ON BANACH SPACES HAVING A RADON-NIKODYM DUAL

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The purpose of this paper is to prove a new characterisation of Banach spaces having a Radon-Nikodym dual, namely that if $E$ is a Banach space, then $E'$ has the Radon-Nikodym property if and only if there exists an equivalent norm on $E$ such that for each $E$-valued measure $m$ of bounded variation, there exists an $E'$-valued function $f$ with norm 1 $|m|$-a.e. such that $m(A) = \int_A f \, dm$ for each $A$ in $\mathcal{A}$.

1. Introduction. In [1], we have proved that if $E$ is a Banach space, $m$ an $E$-valued measure defined on a $\sigma$-algebra $\mathcal{A}$ of subsets of a set $T$, with bounded variation $|m|$, and if $\varepsilon$ is any positive number, then there exists an $E'$-valued strongly measurable function $f$ defined on the set $T$, such that $\|f\| < 1 + \varepsilon$ and $|m|(A) = \int_A f \, dm$ for each $A$ in $\mathcal{A}$.

A very natural question which arises is the following: Does there always exist an $E'$-valued strongly measurable function with norm 1 such that $|m|(A) = \int_A f \, dm$ for each $A$ in $\mathcal{A}$? Following the example given in [1], this seems to be possible.

Finally, an answer to that question was provided by F. Delbaen who proved the following unpublished theorem: If $E$ is a Banach space, the following are equivalent:

(a) $E'$ has the Radon-Nikodym property
(b) For each equivalent norm on $E$, for each $E$-valued measure $m$ of bounded variation defined on a $\sigma$-algebra $\mathcal{A}$ of subsets of a set $T$, there exists a $|m|$-strongly measurable function $f$ from $T$ to $E'$ such that $\|f\| = 1$ $|m|$-a.e. and $|m|(A) = \int_A f \, dm$ for each $A$ in $\mathcal{A}$.

The purpose of this paper is to provide a positive answer to the following question: Is it possible to weaken assertion (b) by requiring the existence of an equivalent norm on the space having the property instead of assuming it for each equivalent norm on $E$.

2. Proof of the theorem. Before proving our theorem let us recall the Mazur density theorem and prove two lemmas.
THEOREM (Mazur density theorem [5] p. 171). If \( E \) is a separable Banach space, then for each equivalent norm on \( E \), the set of smooth points of the unit sphere of \( E \) is dense in the unit sphere.

**Lemma 1.** Let \( E \) be a Banach space such that \( E' \) is not separable, \( B \) a dense subset of \( S(E) = \{ x| x \in E, \|x\| = 1 \} \) and \( \varepsilon > 0 \). If we denote by \( \Omega \) the first uncountable ordinal and by \( S \) the set \( \{ i|i < \Omega \} \), then for each \( i \) in \( S \), there exists \( x_i \) in \( B \) and \( x'_i \) in \( S(E') \), the unit sphere of \( E' \) such that \( x'_i(x_i) = 1 \) and \( \|x'_i - x'_j\| > 1 - \varepsilon \) if \( i \neq j \).

**Proof.** Let \( i \) in \( S \) and suppose that the families \( (x_j) \) and \( (x'_j) \) are chosen for \( j < i \).

As \( E' \) is not separable, \( \bigcap_{j<i} \text{Ker} \ x'_j \neq \{0\} \).

Let \( x \in S(E) \cap \bigcap_{j<i} \text{Ker} \ x'_j \) and choose \( x_i \) in \( B \) such that \( \|x - x_i\| < \varepsilon \). Now, if we choose \( x'_i \) in \( S(E') \) such that \( x'_i(x_i) = 1 \) it is easy to see that we are done.

\[ \|x'_i - x'_j\| > 1 - \varepsilon \] follows from the fact that if \( j < i \), \( (x'_i - x'_j)(x_i) > 1 - \varepsilon \).

**Lemma 2.** For the same set \( S \) as in Lemma 1, there exists a positive scalar measure \( \mu \) on the \( \sigma \)-algebra \( \mathcal{P}(S) \) of the subsets of \( S \) such that \( \mu(S) = 1 \) and \( \mu(A) = 0 \) if \( A \) is countable.

**Proof.** Let \( i \) in \( S \) and define \( \mu_i \) as the evaluation measure at the point \( i \). As the set of measures on the \( \sigma \)-algebra of the subsets of \( S \) is the dual of the space of continuous bounded functions on \( S \) for a locally convex topology, the family of measures has a cluster point which is a measure satisfying our requirement.

We are now ready for the proof of the following

**Theorem.** For any Banach space \( E \), the following are equivalent:

1. \( E' \) has the Radon-Nikodym property.
2. For each equivalent norm on \( E \), for each \( E \)-valued measure \( m \) of bounded variation defined on a \( \sigma \)-algebra \( \mathcal{A} \) of subsets of a set \( T \), there exists a function \( f \) from \( T \) into \( E' \) \( |m| \)-strongly measurable such that \( \|f(t)\| = 1 \) \( |m| \)-a.e. and \( m(A) = \int_A f \ dm \) for each \( A \) in \( \mathcal{A} \).
3. There exists an equivalent norm on \( E \) such that for each \( E \)-valued measure \( m \) of bounded variation defined on a \( \sigma \)-algebra \( \mathcal{A} \) of subsets of a set \( T \), there exists a function \( f \) from \( T \) into \( E' \) \( |m| \)-strongly measurable such that \( \|f(t)\| = 1 \) \( |m| \)-a.e. and \( m(A) = \int_A f \ dm \) for each \( A \) in \( \mathcal{A} \).
Proof. (1) ⇒ (2) It follows from the theorem we proved in [1] that for each integer \( n \), there exists a function \( f_n \) from \( T \) into \( E' \) such that \( f_n \) is \(|m|-\)strongly measurable, \( 1 \leq \|f_n(t)\| < 1 + 1/n \) and \( |m|(A) = \int_A f_n dm \) for each \( A \) in \( \mathcal{A} \).

