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Our main objective is to prove the existence of infinitely many pairs (u_1, u_2) of positive solutions of quasilinear elliptic differential equations

$$(1.1)\qquad \Delta u - q(|x|)u = f(x, u, \nabla u), \qquad x \in \Omega_\alpha,$$

throughout exterior domains $\Omega_\alpha \subset \mathbf{R}^N$, $N \geq 2$, of the type

$$\Omega_\alpha = \{x \in \mathbf{R}^N : |x| > \alpha\}, \qquad \alpha > 0,$$

where $x = (x_1, \ldots, x_N)$, $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$, and $\Delta = \nabla \cdot \nabla$. Each pair has the property that $u_1(x)/u_2(x)$ has uniform limit zero in Ω_α as $|x| \to \infty$. In particular, if $q(t) \equiv 0$ and $N \geq 3$, $u_1(x)$ has limit 0 as $|x| \to \infty$, and $u_2(x)$ is bounded above and below by positive constants in Ω_α.

1. The function q in (1.1) is assumed to be nonnegative and locally Hölder continuous in $\mathbf{R}_+ = [0, \infty)$, and $f\colon \Omega_\alpha \times \mathbf{R}_+ \times \mathbf{R}^N \to \mathbf{R}$ is locally Hölder continuous in $\Omega_\alpha \times \mathbf{R}_+ \times \mathbf{R}^N$ and satisfies a Nagumo condition. Detailed hypotheses are listed in §3.

Specific asymptotic estimates for the growth (decay) of the solutions $u_1(x)$, $u_2(x)$ as $|x| \to \infty$ follow easily from our construction. In particular sufficient conditions are given for the quasilinear equation

$$\Delta u - |x|^{2r} u = \phi(x) u^\gamma + \psi(x)|\nabla u|^\beta, \qquad x \in \Omega_\alpha,$$

for constants $r \geq 0$, $\gamma \geq 0$, $0 \leq \beta \leq 2$, to have positive solutions $u_1(x)$, $u_2(x)$ in Ω_α such that $u_i(x)$ is bounded above and below by positive constant multiples of

$$|x|^{-\lambda} \exp\left[(-1)^i |x|^{r+1}/(r+1)\right], \qquad x \in \Omega_\alpha,\ i = 1, 2,$$

where $\lambda = (N + r - 1)/2$.

By a similar method we also prove the existence of infinitely many positive solutions of the boundary value problem

$$(1.2)\qquad \begin{aligned} &\Delta u - q(|x|)u = f(x, u, \nabla u), \qquad x \in \Omega_\alpha, \\ &u|_{\partial\Omega_\alpha} = 0 \end{aligned}$$

under the same hypotheses as for (1.1).

The sharpness of our results is indicated by known oscillation criteria for semilinear elliptic equations [**6**, **10**]. For example, in the case $\psi(x) = 0$ in equation (3.26), our conditions (3.27)–(3.30) are all known to be necessary and sufficient conditions for the existence of a positive solution of (3.26), of the type described in Corollaries 3.8–3.11, in some exterior domain [**11**].

Our procedure will be to construct solutions of (1.1) or (1.2) which are squeezed between subsolutions and supersolutions. The latter are obtained as spherically symmetric solutions of elliptic equations with $f(x, u, \nabla u)$ in (1.1) replaced by radial majorants. We therefore begin by proving global existence theorems for quasilinear ordinary differential equations (2.1) below. These results have independent interest, and in fact are of quite general nature. The procedure will be to find solutions of integro-differential equations as fixed points of associated operators from closed convex subsets $\mathscr{Y}$ of $C^1[t_0, \infty)$ into subsets of $\mathscr{Y}$ with compact closure.

In the case of *semilinear* equations $\Delta u = f(x, u)$, in which $q(t) \equiv 0$, global existence theorems for boundary value problems in exterior domains have been obtained by Kawano and Naito [**2**], Kenig and Ni [**3**], and Noussair and Swanson [**8**]. Existence of positive bounded solutions in the entire space $\mathbf{R}^N$, $N \geq 3$, have been proved by Kawano [**1**], Kusano and Oharu [**4**], and Ni [**5**].

2. Positive solutions of quasilinear ordinary differential equations. Existence theorems will be proved for quasilinear ordinary differential equations of the type

$$(2.1) \qquad Ly = h(t, y, y'), \qquad t \geq t_0 > 0,$$

where L is the linear differential operator defined by

$$(2.2) \qquad Lz = \frac{1}{p_2(t)} \frac{d}{dt}\left[\frac{1}{p_1(t)} \frac{d}{dt}\left(\frac{z}{p_0(t)}\right)\right], \qquad z \in C^2[t_0, \infty).$$

ASSUMPTIONS.

