SPECTRAL SETS AS BANACH MANIFOLDS

Angel Rafael Larotonda and Ignacio Zalduendo
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Let $A$ be a commutative Banach algebra, $X$ its spectrum, and $M$ a closed analytic submanifold of an open set in $C^n$. We may consider the set of germs of holomorphic functions from $X$ to $M$, $\mathcal{O}(X, M)$. Now let $\nu$ be the functional calculus homomorphism from $\mathcal{O}(X, C^n)$ to $A^n$, and $A_M = \nu(\mathcal{O}(X, M))$.

It is proven that $A_M$ is an analytic submanifold of $A^n$, modeled on projective $A$-modules of rank $= \dim M$.

1. Introduction. Let $A$ be a commutative complex Banach algebra with identity, and let $X$ be the set of all characters of $A$, considered as a compact subset of the topological dual $A'$ with the weak*-topology.

If $U$ is an open neighborhood of $X$, and $B$ a complex Banach space a map $f: U \to B$ will be called holomorphic if it is locally bounded and all its complex directional derivatives exist. The set of all such functions which are also bounded on $U$ will be denoted by $H^\infty(U, B)$, or simply $H^\infty(U)$, when $B$ is the complex field. These are locally convex spaces with the topology of uniform convergence. We define $\mathcal{O}(X, B)$ and $\mathcal{O}(X)$ to be the inductive limit of these spaces as $U$ ranges over all open neighborhoods of $X$. $\mathcal{O}(X)$ is then a topological algebra. We recall (see [2] or [7]) that there exists a continuous algebra epimorphism, the holomorphic functional calculus

$$\nu: \mathcal{O}(X) \to A$$

such that: the composition of $\nu$ and the Gelfand map

$$\mathcal{O}(X) \to A \to C(X)$$

is the restriction map $f \mapsto f|_X$, and the composition of the linear map $a \mapsto \bar{a}$ and $\nu$

$$A \to \mathcal{O}(X) \to A$$

is the identity map of $A$. Here $\bar{a}$ denotes the germ of the holomorphic map defined on $A'$ by $\gamma \mapsto \gamma(a)$.

In [6], Raeburn has generalized previous results of Taylor and Novodvorskii ([7],[5]). He uses a generalization of the morphism $\nu$, extending the holomorphic functional calculus to a linear map

$$S: \mathcal{O}(X, B) \to A \hat{\otimes} B.$$
If $M \subset B$ denotes a Banach submanifold, $\mathcal{O}(X, M)$ is defined and so is the set

$$A_M = \{ S(f) : f \in \mathcal{O}(X, M) \} \subset A \hat{\otimes} B.$$  

Raeburn shows that if $M$ is a discrete union of Banach homogeneous spaces the set $A_M$ is locally path connected and the generalized Gelfand map

$$A_M \to C(X, M)$$

induces a bijection on the set of components

$$[A_M] \to [X, M].$$

In this note, in §3, we take $B = C^n$ and $M$ a closed submanifold of an open set of $C^n$, and prove that the set $A_M$ is in fact an analytic submanifold of $A^n$. This was first stated by Taylor in [8]. $A_M$ is modeled on projective $A$-modules of rank $= \dim M$. We also prove that $A_M$ and $A^M = \{ a \in A^n : \text{sp}(a) \subset M \}$ have the same homotopy type. Note that with $B = C^n$, we have $S = \nu \times \cdots \times \nu$ and $A \hat{\otimes} B = A^n$.

In order to do this we first prove in §2 a version of the constant rank theorem.

2. The constant rank theorem. In this paragraph we give a version of the constant rank theorem valid for $A$-modules; the whole paragraph is an adaptation of the results in [4].

We will be dealing with submodules of the free module $A^n$, and $A$-module morphisms $T : A^n \to A^m$. A submodule $E$ of $A^n$ will be called $A$-direct if it is closed and there is another closed submodule $E'$ of $A^n$ such that $A^n = E \oplus E'$; obviously, this is equivalent to the fact: $E = \ker p$ (resp: $E = \text{Im} p$), for some continuous $A$-linear projector $p : A^n \to A^n$.

Note that in this case $E$ is a projective module, but not necessarily free.

If $T : A^n \to A^m$ is an $A$-module morphism, we say that $T$ is $A$-direct (also called "split") if $\ker T$ and $\text{im} T$ are $A$-direct.

Assume that

$$A^n = E_1 \oplus E_2, \quad F_1 \oplus F_2 = A^m$$

for some closed submodules $E_1, E_2, F_1, F_2$; if $T : A^n \to A^m$ is an $A$-morphism we shall use the notation

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \to \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$
with \( T_{ij} \in \text{Hom}_A(E_j, F_i) \) \((i, j = 1, 2)\), meaning that if

\[
x = x_1 + x_2 \quad (x_1 \in E_1, x_2 \in E_2),
\]

then

\[
T(x) = [T_{11}(x_1) + T_{12}(x_2)] + [T_{21}(x_1) + T_{22}(x_2)]
\]
is the expression of \( T(x) \) as a sum of elements in \( F_1 \) and \( F_2 \).