Let \( G \) be the Banach subspace of \( E' \) generated by \( \bigcup_{n=1}^{\infty} f_n(T) \).

As \( G \) is separable and \( E' \) has Radon-Nikodym property, there exists a Banach space \( F \) such that \( F' \) is separable and \( G \subseteq F' \) ([3]). Let \( f \) be a pointwise \( \sigma(F', F) \)-cluster point of the sequence \( (f_n) \). \( f \) is \( G \)-valued, thus \( E' \)-valued.

It is clear that \( \|f\| \leq 1 \) and that \( f \) is \( \sigma(F', F) \)-measurable. As \( |m|(A) = \int_A f dm \) for each \( A \) in \( \mathcal{A} \), if we prove that \( f \) is strongly measurable, the norm of \( f \) will be greater than 1 and our assertion will be proved.

Let \( m_0 \) from \( \mathcal{A} \) into \( F' \) defined by \( m_0(A)(y) = \int_A \langle f, y \rangle d|m| \).

It is clear that \( m_0 \) is a measure with finite variation and that \( |m_0| = |m| \).

As \( F \) has the Radon-Nikodym property, there exists a measurable function \( g \) from \( T \) into \( F' \) such that \( m_0(A) = \int_A g d|m| \) for each \( A \) in \( \mathcal{A} \).

It follows that if \( y \in F \), \( m_0(A)(y) = \int_A \langle g, y \rangle d|m| \) which shows that \( \langle g, y \rangle = \langle f, y \rangle, |m|-\text{a.e.} \) for each \( y \) in \( F \).

As \( F \) is separable, it follows that \( f = g \ |m|-\text{a.e.} \) and that \( f \) is strongly measurable which proves the first assertion.

As (2) ⇒ (3) is obvious, it remains to show that

(3) ⇒ (1) It is easy to prove that if property (3) is satisfied for \( E \) it is also satisfied for each Banach subspace of \( E \). Now we have to prove that each separable subspace of \( E \) has a separable dual, we only have to prove that if a separable Banach space satisfies (3), it has a separable dual.

Let us suppose that there exists a separable Banach space \( E \) satisfying property (3) and such that \( E' \) is not separable. Let \( B \) be the set of smooth points of the unit sphere \( S(E) \) of \( E \) which is dense in \( S(E) \) by Mazur density theorem, \( \varepsilon = 1/4 \) and apply Lemma 1.

We define the function \( f \) from \( S \) to \( E \) by \( f(i) = x_i \). If \( \mathcal{A} \) is defined as the set of inverse images by \( f \) of the open subsets of \( S(E) \), the function \( f \) is strongly measurable. Let us choose on \( \mathcal{A} \) a positive scalar measure \( \mu \) such that \( \mu(S) = 1 \) and \( \mu(A) = 0 \) if \( A \) is countable. Such a \( \mu \) exists by Lemma 2. Now we define \( m \) from \( \mathcal{A} \) to \( E \) by \( m(A) = \int_A f d\mu \).

\( m \) is clearly a measure of bounded variation and \( |m| = \mu \). So there exists a function \( g \) from \( S \) into \( E' \) which is \( \mu \)-strongly measurable, \( \|g\| = 1 \) \( \mu \)-a.e. and \( \mu(A) = \int_A g dm \) for each \( A \) in \( \mathcal{A} \).

It follows that \( \mu(A) = \int_A \langle f, g \rangle d\mu \) for each \( A \) in \( \mathcal{A} \) and that \( \langle f, g \rangle = 1 \) \( \mu \)-a.e. 

Proof. (1) ⇒ (2) It follows from the theorem we proved in [1] that for each integer \( n \), there exists a function \( f_n \) from \( T \) into \( E' \) such that \( f_n \) is \(|m|-\)strongly measurable, \( 1 \leq \|f_n(t)\| < 1 + 1/n \) and \( |m|(A) = \int_A f_n dm \) for each \( A \) in \( \mathcal{A} \).
So there exists a $\mu$-negligible subset $N$ of $S$ such that $g(i)(f(i)) = 1$ if $i \notin N$ and $g(S \setminus N)$ is separable. If $i \notin N$, $g(i)(f(i)) = g(i)(x_i) = 1$.

As $x_i$ is a smooth point and $\|g(i)\| = 1$, $g(i) = x_i$.

It follows that $\|g(i) - g(j)\| \geq 1 - \epsilon = 3/4$ for $i \neq j$ in $S \setminus N$ which shows that $g(S \setminus N)$ is discrete.

As it is separable, it has to be countable. So $S \setminus N$ has to be countable which is impossible.

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