoplus_{\lambda\in\Lambda} Q_\lambda$, (b) R is right nonsingular and every nonsingular injective right R-module is a direct product of pairwise independent semisimple and homogeneous modules (we say that two semisimple right R-modules L, M are *independent* if $\mathrm{Hom}_R(L, M) = 0$, i.e. if L does not contain a simple submodule which is isomorphic to some submodule of M).

Throughout, all rings will be associative with identity, all modules will be unitary and all maps between modules will be module homomorphisms. For a given ring R, we shall denote with Mod-R the category of all right R-modules. If M is a given right R-module, we shall denote with $E(M)$, $Z(M)$, $J(M)$ and $\mathrm{Soc}\, M$ resp. the injective envelope, the singular submodule, the Jacobsen radical and the socle of M; if $\mathscr{A}$ is a set of pairwise non-isomorphic simple right R-modules, then $\mathrm{Soc}_{\mathscr{A}}(M)$ will denote the $\mathscr{A}$-homogeneous component of $\mathrm{Soc}\, M$ (we shall write $\mathrm{Soc}_P(M)$ in case $\mathscr{A} = \{P\}$); the notation $N \leq M_R$ (resp. $N \trianglelefteq M_R$) will mean that N is an R-submodule (resp. an essential R-submodule) of M. Given a subset $X \subset M$, $r_R(X)$ will be the right annihilator of X in R; similarly, if M is a left R-module, then $l_R(X)$ will be the left annihilator of X in R. We assume the reader familiar with elementary facts about torsion theories, in particular the Goldie torsion theory (see e.g. [**6**] and [**12**]).

We proceed to give first several preliminary results concerning the projective components of the socle of a ring; these results are mainly based on the following one, which was proved in [**2**, Proposition 1.4 and Corollary 1.5].

PROPOSITION 1. *Let R be a given ring, let $\mathscr{P}$ be a set of representatives of the simple projective right R-modules and let K be a two-sided ideal contained in* $\mathrm{Soc}\, R_R$. *Then the following conditions are equivalent*:

(1) $K^2 = K$.

(2) ${}_R(R/K)$ *is flat.*

(3) *There is a subset $\mathscr{A} \subset \mathscr{P}$ such that $K = \mathrm{Soc}_{\mathscr{A}}(R_R)$.*

If these conditions hold, then for each module M_R we have $\mathrm{Soc}_{\mathscr{A}}(M) = MK$. □

By a *right full linear ring* we mean a ring which is isomorphic to the endomorphism ring of a right vector space over some division ring. It is well known that R is a right full linear ring if and only if R is a prime von Neumann regular right self-injective ring with essential socle (see [**12**, Ch. XII, Corollary 1.5, page 246]); if it is the case, then R is a right QF 3 ring (see Tachikawa [**13**, page 43, 44]). The following proposition tells us that prime right QF 3 rings can be characterized as special subrings of right full linear rings (see however [**13**, Proposition 4.3]). We need a lemma.

LEMMA 2. *Let P be a minimal right ideal of the ring R and let e be an idempotent such that $P \trianglelefteq eR_R$. Then either $eR = P$ or $P^2 = 0$.*

Proof. If $P^2 \neq 0$, then, by the modular law, P is a direct summand of eR and hence equals eR. □

PROPOSITION 3. *Given a ring R, the following conditions are equivalent*:
(1) *R has a simple injective, projective and faithful right module.*
(2) *R is a prime right* QF 3 *ring.*
(3) *R is a subring of a right full linear ring Q and* Soc $Q \subset R$.

Proof. (1) ⇒ (2) is clear from [**5**, Theorem 1].

(2) ⇒ (3). It follows from (2) that R has a nonzero homogeneous projective essential socle S. Moreover, since R is right QF 3, there is an idempotent $e \in R$ such that eR_R is faithful, injective with a simple essential socle P. Inasmuch as P is prime, then $P^2 = P$ and hence $P = eR$ by Lemma 2, so all minimal right ideals of R are injective. Let Q be the maximal right quotient ring of R. It is well known that $Q \cong \text{End}\, S_R \cong E(R_R)$ and Q is a right full linear ring (see e.g. [**12**, page 249]). Now if N is a minimal right ideal of R, then, by the above, $R \supset N = E(N_R) = NQ$. The latter equality tells us that $\text{Soc}\, Q_Q = SQ \subset R$.