We shall need the following elementary lemma, which we state without proof.

**Lemma 2.1.** Let \( P_1, P_2 \) be \( A \)-direct submodules of \( A^n \) of the same rank. Then \( P_1 \subset P_2 \) implies \( P_1 = P_2 \).

**Theorem 1.** Suppose \( T_0: A^n \rightarrow A^m \) is an \( A \)-direct morphism and let \( E_1 \) and \( F_2 \) be closed submodules of \( A^n \) and \( A^m \) respectively such that

\[
A^n = E_1 \oplus \text{Ker} \ T_0, \quad \text{Im} \ T_0 \oplus F_2 = A^m
\]

If

\[
T = \begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix}
\begin{bmatrix}
E_1 \\
\text{Ker} \ T_0
\end{bmatrix} \rightarrow
\begin{bmatrix}
\text{Im} \ T_0 \\
F_2
\end{bmatrix}
\]

then the following are equivalent

(i) \( T \) is \( A \)-direct, \( A^n = E_1 \oplus \text{Ker} \ T \) and \( A^m = \text{Im} \ T \oplus F_2 \).

(ii) \( \alpha \in \text{Iso}(E_1, \text{Im} \ T_0) \) and \( \delta = \gamma \alpha^{-1} \beta \).

(iii) There exist \( A \)-linear automorphisms \( u: A^n \rightarrow A^n, v: A^m \rightarrow A^m \) such that \( T_0 = v Tu \) and

\[
\begin{align*}
u|_{E_1} &= \text{id}_{E_1} \\
v|_{F_2} &= \text{id}_{F_2}.
\end{align*}
\]

(iv) \( T \) is \( A \)-direct, \( \alpha \in \text{Iso}(E_1, \text{Im} \ T_0) \) and \( \text{rk}(\text{Im} \ T_0) = \text{rk}(\text{Im} \ T) \).

**Proof:** Suppose (i) and consider the diagram

\[
\begin{array}{ccc}
E_1 \times \text{Ker} \ T & \xrightarrow{w} & \text{Im} \ T \times F_2 \\
\downarrow \phi & & \downarrow \psi \\
A^n = E_1 \oplus \text{Ker} \ T_0 & \xrightarrow{T} & \text{Im} \ T_0 \oplus F_2 = A^m
\end{array}
\]

where \( \phi \) is the isomorphism \( v \rightarrow (v_1, v_2) \); here \( v_1 \) (resp: \( v_2 \)) is the projection of \( v \) onto \( E_1 \) (resp: \( \text{Ker} \ T \)) with kernel \( \text{Ker} \ T \) (resp. \( E_1 \)). We define \( \psi \)
in a similar way. Then we have

$$\phi = \begin{bmatrix} 1 & \tau \\ 0 & \theta \end{bmatrix} : \begin{bmatrix} E_1 \\ \text{Ker } T_0 \end{bmatrix} \rightarrow \begin{bmatrix} E_1 \\ \text{Ker } T \end{bmatrix}$$

and

$$\psi = \begin{bmatrix} \mu & 0 \\ \nu & 1 \end{bmatrix} : \begin{bmatrix} \text{Im } T \\ F_2 \end{bmatrix} \rightarrow \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix}$$

with \( \tau \in \text{Hom}_A(\text{Ker } T_0, E_1) \), \( \nu \in \text{Hom}_A(\text{Im } T, F_2) \) and \( \theta \in \text{Iso}_A(\text{Ker } T_0, \text{Ker } T) \), \( \mu \in \text{Iso}_A(\text{Im } T, \text{Im } T_0) \). On the other hand we also have

$$w = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ \text{Ker } T \end{bmatrix} \rightarrow \begin{bmatrix} \text{Im } T \\ F_2 \end{bmatrix}$$

with \( \lambda \in \text{Iso}_A(E_1, \text{Im } T) \).