(A_1) Each $p_i(t)$ is positive in $[t_0, \infty)$, $p_i \in C^{2-i}[t_0, \infty)$, $i = 0, 1, 2$, and $\lim_{t\to\infty} P(t) = +\infty$, where

$$P(t) = \int_{t_0}^{t} p_1(s)\, ds.$$

(A_2) h: $[t_0, \infty) \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$, $\mathbf{R}_+ = [0, \infty)$, is continuous and satisfies

$$|h(t, y, z)| \leq H(t, |y|, |z|)$$

for all $t \in [t_0, \infty)$, $y \in \mathbf{R}_+$, $z \in \mathbf{R}$, where $H(t, u, v)$ is continuous in $[t_0, \infty) \times \mathbf{R}_+ \times \mathbf{R}_+$, nondecreasing in u for each t, v, and nondecreasing in v for each t, u.

The notation below will be used throughout:

$$p(t) = p_0(t)p_1(t)/P(t) + |p_0'(t)|. \tag{2.3}$$

For $t > t_0$, (A_1), (A_2) and (2.2) show that the linear equation $Lz = 0$ has linearly independent positive solutions

$$z_1(t) = p_0(t), \qquad z_2(t) = p_0(t)P(t),$$

which are *asymptotically ordered*, i.e., $\lim_{t\to\infty} z_1(t)/z_2(t) = 0$. The objective of this section is to establish sufficient conditions for equation (2.1) to have solutions $y_1(t)$ and $y_2(t)$ on the entire interval $[t_0, \infty)$ with the same asymptotic behavior as $z_1(t)$ and $z_2(t)$, respectively, as $t \to \infty$. We also prove the existence of a positive solution $y(t)$ of (2.1) in (t_0, ∞) satisfying the boundary condition $y(t_0) = 0$.

It is well known **[12]** that every nonoscillatory linear differential operator L of second order in $(0, \infty)$ can be written in the factorized form (2.2), where

$$p_0 = z_1, \quad p_1 = (z_2/z_1)', \quad p_2 = 1/p_0p_1 \tag{2.4}$$

for linearly independent, asymptotically ordered solutions $z_1(t)$ and $z_2(t)$ of $Lz = 0$, provided t_0 is large enough so that these solutions are positive in $[t_0, \infty)$. The functions (2.4) are unique up to multiplicative constants.

The following hypotheses are used in the theorem below:

$$\int_{t_0}^{\infty} p_2(t)P(t)H(t, ap_0(t), bp(t))\, dt < \infty; \tag{2.5}$$

$$\int_{t_0}^{\infty} p_2(t)H(t, ap_0(t)P(t), bp(t)P(t))\, dt < \infty \tag{2.6}$$

for some positive constants a and b.

THEOREM 2.1. *In addition to* (A_1), (A_2) *suppose that* $\lambda^{-1}H(t, \lambda u, \lambda v)$ *is a nondecreasing function of* $\lambda \in (0, \infty)$ *and*

$$\lim_{\lambda\to 0+} \lambda^{-1}H(t, \lambda u, \lambda v) = 0 \tag{2.7}$$

for each fixed $(t, u, v) \in [t_0, \infty) \times \mathbf{R}_+ \times \mathbf{R}_+$.

(i) *Condition* (2.5) *implies that equation* (2.1) *has infinitely many positive* (*negative*) *solutions* $y(t)$ *in* $[t_0, \infty)$ *such that* $\lim_{t\to\infty} y(t)/z_1(t)$ *exists and is positive* (*negative, respectively*).

(ii) *Condition* (2.6) *implies that equation* (2.1) *has infinitely many positive* (*negative*) *solutions* $y(t)$ *in* (t_0, ∞) *such that* $\lim_{t\to\infty} y(t)/z_2(t)$ *exists and is positive* (*negative, respectively*).

Proof of (i). Let $m = \min\{a, b\}$. The nondecreasing hypothesis on $\lambda^{-1}H(t, \lambda u, \lambda v)$ shows that

$$\begin{aligned}(2.8)\qquad &\lambda^{-1}p_2(t)P(t)H(t, \lambda p_0(t), \lambda p(t))\\ &\qquad\le m^{-1}p_2(t)P(t)H(t, mp_0(t), mp(t))\end{aligned}$$

for $0 < \lambda \le m$ and for all $t \in [t_0, \infty)$. By (A_2) and (2.5), the right member of (2.8) is integrable in (t_0, ∞), and by (2.7) the left member has limit zero as $\lambda \to 0+$ at every $t \in [t_0, \infty)$. The dominated convergence theorem then implies that

$$\lim_{\lambda\to 0+} \int_{t_0}^{\infty} \lambda^{-1}p_2(t)P(t)H(t, \lambda p_0(t), \lambda p(t))\, dt = 0,$$

and therefore a sufficiently small constant $k > 0$ exists such that

$$(2.9)\qquad \int_{t_0}^{\infty} p_2(t)P(t)H(t, 3kp_0(t), 3kp(t))\, dt \le k.$$

With this choice of k we consider

$$\begin{aligned}(2.10)\qquad \mathscr{Y} = \{y \in C^1[t_0, \infty)\colon kp_0(t) \le y(t) \le 3kp_0(t),&\\ |y'(t)| \le 3kp(t), t \ge t_0\},&\end{aligned}$$

where $C^1[t_0, \infty)$ denotes the locally convex space of all continuously differentiable functions in $[t_0, \infty)$ with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $[t_0, \infty)$. It is clear that $\mathscr{Y}$ is a closed convex subset of $C^1[t_0, \infty)$. Define the integro-differential operator M by

$$\begin{aligned}(2.11)\quad &(My)(t)\\ &= p_0(t)\left[2k + \int_t^{\infty} \left(\int_t^s p_1(\sigma)\, d\sigma\right) p_2(s)h(s, y(s), y'(s))\, ds\right],\\ &\qquad t \ge t_0.\end{aligned}$$