(3) ⇒ (1). Suppose that $\text{Soc}\, Q_Q \subset R \subset Q$, where Q is a right full linear ring. Then R is right primitive, $\text{Soc}\, R = \text{Soc}\, Q_Q$ and Q is the maximal right quotient ring of R. If N is a minimal right ideal of R, then N_R is faithful, projective, and, as in the proof of the implication (2) ⇒ (3), $E(N_R) = NQ$, therefore N is essential in NQ_R. Since the latter is semi-simple, it follows that $N = NQ$ and hence N_R is injective. □

COROLLARY 4. *A ring R is a prime* QF 3 *ring if and only if R is simple artinian.*

Proof. The "if" part is obvious. Assume that R is prime and QF 3. Then R has both a right and a left simple injective, projective and faithful module by Proposition 3. It follows from Jans [**9**, corollary 2.2] that R is simple artinian. □

In what follows we fix a simple projective right R-module P and we set $L = l_R(\mathrm{Soc}_P(R_R))$. Then, in view of Proposition 1, we have $\mathrm{Soc}_P(R_R) \cdot L \subset \mathrm{Soc}_P(R_R) \cap L = L \cdot \mathrm{Soc}_P(R_R) = 0$, so that P may be regarded as a simple right R/L-module. The proof of the following lemma is left to the reader.

LEMMA 5. *With the above notations, R/L is a right nonsingular ring with essential and homogeneous right socle; to be precise, the canonical map $R \to R/L$ induces an isomorphism* $\mathrm{Soc}_P(R_R) \cong \mathrm{Soc}(R/L)_{R/L}$. □

LEMMA 6. *With the above notations, the following conditions are equivalent*:

(1) *P_R is injective.*

(2) *$P_{R/L}$ is injective.*

(3) *R/L is a prime right* QF 3 *ring.*

If any of the above conditions holds, then $L = r_R(P)$.

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (3). It follows from Lemma 5 that $\mathrm{Soc}(R/L)_{R/L}$ is homogeneous and essential in R/L and, since $P_{R/L}$ is injective, we have $J(R/L) = 0$. Thus R/L is primitive and $P_{R/L}$ is a simple faithful, injective and projective module, therefore R/L is right QF 3 by Proposition 3.

(3) ⇒ (1). If R/L is a prime right QF 3 ring, then, again by Proposition 3, R/L is primitive with P as a simple faithful injective and projective right R/L-module. This implies $J(R) \subset L$ and, taking Proposition 1 into account, we get $J(R) \cap \mathrm{Soc}_P(R_R) = J(R)\mathrm{Soc}_P(R_R) = 0$. We may now apply [3, Theorem 1.3, equivalence of conditions (1) and (8)] and we infer that $E((\mathrm{Soc}_P(R_R))_{R/L})$ is R-injective. From that, since $P_{R/L}$ is injective, we conclude that P_R is injective.

Finally, the arguments in the proof of the last implication together with [3, Theorem 1.3], show the last part of our lemma. □

If P_R is injective, then $J(R) \cap \mathrm{Soc}_P(R_R) = 0$ and [3, Theorem 1.3] implies that $\mathrm{Soc}_P(R_R) = \mathrm{Soc}_{P'}({}_RR)$, where P' is some simple projective left R-module (to be precise, $P' = \mathrm{Hom}_R(P, R)$); moreover $L = r_R(P) = l_R(P')$. The condition that ${}_RP'$ also is injective is very sharp, as it is shown by the following corollary.

COROLLARY 7. *With the above notations, the following conditions are equivalent*:

(1) *P_R and its dual ${}_RP' = \mathrm{Hom}_R(P, R)$ are injective.*

(2) *R/L is a simple artinian ring.*

(3) $\mathrm{Soc}_P(R_R) = eR$ *for a central idempotent $e \in R$.*

Proof. (1) $\Rightarrow$ (2). As we observed before, the injectivity of P_R implies that $L = r_R(P) = l_R(P')$. Thus, according to Lemma 6, (1) implies that R/L is a prime QF 3 ring; hence R/L is simple artinian by Corollary 4.