The commutativity of the diagram implies

$$\begin{bmatrix} \mu & 0 \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \tau \\ 0 & \theta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

hence \( \mu \lambda = \alpha \) (which implies that \( \alpha \) is an isomorphism) and \( \delta = \nu \lambda \tau = \nu \lambda (\lambda^{-1}\mu^{-1}) \mu \lambda = \gamma \alpha^{-1} \beta \), and we have (ii). Now assume (ii): if

$$T_0 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ \text{Ker } T_0 \end{bmatrix} \rightarrow \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix}$$

with \( \lambda \in \text{Iso}_A(E_1, \text{Im } T_0) \) we define

$$u = \begin{bmatrix} 1 & -\alpha^{-1} \beta \\ 0 & 1 \end{bmatrix} : \begin{bmatrix} E_1 \\ \text{Ker } T_0 \end{bmatrix} \rightarrow \begin{bmatrix} E_1 \\ \text{Ker } T_0 \end{bmatrix}$$

and

$$v = \begin{bmatrix} \lambda \alpha^{-1} & 0 \\ -\gamma \alpha^{-1} & 1 \end{bmatrix} : \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix} \rightarrow \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix}$$

and a routine calculation gives (iii).

Now suppose we have (iv) and define the automorphism \( S : A^m \rightarrow A^m \) by

$$S = \begin{bmatrix} 1 & 0 \\ -\gamma \alpha^{-1} & 1 \end{bmatrix} : \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix} \rightarrow \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix}.$$
Then we have the composition
\[ T' = ST = \begin{bmatrix} \alpha & \beta \\ 0 & \delta - \gamma \alpha^{-1} \beta \end{bmatrix} : \begin{bmatrix} E_1 \\ \text{Ker } T_0 \end{bmatrix} \to \begin{bmatrix} \text{Im } T_0 \\ F_2 \end{bmatrix} \]
which is also \( A \)-direct. Note that \( \text{Im}(T') = S(\text{Im } T) \), hence \( \text{Im}(T') \) and \( \text{Im}(T) \) have the same rank; from this it follows that \( \text{rk}(\text{Im } T') = \text{rk}(\text{Im } T_0) \).

But \( \text{Im}(T') \supset \alpha(E_1) = \text{Im}(T_0) \); Lemma 2.1 gives \( \text{Im}(T') = \text{Im}(T_0) \) and this fact implies \( \delta - \gamma \alpha^{-1} \beta = 0 \). This proves (ii)

(iii) \( \Rightarrow \) (i) is simple; in fact, it is obvious that \( T \) is \( A \)-direct. It is also clear that \( u(\text{Ker } T_0) = \text{Ker } T \), hence
\[ A^n = u(\text{Ker } T_0 \oplus E_1) = u(\text{Ker } T_0) \oplus E_1 = \text{Ker } T \oplus E_1. \]

In order to complete the proof, we only need the inference (i) \( \Rightarrow \) (iv): \( \alpha \in \text{Iso}(E_1, \text{Im } T_0) \) as in (i) \( \Rightarrow \) (ii). The rest is obvious, so the proof is complete.

We shall be concerned now with a generalization of the results in §1 of [6], we shall follow the definitions of this reference.

Let \( \Omega \) be an open set in \( A^n \), \( F: \Omega \to A^m \) an holomorphic map, and \( a \in \Omega \); we denote the differential of \( F \) at \( a \) by \( DF(a) \).

A linear representation of \( F \) in \( a \) is an object \( (u, U, v, V, T) \) where
\( (i) \) \( U \) is a neighborhood of \( 0 \in A^n \), \( u \) is biholomorphic from \( U \) onto \( u(U) \), a neighborhood of \( a \) contained in \( \Omega \), and \( u(0) = a \).
\( (ii) \) \( V \) is a neighborhood of \( 0 \in A^m \), \( v \) is biholomorphic from \( V \) onto \( v(V) \), a neighborhood of \( F(a) \) and \( v(0) = F(a) \).
\( (iii) \) \( T: U \to A^m \) is the restriction of an \( A \)-linear map, and \( v^{-1} Fu = T. \)
\( (iv) \) \( Du(x) \) and \( Dv(y) \) are \( A \)-linear maps if \( x \in U, y \in V. \)

We will say that the holomorphic map \( F: \Omega \to A^m \) is locally \( A \)-direct at \( a \in \Omega \) if there are closed sub-modules \( E_1 \subset A^n, F_2 \subset A^m \) and a neighborhood \( U \) of \( a \) such that, for all \( x \in U, \)
\( (i) \) \( DF(x) \) is \( A \)-linear
\( (ii) \) \( A^n = E_1 \oplus \text{Ker } DF(x) \)
\( (iii) \) \( A^m = \text{Im } DF(x) \oplus F_2. \)
We have now the following:

**Lemma 2.2.** Let \( \Omega \) be an open set in \( A^n \), \( F: \Omega \to A^m \) holomorphic and \( a \in \Omega \). If \( F \) is locally \( A \)-direct at \( a \), then there is a linear representation \( (u, U, v, V, T) \) of \( F \) in \( a \), with \( TA \)-direct.
Proof. Without loss of generality we can assume that \( a = 0 \) and \( F(a) = 0 \); then there exist a neighborhood \( \Omega_0 \subset \Omega \) of \( 0 \in A^n \) and closed submodules \( E_1 \subset A^n \), \( F_2 \subset A^m \) such that
\[
A^n = E_1 \oplus \ker DF(x), \quad A^m = \text{Im} \ DF(x) \oplus F_2
\]
for all \( x \in \Omega_0 \). Also, \( DF(x) \) is \( A \)-linear if \( x \in \Omega_0 \).