If $y \in \mathscr{Y}$, (2.9)–(2.11) show that

$$\left|\frac{(My)(t)}{p_0(t)} - 2k\right| \le \int_{t_0}^{\infty} p_2(s)P(s)H(s, 3kp_0(s), 3kp(s))\, ds \le k$$

and

$$|(My)'(t)| \le |p_0'(t)|\left[2k + \int_{t_0}^{\infty} p_2(s)P(s)H(s, 3kp_0(s), 3kp(s))\, ds\right]$$
$$+p_0(t)p_1(t)\int_t^{\infty} p_2(s)H(s, 3kp_0(s), 3kp(s))\, ds$$
$$\le 3k|p_0'(t)| + \frac{p_0(t)p_1(t)}{P(t)}\int_t^{\infty} p_2(s)P(s)H(s, 3kp_0(s), 3kp(s))\, ds$$
$$\le 3k|p_0'(t)| + kp_0(t)p_1(t)/P(t) < 3kp(t).$$

Therefore M maps $\mathscr{Y}$ into $\mathscr{Y}$. The continuity of M follows from (2.9) and the dominated convergence theorem. The fact that $M\mathscr{Y}$ has compact closure can be easily checked via Ascoli's theorem. It then follows from the Schauder-Tychonoff fixed point theorem that M has a fixed point $y \in \mathscr{Y}$. Differentiation of the equation $y = My$ twice shows that $y(t)$ is a positive solution of (2.1) in $[t_0, \infty)$, and clearly $\lim_{t\to\infty} y(t)/p_0(t) = 2k$ by (2.11). Since infinitely many distinct choices of k are possible in (2.9), equation (2.1) has infinitely many positive solutions $y(t)$ such that $\lim_{t\to\infty} y(t)/p_0(t)$ exists and is positive. A slight modification establishes the analogous statement for negative solutions.

Proof of (ii). It follows as in part (i) that

$$\lim_{\lambda\to 0+} \int_{t_0}^{\infty} \lambda^{-1} p_2(t)H(t, \lambda p_0(t)P(t), \lambda p(t)P(t))\, dt = 0,$$

from which a sufficiently small constant $k > 0$ exists such that

$$(2.12) \qquad \int_{t_0}^{\infty} p_2(t)H(t, 3kp_0(t)P(t), 3kp(t)P(t))\, dt \le k.$$

With k as in (2.12), we define analogues $\tilde{\mathscr{Y}}$ and $\tilde{M}$ of (2.10) and (2.11), respectively, by

$$(2.13) \quad \tilde{\mathscr{Y}} = \{y \in C^1[t_0, \infty): kp_0(t)P(t) \le y(t) \le 3kp_0(t)P(t),$$
$$|y'(t)| \le 3kp(t)P(t),\ t \ge t_0\},$$

$$(2.14) \quad (\tilde{M}y)(t)$$
$$= p_0(t)\left[2kP(t) - \int_{t_0}^{t} p_1(s)\int_s^{\infty} p_2(\sigma)h(\sigma, y(\sigma), y'(\sigma))\, d\sigma\, ds\right],$$
$$t \ge t_0.$$

If $y \in \tilde{\mathcal{Y}}$, one finds from (2.12) and (2.14) upon interchanging the order of integration that

$$\begin{aligned}&\left|\frac{(\tilde{M}y)(t)}{p_0(t)} - 2kP(t)\right|\\ &\qquad \le P(t)\int_{t_0}^{\infty} p_2(\sigma)H(\sigma, 3kp_0(\sigma)P(\sigma), 3kp(\sigma)P(\sigma))\,d\sigma\\ &\qquad \le kP(t),\end{aligned}$$

and

$$\begin{aligned}&|(\tilde{M}y)'(t)|\\ &\le |p_0'(t)|\left[2kP(t) + P(t)\int_{t_0}^{\infty} p_2(\sigma)H(\sigma, 3kp_0(\sigma)P(\sigma), 3kp(\sigma)P(\sigma))\,d\sigma\right]\\ &\qquad + p_0(t)p_1(t)\left[2k + \int_{t_0}^{\infty} p_2(\sigma)H(\sigma, 3kp_0(\sigma)P(\sigma), 3kp(\sigma)P(\sigma))\,d\sigma\right]\\ &\le 3k|p_0'(t)|P(t) + 3kp_0(t)p_1(t) = 3kp(t)P(t), \qquad t \ge t_0.\end{aligned}$$

This shows that $\tilde{M}$ maps $\tilde{\mathcal{Y}}$ into $\tilde{\mathcal{Y}}$. The remainder of the proof closely parallels that of part (i) and will be deleted.

