A GALOIS-CORRESPONDENCE FOR GENERAL LOCALLY
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We give a characterization in terms of $G$ of those parts in the unitary dual of a locally compact group $G$, which correspond to closed normal subgroups of $G$. These are exactly the sets $S \subset \hat{G}$, which have the property that for all $\pi, \rho \in S$ the support of $\pi \otimes \bar{\rho}$ is contained in $S$ and which are closed in a topology on $\hat{G}$, which is in general weaker than the standard topology on $\hat{G}$, and which we call the $L^1$-hull-kernel-topology. As an easy consequence we obtain that for $*$-regular groups $G$ the mapping $N \to N^\perp = \{ \pi \in \hat{G} | \pi|_N = 1 |_{\pi^*} \}$ is a bijection from the set of closed normal subgroups of $G$ onto the set of closed subsets $S \subset \hat{G}$ with the property that $\pi \otimes \bar{\rho}$ has support in $S$ for all $\pi, \rho \in S$. This generalizes and unifies results of Pontryagin, Helgason and Hauenschild, with a considerably simplified proof. Furthermore we prove that $*$-regular groups have the weak Frobenius property (TP 1), i.e. $1_G$ is weakly contained in $\pi \otimes \bar{\pi}$ for all unitary representations $\pi$ of $G$, generalizing a result of E. Kaniuth.

Let $G$ be a locally compact group with unitary dual $\hat{G}$ and let $\mathcal{N}_G$ denote the set of closed normal subgroups of $G$. To every $N \in \mathcal{N}_G$ corresponds a canonical subset of $\hat{G}$, namely the annihilator $N^\perp = \{ \pi \in \hat{G} | \pi|_N = 1 |_{\pi^*} \}$ of $N$. By the Gelfand-Raikov theorem $N \to N^\perp$ is an injective mapping from $\mathcal{N}_G$ into the subsets of $\hat{G}$ and it is an important problem in harmonic analysis to describe the image of this mapping in terms of $\hat{G}$.

**Definition.** A nontrivial subset $S$ of $\hat{G}$ is called a subdual of $\hat{G}$, if for all $\pi, \rho \in S$ the tensor product $\pi \otimes \bar{\rho}$ of $\pi$ and the conjugate $\bar{\rho}$ of $\rho$ has support in $S$. We denote by $\mathcal{S}_G$ the set of closed subduals of $\hat{G}$.

It is clear that $N \to N^\perp$ is an injective mapping from $\mathcal{N}_G$ into $\mathcal{S}_G$. Let $[H]$ be the class of locally compact groups $G$, for which $N \to N^\perp$ is a surjection onto $\mathcal{S}_G$.

As a well known consequence of the duality theorem of Pontryagin one obtains that all abelian locally compact groups belong to $[H]$ (see for example [6], Chap. II, §1.7). S. Helgason proved in [8], Theorem 1, that all compact groups belong to $[H]$. It was then W. Hauenschild, who generalized and unified these results in [7], and proved that all Moore groups,
i.e. all locally compact groups $G$, which have only finite dimensional irreducible unitary representations, belong to the class $[H]$. On the other hand the support $\hat{G}_r$ of the left regular representation $\lambda_G$ of a locally compact group $G$ is clearly a closed subdual of $\hat{G}$. If $\hat{G}_r = N^\perp$ for some $N \in \mathcal{M}_G$, then $N = \{e\}$ and $\hat{G}_r = \hat{G}$. Therefore every group $G$, which belongs to $[H]$, has to be amenable.

We recall that the (standard) topology on $\hat{G}$ is induced by the Jacobson topology on the primitive ideal space $\text{Prim}(G)$ of the group $C^*$-algebra $C^*(G)$ of $G$ via the mapping $\pi \mapsto \ker_{C^*(G)} \pi$. Let $\text{Prim}_* L^1(G)$ denote the space of kernels in $L^1(G)$ of topologically irreducible *-representations of $L^1(G)$ in Hilbert spaces. $\text{Prim}_* L^1(G)$ is also a topological space with the Jacobson topology and the mapping $\pi \mapsto \ker_{L^1(G)} \pi$ defines a second topology on $\hat{G}$, which we call the $L^1$-hull-kernel-topology. This topology is weaker than the standard one and in general both topologies are different. Both topologies coincide if and only if the canonical continuous and surjective mapping $\Psi: \text{Prim}(G) \to \text{Prim}_* L^1(G)$, given by $\Psi(I) = I \cap L^1(G)$, is a homeomorphism, i.e. if $G$ is *-regular.