Let \( E_2 = \ker DF(0), F_1 = \text{Im} \ DF(0) \); we denote \( x_1, x_2 \) (resp. \( y_1, y_2 \)) the components of \( x \in A^n \) (resp. \( y \in A^m \)) in the decomposition \( E_1 \oplus E_2 \) (resp. \( F_1 \oplus F_2 \)). In a similar way we write \( F(x) = f_1(x) + f_2(x) \), with \( f_1(x) \in F_1 \) and \( f_2(x) \in F_2 \).

We have
\[
DF(x) = \begin{bmatrix} D_1 f_1(x) & D_2 f_1(x) \\ D_1 f_2(x) & D_2 f_2(x) \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \rightarrow \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]
and so we can simplify the notation writing \( \alpha_{ij}(x) = D_i f_j(x) \) (\( i, j = 1, 2 \)). Recall that Theorem 1 gives
\begin{enumerate}[(a)]  
  \item \( \alpha_{11}(x) : E_1 \rightarrow F_1 \) is an isomorphism, and  
  \item \( \alpha_{22}(x) = \alpha_{12}(x) \alpha_{11}^{-1}(x) \alpha_{21}(x) \) for all \( x \in \Omega_0 \).
\end{enumerate}

Define the following \( A \)-linear maps
\[
S : E_1 \rightarrow F_1, \quad S = \alpha_{11}(0),  
T : A^n \rightarrow A^m, \quad T(x) = S(x_1),  
c : A^m \rightarrow A^n, \quad c(y) = S^{-1}(y_1),  
P : A^n \rightarrow A^n, \quad P(x) = x_2,  
Q : A^m \rightarrow A^m, \quad Q(y) = y_2.
\]

Now define the holomorphic map \( h : \Omega_0 \rightarrow A^n \) by
\[
h = cF + P.
\]
We have: \( Dh(x) \) is an \( A \)-linear map if \( x \in \Omega_0 \). In fact,
\[
Dh(x) = \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{11}(x) & \alpha_{21}(x) \\ \alpha_{12}(x) & \alpha_{22}(x) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
= \begin{bmatrix} S^{-1} \alpha_{11}(x) & S^{-1} \alpha_{21}(x) \\ 0 & 1 \end{bmatrix},
\]
hence by the inverse function theorem \( h : \Omega_1 \rightarrow \Omega_2 \) is biholomorphic for suitable neighborhoods of \( 0 \in A^n \).

Note that the differential of the map \( Fh^{-1}P : P^{-1}(\Omega_2) \rightarrow A^m \) vanishes identically, that is
\[
D(Fh^{-1}P)(x) = 0 \quad (x \in P^{-1}(\Omega_2)).
\]
In fact we can compute this differential as the composition $DF(h^{-1}P(x))Dh(h^{-1}P(x))^{-1}P$; the calculation leads (with $x' = h^{-1}P(x)$) to

$$\left[ \begin{array}{cc} \alpha_{11}(x') & \alpha_{21}(x') \\ \alpha_{12}(x') & \alpha_{22}(x') \end{array} \right] \left[ \begin{array}{cc} \alpha_{11}(x')^{-1}S & -\alpha_{11}(x')^{-1}\alpha_{21}(x) \\ 0 & 1 \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \left[ \begin{array}{cc} S & 0 \\ \alpha_{12}(x')\alpha_{11}(x')^{-1}S & 0 \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = 0,$$

where we use the identity $\alpha_{22} = \alpha_{12}\alpha_{11}^{-1}\alpha_{21}$.

Hence we have proved

(c) $Fh^{-1}P$ vanishes identically in a neighborhood of 0 (for instance, in the connected component of 0 in $P^{-1}(\Omega_2)$).

Finally we define the holomorphic mapping $g: c^{-1}(\Omega_2) \to A^m$

$$g = Fh^{-1}c + Q.$$

Then if $x = h^{-1}c(y)$ we compute

$$Dg(y) = \left[ \begin{array}{c} 1 \\ \alpha_{12}(x)\alpha_{11}(x)^{-1} \\ 0 \end{array} \right]$$

and this shows that $g: \Omega_1' \to \Omega_2'$ is a biholomorphic map, where $\Omega_1'$ and $\Omega_2'$ are small enough neighborhoods of $0 \in A^m$. Also $Dg(y)$ is $A$-linear for every $x \in \Omega_1$.