THEOREM 2.2. *In addition to* (A_1) *and* (A_2) *suppose that* $\mu^{-1}H(t, \mu u, \mu v)$ *is a nonincreasing function of* $\mu \in (0, \infty)$ *and*

$$\lim_{\mu\to\infty} \mu^{-1}H(t, \mu u, \mu v) = 0 \tag{2.15}$$

for each fixed $(t, u, v) \in [t_0, \infty) \times \mathbf{R}_+ \times \mathbf{R}_+$. *Then both conclusions* (i) *and* (ii) *of Theorem* 2.1 *hold verbatim et literatim.*

Proof. Let $C = \max\{a, b\}$. Then, in analogy with (2.8),

$$\mu^{-1}p_2(t)P(t)H(t, \mu p_0(t), \mu p(t)) \le C^{-1}p_2(t)P(t)H(t, Cp_0(t), Cp(t))$$

for all $\mu \ge C$ and $t \in [t_0, \infty)$. It follows from (A_2), (2.5) and (2.15) exactly as in Theorem 2.1 that

$$\lim_{\mu\to\infty} \int_{t_0}^{\infty} \mu^{-1}p_2(t)P(t)H(t, \mu p_0(t), \mu p(t))\,dt = 0,$$

and hence there exists a sufficiently *large* positive constant k such that (2.9) is satisfied. Proceeding exactly as in the proof of part (i) of Theorem 2.1, we obtain a positive solution $y(t)$ of (2.1) in $[t_0, \infty)$ with $\lim_{t\to\infty} y(t)/p_0(t) = 2k > 0$ as a fixed point of the operator M in the set $\mathcal{Y}$.

To prove part (ii), it suffices to observe that

$$\lim_{\mu \to \infty} \mu^{-1} \int_{t_0}^{\infty} p_2(t) H(t, \mu p_0(t) P(t), \mu p(t) P(t))\, dt = 0.$$

This implies that (2.12) holds for sufficiently *large* positive constants k, and the proof is completed as in Theorem 2.1.

The proof of Theorem 2.1 (ii) also establishes the existence of infinitely many positive solutions of the quasilinear boundary value problem

$$\begin{aligned} &Ly = h(t, y, y'), \qquad t \geq t_0, \\ &y(t_0) = 0. \end{aligned} \tag{2.16}$$

THEOREM 2.3. *Suppose that* (A_1) *and* (A_2) *hold. Let H be as in either Theorem* 2.1 *or Theorem* 2.2. *Then the condition* (2.6) *is sufficient for the boundary value problem* (2.16) *to have infinitely many positive solutions in* (t_0, ∞).

In fact, for infinitely many admissible constants k in (2.6), Theorem 2.1 (ii) or Theorem 2.2 (ii) shows that equation (2.1) has a solution $y \in \tilde{\mathscr{Y}}$ satisfying $y(t) = (\tilde{M}y)(t)$ for $t \geq t_0$. Since $P(t_0) = 0$ by (A_1), $y(t)$ is a positive solution of (2.16) in (t_0, ∞) by (2.13) and (2.14).

3. Pairs of positive solutions of quasilinear elliptic equations. The radial component of the linear part of equation (1.1) (i.e., f replaced by 0) is

$$Lz = t^{1-N} \frac{d}{dt}\left(t^{N-1} \frac{dz}{dt}\right) - q(t) z = 0, \qquad t \geq \alpha. \tag{3.1}$$

Since (3.1) is nonoscillatory at ∞, there exist linearly independent positive solutions $z_1(t)$ and $z_2(t)$ of (3.1) on some fixed interval $[t_0, \infty)$, relabelled as $[\alpha, \infty)$, such that $\lim_{t \to \infty} z_1(t)/z_2(t) = 0$. Then the operator L has the factorized form (2.2) in $[\alpha, \infty)$, where p_0, p_1 and p_2 are defined by (2.4).

By applying the results of §2, we intend to find sufficient conditions for equation (1.1) to have positive solutions $u_1(x)$ and $u_2(x)$ in the entire domain Ω_α with the same asymptotic behavior at ∞ as the functions $z_1(|x|)$ and $z_2(|x|)$, respectively. The standing hypotheses on equation (1.1) are listed below.

ASSUMPTIONS.

(B_1) $q(t)$ is nonnegative and locally Hölder continuous in $\mathbf{R}_+$, and $f(x, u, p)$ is locally Hölder continuous in $\Omega_\alpha \times \mathbf{R}_+ \times \mathbf{R}^N$.

(B_2) For any bounded subdomain G of Ω_α and any constant $C > 0$, there corresponds a constant $\rho(G, C)$ such that

$$|f(x, u, p)| \leq \rho(G, C)\left(1 + |p|^2\right)$$

for all $x \in G$, $0 \leq u \leq C$, and $p \in \mathbf{R}^N$ (Nagumo condition).

(B_3) There exists a continuous function F: $\mathbf{R}^3_+ \to \mathbf{R}_+$ such that $F(t, u, v)$ is nondecreasing in u and v and

$$|f(x, u, p)| \leq F(|x|, |u|, |p|), \qquad x \in \Omega_\alpha, u \in \mathbf{R}_+, p \in \mathbf{R}^N.$$

(B_4) For each fixed $(t, u, v) \in [\alpha, \infty) \times \mathbf{R}_+ \times \mathbf{R}_+$, $\lambda^{-1}F(t, \lambda u, \lambda v)$ is a nondecreasing function of $\lambda \in (0, \infty)$ with $\lim_{\lambda \to 0+} \lambda^{-1}F(t, \lambda u, \lambda v) = 0$ (superlinearity).