(2) $\Rightarrow$ (3). If (2) holds, then $J(R) \subset L$ and hence $J(R) \cap \text{Soc}_P(R_R) = 0$. According to the above remarks, there is a simple projective left R-module P' such that $\text{Soc}_P(R_R) = \text{Soc}_{P'}({}_RR)$. Taking Lemma 5 into account, we see that $R/L \cong \text{Soc}_P(R_R) = \text{Soc}_{P'}({}_RR)$, therefore R/L is projective both as a right and a left R-module. We conclude that $L = fR$ for a central idempotent $f \in R$ and (3) holds with $e = 1 - f$.

(3) $\Rightarrow$ (1) is a consequence of [**2**, Theorem 2.7]. □

Recall that the ring R is *semiprime* if it has no non-zero nilpotent right (and hence left) ideals. Without any hypothesis on R, if N is a minimal right ideal of R, then either $N^2 = 0$ or $N = eR$ for some idempotent $e \in R$. Thus, if R is semiprime, it follows from Proposition 1 that $\text{Soc}\, R_R = \text{Soc}_{\mathscr{P}}(R_R)$ and every two-sided ideal contained in $\text{Soc}\, R_R$ is of the form $\text{Soc}_{\mathscr{A}}(R_R)$ for some subset $\mathscr{A} \subset \mathscr{P}$; moreover, it was proved by Jacobson (see [**8**, Ch. IV, n. 3, Theorem 1, page 65]) that every homogeneous component of $\text{Soc}\, R_R$ is also a homogeneous component of $\text{Soc}_R R$ and conversely, so that $\text{Soc}\, R_R = \text{Soc}_R R$.

LEMMA 8. *Let Q be a ring with essential and projective right socle S and let R be a subring of Q containing S. Then the following are true*:

(1) $S = \text{Soc}\, R_R = \text{Soc}\, Q_R$.

(2) S_R *is projective.*

(3) $S \trianglelefteq R_R \trianglelefteq Q_R$.

Moreover, if Q is semiprime, then R is semiprime as well.

Proof. Let U be a minimal right ideal of Q and let $0 \neq x \in U$. Taking Proposition 1 into account we have $U = xQ = xS \subset xR \subset U$, hence $xR = U$. This shows that $S \subset \text{Soc}\, R_R$. Since $S \trianglelefteq Q_Q$, then $xS \neq 0$ for each non-zero $x \in Q$ and therefore $S \trianglelefteq R_R$. We infer that $S = \text{Soc}\, R_R$ and S_R is projective since $S^2 = S$. Moreover $S \trianglelefteq Q_R$, so $S = \text{Soc}\, Q_R$. If Q is semiprime, then every minimal right ideal of Q is generated by an idempotent. This fact, together with $S \trianglelefteq R_R$, implies easily that R is semiprime. □

Following L. Levy [**11**], we say that the ring R is an *irredundant subdirect product* of a family $(R_\lambda)_{\lambda \in \Lambda}$ of rings if:

(a) R is a subdirect product of the R_λ's,

(b) canonically identifying R with a subring and each R_λ with a two-sided ideal of $\prod_{\lambda \in \Lambda} R_\lambda$, we have $R \cap R_\lambda \neq 0$.

LEMMA 9. *Given a ring R, the following conditions are equivalent*:

(1) *R is semiprime with essential socle.*

(2) *R is an irredundant subdirect product of a family $(R_\lambda)_{\lambda\in\Lambda}$ of prime rings each with a non-zero socle S_λ.*

(3) *R is a subdirect product of a family $(R_\lambda)_{\lambda\in\Lambda}$ of prime rings, each with a non-zero socle S_λ, and, canonically identifying R with a subring and each R_λ with a two-sided ideal of $\prod_{\lambda\in\Lambda} R_\lambda$, the equality* $\operatorname{Soc} R_R = \bigoplus_{\lambda\in\Lambda} S_\lambda$ *holds.*

Proof. (1) $\Rightarrow$ (3). Inasmuch as R is semiprime, $\operatorname{Soc} R_R$ is projective. Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of representatives of all simple projective right R-modules and, for each $\lambda \in \Lambda$, let us write $L_\lambda = r_R(P_\lambda)$ and $R_\lambda = R/L_\lambda$. It follows from [**3**, Theorem 1.3] that $L_\lambda = l_R(\operatorname{Soc}_{P_\lambda}(R_R))$, hence R is a subdirect product of the family $(R_\lambda)_{\lambda\in\Lambda}$ by Gordon [**7**, Theorem 2.3]; moreover each R_λ has essential right socle S_λ and is prime by the above. Let us identify R with a subring and each R_λ with a two-sided ideal of the ring $\prod_{\lambda\in\Lambda} R_\lambda$ and let $p_\lambda\colon R \to R_\lambda$ be the canonical projection. Then $\operatorname{Soc}_{P_\lambda}(R_R)$ is canonically identified with S_λ via p_λ (see Lemma 5). It follows that $S_\lambda \subset R \cap R_\lambda$ and hence $\operatorname{Soc} R = \bigoplus_{\lambda\in\Lambda} S_\lambda$.