In order to complete the proof, set $u = h^{-1}$ and $v = g$; we must show that the identity

$$gTh = F$$

holds in some neighborhood of $0 \in A^n$; but this follows from (c) and the computation

$$gTh = (Fh^{-1}c + Q)T(cF + P) = Fh^{-1}cQF$$

$$= Fh^{-1}cF = Fh^{-1}(h - P) = F - Fh^{-1}P.$$

**Theorem 2.** Let $\Omega$ be an open subset of $A^n$, and $F: \Omega \to A^n$ an holomorphic retraction that is locally $A$-direct at $x$ for all $x \in \Omega$. Then Im $F$ is a Banach analytic manifold, and for all $x \in \text{Im } F$ the tangent space $T_x(\text{Im } F)$ at $x$ is $\text{Im } DF(x)$.

**Proof.** For every $F(x) \in \text{Im } F$ there is, by Lemma 2.2, a linear representation $(u_x, U_x, v_x, V_x, T_x)$ of $F$ with $T_x A$-direct.
For all $x' \in U_x$,
\[
T_x = DT_x(x') = Dv_x^{-1}(Fu_x(x')) \cdot DF(u_x(x')) \cdot Du_x(x')
\]
\[
= \left[Dv_x(T_x(x'))\right]^{-1} \cdot DF(u_x(x')) \cdot Du_x(x').
\]
$Dv_x(Z)$ and $Du_x(Z')$ are $A$-linear isomorphisms, so $\text{Im } T_x = \text{Im } DF(u_x(x'))$, for all $x' \in U_x$. But $F$ is $A$-direct at $x$, so there is a neighborhood of $x$ where $\text{Im } DF(a) = \text{Im } DF(b)$, for $a, b$ in this neighborhood. Hence the Im $T_z$ for $z$ in this neighborhood are all $A$-isomorphic to a fixed $A$-module $P$. Call $h_z : \text{Im } T_z \to P$ these $A$-isomorphisms. For every $x \in \text{Im } F$, $x = F(x)$, and $U_x, V_x$ may be chosen so that $u_x(U_x) = v_x(V_x)$. Then $v_x : V_x \cap \text{Im } T_x \to v_x(V_x) \cap \text{Im } F$ is a bijection: it is one to one over all of $V_x$, and if $v_x(z) \in \text{Im } F$, say $v_x(z) = u_x(z')$,
\[
v_x(z) = Fu_x(z) = Fu_x(z') = v_x T_x u_x^{-1}(u_x(z')) = v_x(T_x(z'))
\]
so $v_x(z) \in v_x(V_x \cap \text{Im } T_x)$.

Now define the chart near $x \in \text{Im } F$: $(v_x(V_x) \cap \text{Im } F, h_x v_x^{-1})$. These charts are compatible. To see this, suppose
\[
U_{xy} = v_x(V_x) \cap v_y(V_y) \cap \text{Im } F \neq \emptyset
\]
we then have
\[
(h_y v_y^{-1}) h_x v_x^{-1}(U_{xy}) \to h_y v_y^{-1}(U_{xy}).
\]
But $(h_y v_y^{-1}) h_x v_x^{-1} = h_y v_y^{-1} v_x h_x^{-1}$ is holomorphic. The same goes for the other composition. The tangent space $T_x(\text{Im } F)$ is given by
\[
\text{Im } (Dv_x(0) h_x^{-1}) = Dv_x(0)(\text{Im } T_x) = \text{Im } (Dv_x(0) T_x) = \text{Im } D(v_x T_x)(0)
\]
\[
= \text{Im } D(Fu_x)(0) = \text{Im } (DF(u_x(0)) Du_x(0)) = \text{Im } DF(x).
\]

3. $A_M$ as an analytic manifold. Here we will apply the results in the preceding paragraph to Taylor’s $A_M$ [7] where $M$ is a closed submanifold of an open set of $C^n$.

For $a \in A^n$, let $\dot{a}$ denote the function $A' \to C^n$ defined by $\dot{a}(\gamma) = (\gamma(a_1), \ldots, \gamma(a_n))$ for all $\gamma \in A'$. Note that with the supremum norm in both $A^n$ and $C^n$, $|\dot{a}(\gamma)| \leq \|\gamma\| \|a\|$. We will sometimes write $\phi^n$ for $\phi \times \cdots \times \phi$. We denote by $\theta_a$ the classical holomorphic functional calculus of Arens and Calderón [1]. All other functional calculus morphisms and their restrictions will be denoted by $\nu$.

We will need the following lemma.