A companion theorem will also be proved with (B_4) replaced by the sublinear hypothesis (B_4^*) below:

(B_4^*) For each fixed $(t, u, v) \in [\alpha, \infty) \times \mathbf{R}_+ \times \mathbf{R}_+$, $\mu^{-1}F(t, \mu u, \mu v)$ is a nonincreasing function of $\mu \in (0, \infty)$ with $\lim_{\mu \to \infty} \mu^{-1}F(t, \mu u, \mu v) = 0$.

The existence theorems below are proved under one of the following hypotheses:

$$\int_\alpha^\infty p_2(t) \frac{z_2(t)}{z_1(t)} F(t, az_1(t), bp(t))\, dt < \infty, \tag{3.2}$$

$$\int_\alpha^\infty p_2(t) F\left(t, az_2(t), \frac{bp(t) z_2(t)}{z_1(t)}\right) dt < \infty \tag{3.3}$$

for some positive constants a and b, where $z_1(t)$ and $z_2(t)$ are positive solutions of (3.1) as described above, and

$$p_2(t) = \left[z_1(t)\left(\frac{z_2}{z_1}\right)'(t)\right]^{-1},$$

$$p(t) = z_1(t) \frac{d}{dt} \log\left[\frac{z_2(t)}{z_1(t)}\right] + |z_1'(t)|.$$

These forms of $p_2(t)$ and $p(t)$ follow from (2.3) and (2.4).

THEOREM 3.1. *If* (B_1)–(B_3) *and either* (B_4) *or* (B_4^*) *hold, then* (3.2) *is a sufficient condition for equation* (1.1) *to have infinitely many positive solutions* $u(x)$ *such that* $u(x)/z_1(|x|)$ *is bounded above and below by positive constants in* Ω_α.

THEOREM 3.2. *If* (B_1)–(B_3) *and either* (B_4) *or* (B_4^*) *hold, then* (3.3) *is sufficient for equation* (1.1) *to have infinitely many positive solutions* $u(x)$ *such that* $u(x)/z_2(|x|)$ *is bounded above and below by positive constants in* Ω_α.

THEOREM 3.3. *If* (B_1)–(B_3) *and either* (B_4) *or* (B_4^*) *hold, then* (3.3) *is sufficient for the boundary value problem* (1.2) *to have infinitely many positive solutions in* Ω_α.

Proof of Theorem 3.1. Consider the ordinary differential equations

$$Ly = F(t, y, |y'|), \qquad t \geq \alpha, \tag{3.4}$$

$$LY = -F(t, Y, |Y'|), \quad t \geq \alpha, \tag{3.5}$$

where L is the linear operator given by (3.1). Since (3.4) and (3.5) have the form (2.1), Hypotheses (B_1), (B_3), and (B_4) (respectively, (B_4^*)) show that Theorem 2.1 (respectively, Theorem 2.2) is applicable. By (2.4), $p_1 = (z_2/z_1)'$ and $P = z_2/z_1 + \mathit{constant}$. Then condition (2.5) reduces to (3.2), and it follows from Theorem 2.1 (i) or Theorem 2.2 (i) that equations (3.4) and (3.5) have positive solutions $y(t)$ and $Y(t)$, respectively, in the interval $[\alpha, \infty)$ such that

$$\lim_{t\to\infty} \frac{y(t)}{z_1(t)} = 2k, \qquad \lim_{t\to\infty} \frac{Y(t)}{z_1(t)} = 2K$$

for all sufficiently small positive constants k and K (or all sufficiently large k and K in the case of hypothesis (B_4^*)). These constants can be chosen so that

$$0 < y(t) < Y(t) \quad \text{for all } t \geq \alpha. \tag{3.6}$$

Indeed, in the case of hypothesis (B_4), let $Y(t)$ be determined first satisfying

$$K \leq \frac{Y(t)}{z_1(t)} \leq 3K, \qquad t \geq \alpha,$$

as in the construction of Theorem 2.1 (i). Then choose $0 < k \leq K/3$ and an associated $y(t) = y(t, k)$ satisfying

$$k \leq \frac{y(t)}{z_1(t)} \leq 3k \leq K \leq \frac{Y(t)}{z_1(t)}, \qquad t \geq \alpha. \tag{3.7}$$

The argument establishing (3.6) is similar in the case of hypothesis (B_4^*).