**Definition.** Let $\mathcal{L}_G^* \subset \mathcal{L}_G$ be the set of subduals of $\hat{G}$, which are closed in the $L^1$-hull-kernel-topology.

The main result of our paper will be that $\mathcal{L}_G^*$ is the exact image of the mapping $N \to N^\perp$ for general locally compact groups. The results of Helgasson and Hauenschild will be an easy consequence. But first we need the following

**Proposition.** For every unitary representation of $G$ in a Hilbert space $\mathcal{H}_\pi$ we have $\ker_{L^1(G)} \pi \otimes \bar{\pi} \subset \ker_{L^1(G)} 1_G$.

**Proof.** Let $\overline{\mathcal{H}_\pi}$ be the adjoint space of $\mathcal{H}_\pi$ and denote by $\bar{\eta}$ the vector $\eta \in \mathcal{H}_\pi$ considered as element of $\overline{\mathcal{H}_\pi}$. Then $\bar{\pi}$ is the representation $\pi$ considered as a representation acting in $\overline{\mathcal{H}_\pi}$. We fix a unit vector $\xi \in \mathcal{H}_\pi$ and an orthonormal basis $\{\xi_i\}_{i \in I}$ of $\mathcal{H}_\pi$. Then for all $x \in G$ we have $\langle \bar{\pi}(x) \bar{\xi}, \bar{\xi}_i \rangle = \langle \pi(x) \xi, \xi_i \rangle$ and we obtain for all $x \in G$

$$1 = \langle \xi, \xi \rangle = \langle \pi(x) \xi, \pi(x) \xi \rangle = \sum_{i \in I} \langle \pi(x) \xi, \xi_i \rangle \langle \bar{\pi}(x) \bar{\xi}, \bar{\xi}_i \rangle.$$

Let $\mathcal{F}$ denote the family of all finite sums of the functions

$$\langle \pi(x) \xi, \xi_i \rangle \langle \bar{\pi}(x) \bar{\xi}, \bar{\xi}_i \rangle,$$

which are matrix-coefficients of $\pi \otimes \bar{\pi}$. If $\varphi \in \mathcal{F}$ then $\varphi$ is continuous and $0 \leq \varphi \leq 1$. Furthermore $1 = \sup_{\varphi \in \mathcal{F}} \varphi$. 

Assume now that $f \in \ker L^1(G) \pi \otimes \bar{\pi}$. Then $\int_G f(x) \varphi(x) \, dx = 0$ for all $\varphi \in \mathcal{F}$. Given $\varepsilon > 0$ choose a compact set $\mathcal{K} \subset G$ such that $\int_{\mathcal{K}^c} |f(x)| \, dx \leq \varepsilon/2$. By Dini there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{F}$ (depending on $\mathcal{K}$), such that $1 = \lim_{n \to \infty} \varphi_n$ uniformly on $\mathcal{K}$. Then

$$\left| \int_G f(x) \, dx \right| = \lim_{n \to \infty} \left| \int_{\mathcal{K}} f(x) \varphi_n(x) \, dx + \int_{\mathcal{K}^c} f(x) \, dx \right|$$

$$= \lim_{n \to \infty} \left| \int_{\mathcal{K} \setminus \mathcal{K}} f(x) \, dx - \int_{\mathcal{K} \setminus \mathcal{K}} f(x) \varphi_n(x) \, dx \right|$$

$$\leq 2 \int_{\mathcal{K} \setminus \mathcal{K}} |f(x)| \, dx \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain $\int_G f(x) \, dx = 0$.

The following corollary generalizes a result of E. Kaniuth (see [9], Lemma 1):

**Corollary 1.** Let $G$ be a $*$-regular locally compact group. Then for every unitary representation $\pi$ of $G$ $\pi \otimes \bar{\pi}$ weakly contains the trivial representation, i.e. every $*$-regular group has the property (TP 1) of [9].

**Proof.** For $*$-regular groups $\ker L^1(G) \pi \otimes \bar{\pi} \subset \ker L^1(G) 1_G$ implies that $1_G$ is weakly contained in $\pi \otimes \bar{\pi}$.

**Remark.** Corollary 1 shows that a quite big class of amenable groups has the weak Frobenius property (TP 1). This supports the conjecture that all amenable groups have the property (TP 1).