**Lemma 3.1.** Let $W$ be an open subset of $C^n$. Then $A_W$ is an open subset of $A^n$. 


Proof. Let \( a \in A_w \), and \( f \in \mathcal{O}(X, W) \) such that \( a = \nu(f) \). Since \( f(X) \) is a compact subset of \( W \), there is an \( \varepsilon > 0 \) such that for every \( \phi \in X \), the polydisc \( \{ z \in \mathbb{C}^n : |f(\phi) - z| < \varepsilon \} \) is contained in \( W \). Now let \( U = \{ b \in A^n : \|a - b\| < \varepsilon \} \). \( \hat{b}(X) \subseteq W \), because

\[
|f(\phi) - \hat{b}(\phi)| = |a - \hat{b}(\phi)| \leq \|a - b\| < \varepsilon.
\]

Then \( \hat{b}^{-1}(W) \) is a neighborhood of \( X \) in \( A' \), so \( \hat{b} \in \mathcal{O}(X, W) \), and \( b \in A_w \).

The sets \( A_w \), with \( W \) open, are now appropriate domains for holomorphic functions. We will need to lift holomorphic functions in \( \mathbb{C}^n \) to holomorphic functions in \( A^n \). This will be done as follows. Let \( h : W \to \mathbb{C}^m \) be holomorphic, and define \( A_h : A_w \to A^m \) by \( A_h(a) = \nu(h \circ f) \), if \( a = \nu(f) \).

**Lemma 3.2.** \( A_h \) is a well-defined holomorphic function. For all \( a = \nu(f) \in A_w \), \( DA_h(a) \) is an \( A \)-module homomorphism given by \( \nu(Dh(f)) \).

**Proof.** First, we will see that \( \nu(f) = \nu(g) \) implies \( \nu(h \circ f) = \nu(h \circ g) \).

For this, let \( b_1, \ldots, b_k \in A \) be elements that finitely determine \( f \) and \( g \), in other words, there is an open set \( \Omega \) in \( \mathbb{C}^k \) and there are \( F \) and \( G \) in \( \mathcal{O}(\Omega, W) \) such that the following diagram commutes

\[
\begin{array}{ccc}
\hat{b}^{-1}(\Omega) & \xrightarrow{f(\text{resp. } g)} & W \\
\downarrow \hat{b} & & \downarrow h \\
\Omega & \xrightarrow{F(\text{resp. } G)} & \mathbb{C}^m
\end{array}
\]

\( \nu(f) = \nu(g) \) means that \( \theta_b(F) = \theta_b(G) \), so \( \text{sp}(\theta_b(F)) = \text{sp}(\theta_b(G)) \subseteq W \). Since \( h \in \mathcal{O}(W, \mathbb{C}^m) \), we may write \( \theta_{\hat{b}(F)}(h) = \theta_{\hat{b}(G)}(h) \). Then \( h(F(b)) = h(G(b)) \), so \( \theta_{\hat{b}}(h \circ F) = \theta_{\hat{b}}(h \circ G) \) and \( \nu(h \circ f) = \nu(h \circ g) \).

To prove that \( A_h \) is holomorphic, let \( a \in A_w \), and \( b \in A^n \). It will be enough to prove the existence of

\[
(1) \quad \frac{\partial A_h}{\partial b}(a) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[ A_h(a + \lambda b) - A_h(a) \right].
\]

Let \( a = \nu(f) \), \( b = \nu(g) \). Then \( a + \lambda b = \nu(f + \lambda g) \), and \( (1) \) is \( \lim_{\lambda \to 0} \lambda^{-1}[\nu(h \circ (f + \lambda g) - h \circ f)] \). Since the functional calculus is continuous, the limit \( (1) \) will exist if \( \lim_{\lambda \to 0} \lambda^{-1}[h \circ (f + \lambda g) - h \circ f] \) exists in \( \mathcal{O}(X, \mathbb{C}^m) \). We must see that \( \lambda^{-1}[h \circ (f + \lambda g) - h \circ f] \) converges uniformly over a neighborhood of \( X \) as \( \lambda \to 0 \). For this, set \( \varepsilon > 0 \), and if \( \lambda \in C \) with \( |\lambda| < \varepsilon \) and \( \gamma \in X \), let

\[
S(\lambda, \gamma) = \begin{cases} 
\frac{1}{\lambda} \left[ h(f(\gamma) + \lambda g(\gamma)) - h(f(\gamma)) \right] - \frac{\partial h}{\partial g(\gamma)} f(\gamma), & \text{if } \lambda \neq 0; \\
0 & \text{if } \lambda = 0.
\end{cases}
\]
$h$ is holomorphic, so $\lim_{\lambda \to 0} S(\lambda, \gamma) = 0$ for each $\gamma \in X$. Then there are $\delta_\gamma > 0$ and neighborhoods $V_\gamma$ of $\gamma$ such that $|S(\lambda, \phi)| < \varepsilon$ for $\lambda \in \mathbb{C}$ with $|\lambda| < \delta_\gamma$ and all $\phi \in V_\gamma$. Being $X$ compact, there are $\gamma_1, \ldots, \gamma_p \in X$ such that $V_{\gamma_i}$, $i = 1, \ldots, p$, cover $X$. Let $\delta = \min\{\delta_\gamma : 1 \leq i \leq p\}$, and $V = \bigcup_{i=1}^p V_{\gamma_i}$. Then for all $\lambda \in \mathbb{C}$ with $|\lambda| < \delta$ and all $\gamma \in V$, $S(\lambda, \gamma) < \varepsilon$, so $A_h$ is holomorphic. We shall denote the limit of $\lambda^{-1}[h \circ (f + \lambda g) - h \circ f]$ as $\lambda \to 0$, by $Dh(f)(g)$.