(3) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Let us write $Q = \prod_{\lambda\in\Lambda} R_\lambda$. We may again assume that R is a subring and each R_λ is a two-sided ideal of Q. For each $\lambda \in \Lambda$, since R_λ is prime, every non-zero two-sided ideal of R_λ is essential, thus S_λ is a minimal two-sided ideal of R; since $R \cap R_\lambda \neq 0$, then $S_\lambda \subset R \cap R_\lambda \subset R$, whence $\operatorname{Soc} Q_Q = \bigoplus_{\lambda\in\Lambda} S_\lambda \subset R$. Inasmuch as Q is semiprime, it follows from Lemma 8 that R is semiprime with essential socle. □

We are now in position to state and prove our structure theorem on semiprime $\aleph$-QF 3 rings. Recall that R is a *right* QF 3′ ring if $E(R_R)$ is torsionless. A torsion theory $(\mathcal{T}, \mathcal{F})$ is *jansian* (or "TTF") if $\mathcal{T}$ is closed by direct products; this happens if and only if there is an idempotent two-sided ideal I of R such that $\mathcal{T} = \{L_R \mid LI = 0\}$.

THEOREM 10. *Let R be a given ring, let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of representatives of all simple projective right R-modules and let $\aleph$ be a non-zero cardinal number. Then the following conditions are equivalent*:

(1) *R is a semiprime right $\aleph$-*QF 3 *ring.*

(2) *R is a semiprime* QF 3′ *ring with essential socle and* $\operatorname{Card}(\Lambda) = \aleph$.

(3) *R is a right $\aleph$-*QF 3 *ring without nilpotent minimal right ideals.*

(4) *R is a semiprime ring with essential socle, every simple projective right R-module is injective and* $\operatorname{Card}(\Lambda) = \aleph$.

(5) *R is an irredundant subdirect product of a family* $(R_\lambda)_{\lambda\in\Lambda}$ *of prime right* QF 3 *rings and* $\mathrm{Card}(\Lambda) = \aleph$.

(6) *R is (isomorphic to) a subring of the direct product of a family* $(Q_\lambda)_{\lambda\in\Lambda}$ *of right full linear rings, with* $\mathrm{Card}(\Lambda) = \aleph$, *and* $\oplus_{\lambda\in\Lambda} \mathrm{Soc}\, Q_\lambda \subset R$.

(7) *R is right nonsingular*, $\mathrm{Card}(\Lambda) = \aleph$ *and every nonsingular right R-module is cogenerated by simple projective modules.*

(8) $\mathrm{Card}(\Lambda) = \aleph$ *and a module M_R is singular if and only if* $\mathrm{Hom}_R(M, P_\lambda) = 0$ *for each* $\lambda \in \Lambda$.

Proof. (1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4). Assume that (3) holds. By the definition of a right $\aleph$-QF 3 ring and taking [**4**, Proposition 2.3] into account, we may assume that each P_λ is a minimal right ideal and there is a family $(e_\lambda)_{\lambda\in\Lambda}$ of idempotents of R, with $W_R = \sum_{\lambda\in\Lambda} e_\lambda R$ faithful, such that $e_\lambda R = E(P_\lambda)$ for each $\lambda \in \Lambda$. Our assumption, together with Lemma 2, implies that $e_\lambda R = P_\lambda$ for each $\lambda \in \Lambda$, so that every simple projective right R-module is injective. Moreover $e_\lambda R \cap J(R) = P_\lambda \cap J(R) = 0$, hence $e_\lambda J(R) = 0$ for each $\lambda \in \Lambda$. We infer that $WJ(R) = 0$ and then $J(R) = 0$, being W_R faithful. Thus R is semiprime and has essential socle by [**4**, Theorem 2.4].

(4) $\Rightarrow$ (5). It follows from Lemma 9 that R is an irredundant subdirect product of the family $(R_\lambda)_{\lambda\in\Lambda}$, where $R_\lambda = R/l_R(\mathrm{Soc}_{P_\lambda}(R_R))$ for each $\lambda \in \Lambda$. Moreover every R_λ is a prime right QF 3 ring by Lemma 6.

