A GALOIS-CORRESPONDENCE FOR GENERAL LOCALLY COMPACT GROUPS

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We give a characterization in terms of \( \hat{G} \) of those parts in the
unitary dual of a locally compact group \( G \), which correspond to closed
normal subgroups of \( G \). These are exactly the sets \( S \subset \hat{G} \), which have
the property that for all \( \pi, \rho \in S \) the support of \( \pi \otimes \bar{\rho} \) is contained in \( S \)
and which are closed in a topology on \( \hat{G} \), which is in general weaker than
the standard topology on \( \hat{G} \), and which we call the \( L^1 \)-hull-kernel-topology.
As an easy consequence we obtain that for \( * \)-regular groups \( G \) the
mapping \( N \rightarrow N^\perp = \{ \pi \in \hat{G} | \pi|_{N} = 1 |_{\hat{G}} \} \) is a bijection from the set of
closed normal subgroups of \( G \) onto the set of closed subsets \( S \subset \hat{G} \) with
the property that \( \pi \otimes \bar{\rho} \) has support in \( S \) for all \( \pi, \rho \in S \). This general-
izes and unifies results of Pontryagin, Helgason and Hauenschild, with
a considerably simplified proof. Furthermore we prove that \( * \)-regular
groups have the weak Frobenius property (TP 1), i.e. \( 1_G \) is weakly
contained in \( \pi \otimes \bar{\pi} \) for all unitary representations \( \pi \) of \( G \), generalizing a
result of E. Kaniuth.

Let \( G \) be a locally compact group with unitary dual \( \hat{G} \) and let \( \mathcal{N}_G \)
denote the set of closed normal subgroups of \( G \). To every \( N \in \mathcal{N}_G \)
corresponds a canonical subset of \( \hat{G} \), namely the annihilator \( N^\perp = \{ \pi \in \hat{G} | \pi|_{N} = 1 |_{\hat{G}} \} \) of \( N \). By the Gelfand-Raikov theorem \( N \rightarrow N^\perp \) is an
injective mapping from \( \mathcal{N}_G \) into the subsets of \( \hat{G} \) and it is an important
problem in harmonic analysis to describe the image of this mapping in
terms of \( \hat{G} \).

DEFINITION. A nontrivial subset \( S \) of \( \hat{G} \) is called a subdual of \( \hat{G} \), if for
all \( \pi, \rho \in S \) the tensor product \( \pi \otimes \bar{\rho} \) of \( \pi \) and the conjugate \( \bar{\rho} \) of \( \rho \) has
support in \( S \). We denote by \( \mathcal{S}_G \) the set of closed subduals of \( \hat{G} \).

It is clear that \( N \rightarrow N^\perp \) is an injective mapping from \( \mathcal{N}_G \) into \( \mathcal{S}_G \). Let
\([H]\) be the class of locally compact groups \( G \), for which \( N \rightarrow N^\perp \) is a
surjection onto \( \mathcal{S}_G \).

As a well known consequence of the duality theorem of Pontryagin
one obtains that all abelian locally compact groups belong to \([H]\) (see for
example [6], Chap. II, §1.7). S. Helgason proved in [8], Theorem 1, that all
compact groups belong to \([H]\). It was then W. Hauenschild, who gener-
ralized and unified these results in [7], and proved that all Moore groups,
i.e. all locally compact groups $G$, which have only finite dimensional irreducible unitary representations, belong to the class $[H]$. 

On the other hand the support $\hat{G}_r$ of the left regular representation $\lambda_{G}$ of a locally compact group $G$ is clearly a closed subdual of $\hat{G}$. If $\hat{G}_r = N^\perp$ for some $N \in \mathcal{N}_G$, then $N = \{ e \}$ and $\hat{G}_r = \hat{G}$. Therefore every group $G$, which belongs to $[H]$, has to be amenable.

We recall that the (standard) topology on $\hat{G}$ is induced by the Jacobson topology on the primitive ideal space Prim$(G)$ of the group $C^*$-algebra $C^*(G)$ of $G$ via the mapping $\pi \to \text{ker}_{C^*(G)} \pi$. Let $\text{Prim}_r L^1(G)$ denote the space of kernels in $L^1(G)$ of topologically irreducible $*$-representations of $L^1(G)$ in Hilbert spaces. $\text{Prim}_r L^1(G)$ is also a topological space with the Jacobson topology and the mapping $\pi \to \text{ker}_{L^1(G)} \pi$ defines a second topology on $\hat{G}$, which we call the $L^1$-hull-kernel-topology. This topology is weaker than the standard one and in general both topologies are different. Both topologies coincide if and only if the canonical continuous and surjective mapping $\Psi: \text{Prim}(G) \to \text{Prim}_r L^1(G)$, given by $\Psi(I) = I \cap L^1(G)$, is a homeomorphism, i.e. if $G$ is $*$-regular.