$DA_h(a)$ is more than just a linear morphism. It is $A$-linear. To prove this we must show that the diagram

\[
\begin{array}{ccc}
\mathcal{O}(X, \mathbb{C})^{m \times n} & \times & \mathcal{O}(X, \mathbb{C})^n \\
\downarrow \nu & & \downarrow \nu \\
A^{m \times n} & \times & A^n \\
& \rightarrow & A^m
\end{array}
\]

commutes.

Here the horizontal arrows indicate matrix multiplication.

As all the arrows are continuous, and $P(\hat{A})^k$ is dense in $\mathcal{O}(X, \mathbb{C})^k$ for all $k$, where $P(\hat{A})$ is the algebra of polynomials in Gelfand transforms of elements of $A$, it will be enough to show that $\nu(p \cdot q) = \nu(p) \cdot \nu(q)$, where $p_{ij}, q_j \in P(\hat{A})$. Let

\[
p_{ij} = \sum_{(k)} \tilde{a}^{ij}(k), \quad \text{where } \tilde{a}^{ij}(k) = \tilde{a}^{ij}_{k_1} \cdots \tilde{a}^{ij}_{k_r},
\]

\[
q_j = \sum_{(k')} \tilde{a}^{j}(k')', \quad \text{where } \tilde{a}^{j}(k')' = \tilde{a}^{j}_{k'_1} \cdots \tilde{a}^{j}_{k'_s}.
\]

\[
\nu(p \cdot q) = \nu\left(\sum_{j=1}^n p_{1j} q_j, \ldots, \sum_{j=1}^n p_{mj} q_j\right)
\]

\[
= \nu\left(\sum_{j=1}^n \sum_{(k)} \tilde{a}^{ij}(k) \sum_{(k')} \tilde{a}^{j}(k')', \ldots, \sum_{j=1}^n \sum_{(k)} \tilde{a}^{mij}(k) \sum_{(k')} \tilde{a}^{j}(k')'\right)
\]

\[
= \left(\sum_{j=1}^n \sum_{(k)} \tilde{a}^{ij}(k) \tilde{a}^{j}(k')', \ldots, \sum_{j=1}^n \sum_{(k)} \tilde{a}^{mij}(k) \tilde{a}^{j}(k')'\right).
\]

On the other hand,

\[
(2) \quad \nu(p) \cdot \nu(q) = \left(\sum_{j=1}^n \nu(p)_{1j} \nu(q)_j, \ldots, \sum_{j=1}^n \nu(p)_{mj} \nu(q)_j\right).
\]

But

\[
\nu(p)_{ij} = \nu(p_{ij}) = \nu\left(\sum_{(k)} \tilde{a}^{ij}(k)\right) = \sum_{(k)} \tilde{a}^{ij}(k),
\]
and

$$v(q) = v(q_j) = v\left(\sum_{(k')} \hat{a}^j(k')\right) = \sum_{(k')} a^j(k').$$

So

$$\text{(2) } = \left(\sum_{j=1}^{n} \sum_{(k)} a^{1j}(k) \sum_{(k')} a^j(k'), \ldots, \sum_{j=1}^{n} \sum_{(k)} a^{mj}(k) \sum_{(k')} a^j(k')\right) = v(p \cdot q).$$

Then

$$DA_h(a)(b) = v(Dh(f)(g)) = v(Dh(f)) \cdot v(g) = v(Dh(f))(b).$$

So that $DA_h(a) = v(Dh(f)) \in A^{m \times n}$ is an $A$-module morphism, for all $a \in A_W$.

Note that $A_h$ could have been well-defined by putting $A_h(a) = v(h \circ \hat{a})$, but this definition will not do for our later purposes.

Now let $M$ be a closed submanifold of an open set of $\mathbb{C}^n$, of dimension $k$. We recall that by [3; Ch. VIII, C] there is an open neighborhood $W$ of $M$ and an holomorphic retraction $r: W \to M$. Hence we also have $A_r: A_W \to A_M$, the image of $A_r$, being contained in $A_M$ because $r \circ f \in \mathcal{O}(X, M)$ for all $f \in \mathcal{O}(X, W)$. Of course the image of $A_r$ is exactly $A_M$, for if $a \in A_M$, then $A_r(a) = v(r \circ f)$ where $f \in \mathcal{O}(X, M)$ so $r \circ f = f$, and $A_r(a) = v(r \circ f) = v(f) = a \in \text{Im } A_r$. Now we obtain our main theorem.