We define functions w and v: $\Omega_\alpha \to \mathbf{R}_+$ by

$$w(x) = y(|x|), \quad v(x) = Y(|x|), \qquad x \in \Omega_\alpha.$$

Then (3.1), (3.4), and (3.5) show that w and v satisfy

$$\Delta w - q(|x|)w = F(|x|, w, |\nabla w|), \quad x \in \Omega_\alpha,$$
$$\Delta v - q(|x|)v = -F(|x|, v, |\nabla v|), \quad x \in \Omega_\alpha,$$

respectively, and hence by (B_3) they satisfy the elliptic inequalities

$$\Delta w - q(|x|)w \geq f(x, w, \nabla w), \quad x \in \Omega_\alpha, \tag{3.8}$$

$$\Delta v - q(|x|)v \leq f(x, v, \nabla v), \quad x \in \Omega_\alpha. \tag{3.9}$$

Since $0 < w(x) < v(w)$ throughout Ω_α by (3.6), it follows [7, p. 125] that there exists a solution $u(x)$ of

$$\Delta u - q(|x|)u = f(x, u, \nabla u), \qquad x \in \Omega_\alpha,$$

with locally Hölder continuous second derivatives in Ω_α such that $w(x) \leq u(x) \leq v(x)$ for all $x \in \Omega_\alpha$. As indicated by (3.7), this implies that

$$k \leq \frac{y(|x|)}{z_1(|x|)} \leq \frac{u(x)}{z_1(|x|)} \leq \frac{Y(|x|)}{z_1(|x|)} \leq 9k \tag{3.10}$$

throughout Ω_α.

This completes the proof of Theorem 3.1. The proofs of Theorems 3.2 and 3.3 are virtually the same by application of Theorems 2.1 (ii) and 2.2 (ii).

EXAMPLE 3.4. If $q(t)$ is identically zero, (1.1) reduces to

$$\Delta u = f(x, u, \nabla u), \qquad x \in \Omega_\alpha. \tag{3.11}$$

In this case a fundamental set of asymptotically ordered positive solutions of $Lz = 0$ in $[\alpha, \infty)$ is

$$\begin{aligned} &z_1(t) = 1, && z_2(t) = \log(t/\varepsilon) && \text{if } N = 2, \\ &z_1(t) = t^{2-N}, && z_2(t) = 1 && \text{if } N \geq 3, \end{aligned} \tag{3.12}$$

for $0 < \varepsilon < \alpha$. Condition (3.2) is equivalent to

$$\begin{aligned} &\int_\alpha^\infty t \log t \, F\left(t, a, \frac{b}{t \log t}\right) dt < \infty && \text{if } N = 2, \\ &\int_\alpha^\infty t^{N-1} F(t, a t^{2-N}, b t^{1-N})\, dt < \infty && \text{if } N \geq 3, \end{aligned} \tag{3.13}$$

and condition (3.3) is equivalent to

$$\begin{aligned} &\int_\alpha^\infty tF\left(t, a \log t, \frac{b}{t}\right) dt < \infty && \text{if } N = 2, \\ &\int_\alpha^\infty tF\left(t, a, \frac{b}{t}\right) dt < \infty && \text{if } N \geq 3. \end{aligned} \tag{3.14}$$

Then (3.13) implies that equation (3.11) has infinitely many positive solutions $u(x)$ in Ω_α for any $\alpha > 0$ such that $u(x)/z_1(|x|)$ is bounded above and below by positive constants in Ω_α; and in particular $u(x) \to 0$

as $|x| \to \infty$ in dimensions $N \geq 3$. Similarly (3.14) implies the conclusions of Theorems 3.2 and 3.3, and in particular implies the existence of infinitely many solutions of (3.11) in Ω_α which are uniformly bounded above and below by positive constants if $N \geq 3$, $\alpha > 0$.

EXAMPLE 3.5. Consider the equation

$$\Delta u - \rho^2 |x|^{2r} u = \phi(x) u^\gamma + \psi(x) |\nabla u|^\beta, \qquad x \in \Omega_\alpha, \tag{3.15}$$

where β, γ, ρ, r are constants, $\gamma \geq 0$, $0 \leq \beta \leq 2$, $\rho > 0$, $2r$ is a nonnegative integer, and ϕ, ψ are locally Hölder continuous functions in Ω_α for some $\alpha > 0$. Equation (3.15) is an example of (1.1) in which $q(t) = \rho^2 t^{2r}$ and $f(x, u, p) = \phi(x) u^\gamma + \psi(x)|p|^\beta$. Define

$$\phi^*(t) = \max_{|x|=t} |\phi(x)|, \qquad \psi^*(t) = \max_{|x|=t} |\psi(x)|,$$

$$F(t, u, v) = \phi^*(t) u^\gamma + \psi^*(t) v^\beta, \qquad t \geq \alpha, u \in \mathbf{R}_+, v \in \mathbf{R}_+. \tag{3.16}$$

Then $|f(x, u, p)| \leq F(|x|, |u|, |p|)$ for all $x \in \Omega_\alpha$, $u \in \mathbf{R}_+$, $p \in \mathbf{R}^N$, and hypothesis (B_3) holds for this choice of F.