(5) $\Rightarrow$ (6). Suppose that (5) holds. It follows then from Lemma 8 and 9 that R is a semiprime ring with essential socle and $\mathrm{Soc}\, R = \oplus_{\lambda\in\Lambda} \mathrm{Soc}\, R_\lambda$. Now Proposition 3 tells us that each R_λ is (isomorphic to) a subring of a right full linear ring Q_λ and $\mathrm{Soc}\, Q_\lambda \subset R_\lambda$. This is enough to conclude that R has the properties stated in (6).

(6) $\Rightarrow$ (1). If (6) holds, then it follows from Lemma 9 that R is semiprime and $\mathrm{Soc}\, R = \oplus_{\lambda\in\Lambda} \mathrm{Soc}\, Q_\lambda$. Moreover $E(R_R) = \prod_{\lambda\in\Lambda} Q_\lambda$ (see [**12**, Ch. XII, Proposition 2.4, page 247]). There is a family $(e_\lambda)_{\lambda\in\Lambda}$ of pairwise orthogonal and pairwise non-isomorphic idempotents of R such that $e_\lambda Q_\lambda = e_\lambda R$ is simple and injective. Since $\sum_{\lambda\in\Lambda} e_\lambda Q_\lambda$ is faithful as a right ideal of $\prod_{\lambda\in\Lambda} Q_\lambda$, then it is faithful as a right ideal of R and therefore R is right $\aleph$-QF 3.

(4) $\Rightarrow$ (7). Inasmuch as R is semiprime with essential socle, R must be right (and left) nonsingular. Thus the Lambek torsion theory and the Goldie torsion theory on Mod-R coincide, so that every nonsingular (= torsionfree) right R-module is cogenerated by $E(R_R)$. It follows from the equivalence of conditions (4) and (6) that $E(R_R) = \prod_{\lambda\in\Lambda} Q_\lambda$, where

each Q_λ is a right full linear ring. Since Q_λ is isomorphic to the direct product $P_\lambda^{\Gamma_\lambda}$ for some Γ_λ, we infer that the family $(P_\lambda)_{\lambda\in\Lambda}$ cogenerates $E(R_R)$, hence it cogenerates every nonsingular right R-module.

(7) $\Rightarrow$ (8). This implication is clear, taking into account that, since R is right nonsingular, the Goldie torsion class in Mod-R consists of all singular modules.

(8) $\Rightarrow$ (4). Assume that (8) holds and let us prove first that $Z(R_R) = 0$. Let us denote by S the projective component of Soc R_R. Since $\aleph \neq 0$, (8) implies that $S \neq 0$ and ${}_R(R/S)$ is flat by Proposition 1, so that we may consider the jansian torsion theory $(\mathcal{T}, \mathcal{F})$ associated with the idempotent two-sided ideal S: $\mathcal{T} = \{L_R | LS = 0\}$, $\mathcal{F} = \{M_R | MS \trianglelefteq M\}$ (for the last equality see [**1**, Proposition 1.3]). Now (8) implies that a module M_R is nonsingular iff it has projective and essential socle and, since the latter is given by MS (see Proposition 1), we infer that $(\mathcal{T}, \mathcal{F})$ coincides with the Goldie torsion theory. Moreover (8) implies that the class of all singular right R-modules is a (hereditary) torsion class, whence it must coincide with $\mathcal{T}$. From this we conclude that the Gabriel topology $\{I \leq R_R | S \subset I\}$ associated with $\mathcal{T}$ consists of all essential right ideals, whence $S \trianglelefteq R_R$ and so $Z(R_R) = 0$. Let us prove now that each P_λ is injective. Indeed, since $E(P_\lambda)$ is nonsingular, it follows from (8) that there is a non zero homomorphism $E(P_\lambda) \to P_\mu$ for some $\mu \in \Lambda$. Thus, since P_μ is projective, $E(P_\lambda)$ has a direct summand isomrophic to P_μ, which implies $\lambda = \mu$ and $E(P_\lambda) = P_\lambda$. We conclude from the above that every minimal right ideal of R is idempotent and, since $S = \text{Soc}\, R_R \trianglelefteq R_R$, R must be semiprime.