**Definition.** Let $\mathcal{S}^*_G \subset \mathcal{S}_G$ be the set of subduals of $\hat{G}$, which are closed in the $L^1$-hull-kernel-topology.

The main result of our paper will be that $\mathcal{S}_G^*$ is the exact image of the mapping $N \to N^\perp$ for general locally compact groups. The results of Helgaason and Hauenschild will be an easy consequence. But first we need the following

**Proposition.** For every unitary representation of $G$ in a Hilbert space $\mathcal{H}_\pi$ we have $\text{ker}_{L^1(G)} \pi \otimes \bar{\pi} \subset \text{ker}_{L^1(G)} 1_G$.

**Proof.** Let $\mathcal{H}_\pi^*$ be the adjoint space of $\mathcal{H}_\pi$ and denote by $\bar{\eta}$ the vector $\eta \in \mathcal{H}_\pi$ considered as element of $\overline{\mathcal{H}_\pi}$. Then $\bar{\pi}$ is the representation $\pi$ considered as a representation acting in $\overline{\mathcal{H}_\pi}$. We fix a unit vector $\xi \in \mathcal{H}_\pi$ and an orthonormal basis $\{ \xi_i \}_{i \in I}$ of $\mathcal{H}_\pi$. Then for all $x \in G$ we have $\langle \bar{\pi}(x)\xi, \xi_i \rangle = \langle \pi(x)\xi, \xi_i \rangle$ and we obtain for all $x \in G$

$$1 = \langle \xi, \xi \rangle = \langle \pi(x)\xi, \pi(x)\xi \rangle = \sum_{i \in I} \langle \pi(x)\xi, \xi_i \rangle \langle \bar{\pi}(x)\xi, \bar{\xi}_i \rangle.$$ 

Let $\mathcal{F}$ denote the family of all finite sums of the functions $\langle \pi(x)\xi, \xi_i \rangle \langle \bar{\pi}(x)\xi, \bar{\xi}_i \rangle$, which are matrix-coefficients of $\pi \otimes \bar{\pi}$. If $\varphi \in \mathcal{F}$ then $\varphi$ is continuous and $0 \leq \varphi \leq 1$. Furthermore $1 = \sup_{\varphi \in \mathcal{F}} \varphi$. 
Assume now that $f \in \text{kern}_{L^1(G)} \pi \otimes \overline{\pi}$. Then $\int_G f(x) \varphi(x) \, dx = 0$ for all $\varphi \in \mathcal{F}$. Given $\varepsilon > 0$ choose a compact set $\mathcal{K} \subset G$ such that $\int_{G \setminus \mathcal{K}} |f(x)| \, dx \leq \varepsilon/2$. By Dini there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}$ (depending on $\mathcal{K}$), such that $1 = \lim_{n \to \infty} \varphi_n$ uniformly on $\mathcal{K}$. Then

$$\left| \int_G f(x) \, dx \right| = \lim_{n \to \infty} \left| \int_{\mathcal{K}} f(x) \varphi_n(x) \, dx + \int_{G \setminus \mathcal{K}} f(x) \, dx \right|$$

$$= \lim_{n \to \infty} \left| \int_{G \setminus \mathcal{K}} f(x) \, dx - \int_{G \setminus \mathcal{K}} f(x) \varphi_n(x) \, dx \right|$$

$$\leq 2 \int_{G \setminus \mathcal{K}} |f(x)| \, dx \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\int_G f(x) \, dx = 0$.

The following corollary generalizes a result of E. Kaniuth (see [9], Lemma 1):

**Corollary 1.** Let $G$ be a *-regular locally compact group. Then for every unitary representation $\pi$ of $G$ $\pi \otimes \overline{\pi}$ weakly contains the trivial representation, i.e. every *-regular group has the property (TP 1) of [9].

**Proof.** For *-regular groups $\text{kern}_{L^1(G)} \pi \star \overline{\pi} \subset \text{kern}_{L^1(G)} 1_G$ implies that $1_G$ is weakly contained in $\pi \otimes \overline{\pi}$.

**Remark.** Corollary 1 shows that a quite big class of amenable groups has the weak Frobenius property (TP 1). This supports the conjecture that all amenable groups have the property (TP 1).