In the case of (3.15), equation (3.1) becomes

$$t^{1-N} \frac{d}{dt}\left(t^{N-1} \frac{dz}{dt}\right) - \rho^2 t^{2r} z = 0, \qquad t \geq \alpha, \tag{3.17}$$

which possesses linearly independent solutions with the asymptotic behavior [9, p. 285]

$$z_1(t) \sim t^{-\lambda} \exp\left[-\frac{\rho t^{r+1}}{r+1}\right], \qquad z_2(t) \sim t^{-\lambda} \exp\left[\frac{\rho t^{r+1}}{r+1}\right]$$

as $t \to \infty$, where $\lambda = (N + r - 1)/2$. Evidently these solutions are asymptotically ordered at ∞, and we assume that α has been selected large enough so that they are positive in $[\alpha, \infty)$. Using (2.4) one easily verifies the asymptotic forms

$$p_1(t) \sim 2\rho t^r \exp\left[\frac{2\rho t^{r+1}}{r+1}\right] \qquad \text{as } t \to \infty;$$

$$p_2(t) \sim \frac{t^{\lambda - r}}{2\rho} \exp\left[-\frac{\rho t^{r+1}}{r+1}\right] \qquad \text{as } t \to \infty;$$

$$\frac{p_2(t) z_2(t)}{z_1(t)} \sim \frac{t^{\lambda - r}}{2\rho} \exp\left[\frac{\rho t^{r+1}}{r+1}\right] \qquad \text{as } t \to \infty.$$

We note from (3.16) that hypothesis (B_4) holds if

$$\gamma > 1 \quad \text{and} \quad \beta > 1 \tag{3.18}$$

and hypothesis (B_4^*) holds if

$$0 \leq \gamma < 1 \quad \text{and} \quad 0 \leq \beta < 1. \tag{3.19}$$

In the present example, condition (3.2) is satisfied if both

$$\int_\alpha^\infty t^{\lambda(1-\gamma)-r}\exp\left[\frac{\rho(1-\gamma)t^{r+1}}{r+1}\right]\phi^*(t)\,dt<\infty \tag{3.20}$$

and

$$\int_\alpha^\infty t^{(\lambda-r)(1-\beta)}\exp\left[\frac{\rho(1-\beta)t^{r+1}}{r+1}\right]\psi^*(t)\,dt<\infty, \tag{3.21}$$

and condition (3.3) holds if both

$$\int_\alpha^\infty t^{\lambda(1-\gamma)-r}\exp\left[\frac{\rho(\gamma-1)t^{r+1}}{r+1}\right]\phi^*(t)\,dt<\infty \tag{3.22}$$

and

$$\int_\alpha^\infty t^{(\lambda-r)(1-\beta)}\exp\left[\frac{\rho(\beta-1)t^{r+1}}{r+1}\right]\psi^*(t)\,dt<\infty. \tag{3.23}$$

We conclude from Theorem 3.1 under hypotheses (3.20), (3.21), and either (3.18) or (3.19) that equation (3.15) has infinitely many bounded positive solutions $u(x)$ in Ω_α such that

$$c_1|x|^{-\lambda}\exp\left[-\frac{\rho|x|^{r+1}}{r+1}\right]\le u(x)\le c_2|x|^{-\lambda}\exp\left[-\frac{\rho|x|^{r+1}}{r+1}\right],$$

$x\in\Omega_\alpha$, for some positive constants c_1 and c_2. Furthermore, Theorem 3.2 shows that (3.22), (3.23), and either (3.18) or (3.19) imply the existence of infinitely many unbounded positive solutions $u(x)$ of (3.15) in Ω_α such that

$$c_3|x|^{-\lambda}\exp\left[\frac{\rho|x|^{r+1}}{r+1}\right]\le u(x)\le c_4|x|^{-\lambda}\exp\left[\frac{\rho|x|^{r+1}}{r+1}\right],\qquad x\in\Omega_\alpha,$$

for some positive constants c_3 and c_4. Theorem 3.3 shows that the same conditions are sufficient for the existence of positive solutions of (3.15) in Ω_α satisfying the boundary condition $u|_{\partial\Omega_\alpha}=0$.

Example 3.5 can be extended to mixed sublinear-superlinear equations (3.15), in which $\gamma>1$ and $0\le\beta<1$, via the corollary below. More generally we consider equations (1.1) with the structure

$$\Delta u-q(|x|)u=f_1(x,u,\nabla u)-f_2(x,u,\nabla u),\qquad x\in\Omega_\alpha, \tag{3.24}$$

where each f_i satisfies hypotheses (B_1), (B_2), f_1 satisfies (B_4), (f_2) satisfies (B_4^*), but (B_3) is replaced by (B_3^*) below:

(B_3^*) There exist continuous functions $F_i\colon \mathbf{R}_+^3\to\mathbf{R}_+$, $i=1,2$, such that each $F_i(t,u,v)$ is nondecreasing in u and v and

$$\begin{gathered}0\le f_1(x,u,p)\le F_1(|x|,u,|p|)\\ 0\le -f_2(x,u,p)\le F_2(|x|,u,|p|)\end{gathered} \tag{3.25}$$

for all $x\in\Omega_\alpha$, $u\in\mathbf{R}_+$, $p\in\mathbf{R}^N$.

COROLLARY 3.6. *Suppose that* q *and each* f_i *satisfies* (B_1), (B_2), (B_3^*), *and* (3.2), *and that* f_1, f_2 *satisfy* (B_4), (B_4^*), *respectively, with* F *replaced by* F_1, F_2, *respectively. Then equation* (3.24) *has infinitely many positive solutions* $u(x)$ *such that* $u(x)/z_1(|x|)$ *is bounded above and below by positive constants throughout* Ω_α.