(1) $\Leftrightarrow$ (2). By the equivalence of conditions (1) and (4), a semiprime right $\aleph$-QF 3 ring has essential socle. Since a semiprime ring with essential socle is nonsingular, the equivalence of (1) and (2) follows from [**4**, Theorem 2.11]. □

In the following corollary we characterize those semiprime rings which are $\aleph$-QF 3.

COROLLARY 11. *With the same hypothesis as in Theorem* 10, *the following conditions are equivalent*:

(1) *R is a semiprime $\aleph$-*QF 3 *ring.*

(2) Soc $R_R \trianglelefteq R_R$, *there is a family $(f_\lambda)_{\lambda\in\Lambda}$ of idempotents of R such that the $f_\lambda R$'s are the homogeneous components of* Soc R_R *and* Card$(\Lambda) = \aleph$.

(3) *R is (isomorphic to) a subring of the direct product of a family $(Q_\lambda)_{\lambda\in\Lambda}$ of simple artinian rings, with* Card$(\Lambda) = \aleph$, *and* $\oplus_{\lambda\in\Lambda} Q_\lambda \subset R$.

(4) *R is right nonsingular, every non-zero injective nonsingular right R-module is a direct product of pairwise independent semisimple and homogeneous modules, and* Card$(\Lambda) = \aleph$.

Proof. (1) $\Rightarrow$ (3). In view of Theorem 10, (1) implies that every simple projective right or left R-module is injective; hence it follows from Corollary 6 that $R/l_R(\mathrm{Soc}_{P_\lambda}(R_R))$ is a simple artinian ring for each $\lambda \in \Lambda$. Thus (3) holds with $Q_\lambda = R/l_R(\mathrm{Soc}_{P_\lambda}(R_R))$ (see the proof of the implications (4) $\Rightarrow$ (5) $\Rightarrow$ (6) of Theorem 10).

(3) $\Rightarrow$ (2) is straightforward.

(2) $\Rightarrow$ (4). It follows from (2) that $\mathrm{Soc}\, R_R$ is projective and, taking [**2**, Theorem 2.7] into account, every semisimple, projective and homogeneous right R-module is injective. Assume that $M_R \neq 0$ is injective and nonsingular. Then $\mathrm{Soc}\, M = M(\mathrm{Soc}\, R_R) \trianglelefteq M$ and it follows from (2) that the homogeneous components of $\mathrm{Soc}\, M$ are the Mf_λ ($\lambda \in \Lambda$). Moreover $\bigoplus_{\lambda\in\Lambda} MF_\lambda$ is essential in $\prod_{\lambda\in\Lambda} Mf_\lambda$; indeed, if $0 \neq (x_\lambda) \in \prod_{\lambda\in\Lambda} Mf_\lambda$, then $x_\lambda f_\lambda \neq 0$ for some $\lambda \in \Lambda$, so that $0 \neq (x_\lambda)_\lambda(\bigoplus_{\lambda\in\Lambda} f_\lambda R) \subset \bigoplus_{\lambda\in\Lambda} Mf_\lambda$. Since all Mf_λ's are injective, we conclude that $M = \bigoplus_{\lambda\in\Lambda} Mf_\lambda$.

(4) $\Rightarrow$ (1). Assume that (4) holds. Then one easily checks that every non-singular R-module is cogenerated by simple projective modules, hence R is a semiprime right ℵ-QF 3 ring by Theorem 10. Also, (4) implies that every projective semisimple and homogeneous right R-module is injective, whence every homogeneous component of $\mathrm{Soc}\, R_R$ is generated by a central idempotent (see [**2**, Theorem 2.7]). Inasmuch as R is semiprime, then every homogeneous component of $\mathrm{Soc}\, R_R$ is also a homogeneous component of $\mathrm{Soc}\, {}_RR$ and conversely. From this and again by [**2**, Theorem 2.7] we infer that each simple projective left R-module is injective. Finally, since $\mathrm{Soc}\, R$ is essential both as a right and a left ideal, it follows from Theorem 10 that R is left ℵ-QF 3 as well. □

Remark. The assumption that R is right nonsingular in condition (7) of Theorem 10 and condition (4) of the last corollary cannot be omitted. In fact, if $R = S \times T$, where S is a quasi-Frobenius ring with essential singular ideal and T is a semisimple ring, then R is QF 3 and every nonsingular R-module is semisimple and injective, but R is not semiprime.

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