**Theorem.** For every locally compact group $G$ the mapping $N \to N^\perp$ is a bijection from $\mathcal{N}_G^\perp$ onto $\mathcal{S}_G^\ast$.

**Proof.** As we remarked above, the mapping $N \to N^\perp$ is an injection from $\mathcal{N}_G^\perp$ into $\mathcal{S}_G$. If $N \in \mathcal{N}_G^\perp$, then $N^\perp$ corresponds to the set of topological irreducible *-representations of $L^1(G)$, which are trivial on the kernel of the canonical homomorphisms from $L^1(G)$ onto $L^1(G/N)$. Therefore $N^\perp \in \mathcal{S}_G^\ast$, and we only have to prove that every set $\mathcal{S} \in \mathcal{S}_G^\ast$ is of the form $N^\perp$ for some $N \in \mathcal{G}_G$.

First observe that by the proposition every $S \in \mathcal{S}_G^\ast$ has the following properties:

(i) $S$ contains $1_G$ and $\pi \in S$ implies $\overline{\pi} \in S$.
(ii) for all $\pi, \rho \in S$ the support of $\pi \ast \rho$ is in $S$. 


Furthermore since \( S^\perp = \{ x \in G|\pi(x) = 1|_{\mathcal{X}_G} \text{ for all } \pi \in S \} \) is a closed normal subgroup of \( G \), we can consider \( S \) as a subdual of \( (G/S^\perp)^\vee \), which separates the points of \( G/S^\perp \) and is closed in the \( L^1 \)-hull-kernel-topology.

It is therefore sufficient to prove that a set \( S \in \mathcal{P}_{\hat{G}} \), which separates the points of \( G \), is equal to \( \hat{G} \).

Let \( S \) be such a set and let \( \mathcal{P} \) be the set of unitary representations of \( G \), which have support in \( S \). Since \( S \) is closed in \( \hat{G} \), \( S \) and \( \mathcal{P} \) are weakly equivalent sets of representations of \( G \). Since \( S \) has properties (i) and (ii) above, \( \mathcal{P} \) contains the trivial representation and is closed under the tensor product and under conjugation. It follows by a Stone-Weierstraß argument (see [1], Theorem) that \( \mathcal{P} \) is \( L^1 \)-separating, i.e. if \( f \in L^1(G) \) and \( \pi(f) = 0 \) for all \( \pi \in \mathcal{P} \), then \( f = 0 \). But then also \( S \) is \( L^1 \)-separating, i.e. its kernel in \( L^1(G) \) is the trivial ideal \( \{0\} \). Since \( S \) is closed in the \( L^1 \)-hull-kernel-topology, we obtain \( S = \hat{G} \).

**COROLLARY 2.** A locally compact group belongs to the class \([H]\) if and only if \( \mathcal{P}_G = \mathcal{P}_{\hat{G}}^\ast \). Especially every *-regular locally compact group belongs to \([H]\) and every locally compact group in \([H]\) is amenable.

**REMARK.** Let \( G \) be a locally compact group, such that all quotients \( G/N \) are \( C^* \)-unique, i.e. \( L^1(G/N) \) has a unique \( C^* \) -norm (see [5]). The same arguments as in the proof of the theorem give that \( G \) belongs to \([H]\). We do not know whether this class of groups is really bigger than the class of *-regular groups.

The following is known about *-regular groups:

(A) Every *-regular group is amenable (see [2]).

(B) All groups \( G \) with polynomially growing Haar measure are *-regular (see [2]).

(C) All semidirect product \( G = H \rtimes N \) with abelian \( H \) and \( N \) are *-regular (see [4]).

(D) A connected group \( G \) is *-regular if and only if all \( I \in \text{Prim}(G) \) are polynomially induced (see [3]).

It follows from the classification of Moore groups given by C. C. Moore in [10], that all Moore groups have polynomial growth and so are *-regular by (B). Therefore the result of W. Hauenschild is an immediate consequence of the Corollary 2 and (B). It should be noted that the proofs of the results of Pontryagin, Helgason and Hauenschild depend explicitly or implicitly on the fact that the groups under consideration are *-regular. Besides this they make use of central theorems as the Pontryagin duality theorem, the Peter-Weyl theorem or structure theorems for Moore groups, which are specific for these classes of groups.
Recently E. Kaniuth proved by quite different methods that a big class of amenable groups, including the almost connected amenable groups, belong to $[H]$. (Cf. E. Kaniuth, *Weak containment and tensor products of group representations*. II, Math. Ann., 270 (1985), 1–15.) There seems to be some hope that the class $[H]$ coincides with the class of amenable groups.

**References**


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