**Theorem.** For every locally compact group $G$ the mapping $N \to N^\perp$ is a bijection from $\mathcal{N}_G$ onto $\mathcal{I}_G^\ast$.

**Proof.** As we remarked above, the mapping $N \to N^\perp$ is an injection from $\mathcal{N}_G$ into $\mathcal{I}_G$. If $N \in \mathcal{N}_G$, then $N^\perp$ corresponds to the set of topological irreducible $*$-representations of $L^1(G)$, which are trivial on the kernel of the canonical homomorphisms from $L^1(G)$ onto $L^1(G/N)$. Therefore $N^\perp \in \mathcal{I}_G^\ast$, and we only have to prove that every set $\mathcal{I} \in \mathcal{I}_G^\ast$ is of the form $N^\perp$ for some $N \in \mathcal{N}_G$.

First observe that by the proposition every $S \in \mathcal{I}_G^\ast$ has the following properties:

(i) $S$ contains $1_G$ and $\pi \in S$ implies $\bar{\pi} \in S$.
(ii) for all $\pi, \rho \in S$ the support of $\pi \ast \rho$ is in $S$. 
Furthermore since $S^\perp = \{ x \in G | \pi(x) = 1 |_{\mathcal{F}} \text{ for all } \pi \in S \}$ is a closed normal subgroup of $G$, we can consider $S$ as a subdual of $(G/S^\perp)^\wedge$, which separates the points of $G/S^\perp$ and is closed in the $L^1$-hull-kernel-topology.

It is therefore sufficient to prove that a set $S \in \mathcal{S}_G^*$, which separates the points of $G$, is equal to $\hat{G}$.

Let $S$ be such a set and let $\mathcal{P}$ be the set of unitary representations of $G$, which have support in $S$. Since $S$ is closed in $\hat{G}$, $S$ and $\mathcal{P}$ are weakly equivalent sets of representations of $G$. Since $S$ has properties (i) and (ii) above, $\mathcal{P}$ contains the trivial representation and is closed under the tensor product and under conjugation. It follows by a Stone-Weierstraß argument (see [1], Theorem) that $\mathcal{P}$ is $L^1$-separating, i.e. if $f \in L^1(G)$ and $\pi(f) = 0$ for all $\pi \in \mathcal{P}$, then $f = 0$. But then also $S$ is $L^1$-separating, i.e. its kernel in $L^1(G)$ is the trivial ideal $\{0\}$. Since $S$ is closed in the $L^1$-hull-kernel-topology, we obtain $S = \hat{G}$.

**Corollary 2.** A locally compact group belongs to the class $[H]$ if and only if $\mathcal{S}_G^* = \mathcal{P}_G^*$. Especially every $\ast$-regular locally compact group belongs to $[H]$ and every locally compact group in $[H]$ is amenable.

**Remark.** Let $G$ be a locally compact group, such that all quotients $G/N$ are C*-unique, i.e. $L^1(G/N)$ has a unique C*-norm (see [5]). The same arguments as in the proof of the theorem give that $G$ belongs to $[H]$. We do not know whether this class of groups is really bigger than the class of $\ast$-regular groups.

The following is known about $\ast$-regular groups:

(A) Every $\ast$-regular group is amenable (see [2]).

(B) All groups $G$ with polynomially growing Haar measure are $\ast$-regular (see [2]).

(C) All semidirect product $G = H \rtimes N$ with abelian $H$ and $N$ are $\ast$-regular (see [4]).

(D) A connected group $G$ is $\ast$-regular if and only if all $I \in \text{Prim}(G)$ are polynomially induced (see [3]).

It follows from the classification of Moore groups given by C. C. Moore in [10], that all Moore groups have polynomial growth and so are $\ast$-regular by (B). Therefore the result of W. Hauenschild is an immediate consequence of the Corollary 2 and (B). It should be noted that the proofs of the results of Pontryagin, Helgason and Hauenschild depend explicitly or implicitly on the fact that the groups under consideration are $\ast$-regular. Besides this they make use of central theorems as the Pontryagin duality theorem, the Peter-Weyl theorem or structure theorems for Moore groups, which are specific for these classes of groups.
Recently E. Kaniuth proved by quite different methods that a big class of amenable groups, including the almost connected amenable groups, belong to $[H]$. (Cf. E. Kaniuth, *Weak containment and tensor products of group representations*. II, Math. Ann., 270 (1985), 1–15.) There seems to be some hope that the class $[H]$ coincides with the class of amenable groups.

**REFERENCES**


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