Proof. Consider the elliptic equations

$$\Delta w - q(|x|)w = f_1(x, w, \nabla w)$$
$$\Delta v - q(|x|)v = -f_2(x, v, \nabla v)$$

in Ω_α. By Theorem 3.1, these equations have solutions v, $w \in C_{\text{loc}}^{2+\lambda}(\Omega_\alpha)$ such that

$$k_1 z_1(|x|) \le w(x) \le k_2 z_1(|x|), \quad x \in \Omega_\alpha,$$
$$k_3 z_1(|x|) \le v(x) \le k_4 z_1(|x|), \quad x \in \Omega_\alpha,$$

for all sufficiently small positive constants k_1, $k_2 > k_1$ and sufficiently large positive constants k_3, $k_4 > k_3$. Since each $f_i \ge 0$, w and v satisfy the elliptic differential inequalities

$$\Delta w - q(|x|)w \ge f_1(x, w, \nabla w) - f_2(x, w, \nabla w),$$
$$\Delta v - q(|x|)v \le f_1(x, v, \nabla v) - f_2(x, v, \nabla v),$$

respectively, in Ω_α. We can choose k_3 large enough so that $0 < w(x) \le v(x)$ throughout Ω_α, and it follows [7, p. 125] as in Theorem 3.1 that (3.24) has a solution $u \in C_{\text{loc}}^{2+\lambda}(\Omega_\alpha)$ satisfying $w(x) \le u(x) \le v(x)$ throughout Ω_α.

COROLLARY 3.7. *Let* q *and each* f_i *be as in Corollary* 3.6 *except that* (3.2) *is replaced by* (3.3). *Then equation* (3.24) *has infinitely many positive solutions* $u(x)$ *such that* $u(x)/z_2(|x|)$ *is bounded above and below by positive constants throughout* Ω_α.

The proof, based on Theorem 3.2, is similar to that of Corollary 3.6.

These corollaries give new global existence theorems even for semilinear equations of the type

$$\Delta u = \phi(x)u^\gamma - \psi(x)u^\beta, \qquad x \in \Omega_\alpha, \tag{3.26}$$

where $\gamma > 1$, $0 \le \beta < 1$, and ϕ and ψ are nonnegative locally Hölder continuous functions in Ω_α, $\alpha > 0$. Corollaries 3.6 and 3.7 and Example 3.4 imply the following additional corollaries.

COROLLARY 3.8. *Equation* (3.26) *has a positive solution which is bounded and bounded away from zero in an arbitrary exterior domain* $\Omega \subset \mathbf{R}^2$ *if both*

$$(3.27) \qquad \int_\alpha^\infty t \log t\, \phi^*(t)\, dt < \infty \quad \text{and} \quad \int_\alpha^\infty t \log t\, \psi^*(t)\, dt < \infty$$

for some $\alpha > 0$.

For example, we can choose

$$\alpha = \inf\{|x| : x \in \partial\Omega\}$$

to obtain $\Omega \subset \Omega_\alpha$, and apply the earlier results to Ω_α.

COROLLARY 3.9. *Equation* (3.26) *has a positive solution which is bounded above and below by positive constant multiples of* $\log(|x|/\varepsilon)$ *in an arbitrary domain* $\Omega_\alpha \subset \mathbf{R}^2$, $0 < \varepsilon < \alpha$, *if both*

$$(3.28) \qquad \int^\infty t(\log t)^\gamma \phi^*(t)\, dt < \infty \quad \textit{and} \quad \int^\infty t(\log t)^\beta \psi^*(t)\, dt < \infty.$$

COROLLARY 3.10. *Equation* (3.26) *has a positive solution which is bounded and bounded away from zero in an arbitrary exterior domain* $\Omega \subset \mathbf{R}^N$, $N \geq 3$, *if both*

$$(3.29) \qquad \int^\infty t\, \phi^*(t)\, dt < \infty \quad \textit{and} \quad \int^\infty t\, \psi^*(t)\, dt < \infty.$$

COROLLARY 3.11. *Equation* (3.26) *has a positive solution which is bounded above and below by positive constant multiples of* $|x|^{2-N}$ *in an arbitrary exterior domain* $\Omega \subset \mathbf{R}^N$, $N \geq 3$, *if both*

$$(3.30) \qquad \int^\infty t^\sigma \phi^*(t)\, dt < \infty \quad \textit{and} \quad \int^\infty t^\rho \psi^*(t)\, dt < \infty,$$

where $\sigma = (N-1) - \gamma(N-2)$, $\rho = (N-1) - \beta(N-2)$.

Theorems 3.1–3.3 also apply to quasilinear equations of the type

$$\Delta u - q(|x|)u = \phi(x)u^\gamma + u^\beta \sum_{i,j=1}^N \psi_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j},$$

where ϕ and each ψ_{ij} are locally Hölder continuous in Ω_α, and the matrix $(\psi_{ij}(x))$ is symmetric and positive semidefinite in Ω_α.

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