**Theorem 3.** If $M$ is a closed submanifold of an open set of $\mathbb{C}^n$, of dimension $k$, then $A_M$ is a Banach manifold modeled on projective $A$-modules of rank $k$.

**Proof.** By Theorem 2, it will clearly be enough to verify that $A_r$ is $A$-direct at $a$ for all $a$ in a neighborhood of $A_M$.

Since $r$ is a retraction, $Dr(r(z)) \circ Dr(z) = Dr(z)$ for all $z \in W$. Therefore $\text{Im } Dr(z) \subseteq \text{Im } Dr(r(z))$, but the rank of the matrix $Dr(z)$ is at least that of $Dr(r(z))$ for $z$ near $r(z)$, so that actually $\text{Im } Dr(z) = \text{Im } Dr(r(z))$ for $z$ in an open neighborhood of $M$. This means that $\dim \text{Im } Dr(z) = k$, and $\dim \ker Dr(z) = n - k$ near $M$. $\mathbb{C}^n$ can be written as the direct sum

$$\mathbb{C}^n = \text{Im } Dr(r(z)) \oplus \ker Dr(r(z)) = \text{Im } Dr(z) \oplus \ker Dr(r(z)).$$

Because of the continuity of $Dr$, we may also write $\mathbb{C}^n = \text{Im } Dr(z) \oplus \ker Dr(z)$, for $z$ near $M$. Note also that $Dr(r(z))|\text{Im } Dr(r(z))$ is the identity, so that $Dr(z)|\text{Im } Dr(z)$ is an automorphism of $\text{Im } Dr(z)$ near $M$. We may suppose the neighborhood of $M$ where all this is true to be $W$.
just discard the old $W$. For all $z \in W$, 
\[
\alpha_z = \begin{bmatrix}
Dr(z) & 0 \\
\vdots & \vdots \\
0 & I
\end{bmatrix} : \begin{bmatrix}
\text{Im } Dr(z) \\
\text{Ker } Dr(z)
\end{bmatrix} \to \begin{bmatrix}
\text{Im } Dr(z) \\
\text{Ker } Dr(z)
\end{bmatrix},
\]
is an automorphism of $C^n$. Define $\alpha : W \to GL_n(C)$ by $\alpha(z) =$ the matrix of $\alpha_z$ in the canonical basis of $C^n$. We will show that $\alpha$ is an holomorphic function. For this, let $z_0 \in W$. There is a neighborhood $U$ of $z_0$ and there are holomorphic functions $\upsilon_i : U \to C^n$, $1 \leq i \leq n$, such that $\upsilon_1(z), \ldots, \upsilon_k(z)$ is a basis of $\text{Im } Dr(z)$ and $\upsilon_{k+1}(z), \ldots, \upsilon_n(z)$ is a basis of $\text{Ker } Dr(z)$ for all $z \in U$. Let $\beta_z \in C^{k \times k}$ be the matrix of $Dr(z)|\text{Im } Dr(z)$ in the basis $\upsilon_1(z), \ldots, \upsilon_k(z)$ and let $c(z)$ be the matrix which changes the canonical basis of $C^n$ to $\upsilon_1(z), \ldots, \upsilon_n(z)$. Then
\[
\alpha(z) = c(z)^{-1} \begin{bmatrix}
\beta_z & 0 \\
\vdots & \vdots \\
0 & I
\end{bmatrix} \cdot c(z)
\]
and it will be enough to verify that $\beta_z$ is an holomorphic function of $z$ in $U$, but this follows from the equations
\[
Dr(z)(\upsilon_i(z))_t = \sum_{s=1}^{k} \beta_{z,s} \upsilon_i(z)_s, \quad i \leq i, \quad t \leq k.
\]
We therefore have $A_\alpha : A_W \to A_{GL_n(C)} = GL_n(A)$. But
\[
A_\alpha(x)|_{\text{Im } DA_r(x)} = DA_r(x)|_{\text{Im } DA_r(x)}
\]
for all $x \in A_W$. To see this, let $b = \upsilon(\text{Dr}(g)(h)) \in \text{Im } DA_r(x)$, where $x = \upsilon(g)$. Now $A_\alpha(x)(b) = \upsilon(\alpha \circ g) \cdot \upsilon(\text{Dr}(g)(h)) = \upsilon(\alpha \circ g \cdot \text{Dr}(g)(h))$, but for all $\gamma$ near $X$,
\[
\alpha(g(\gamma))|_{\text{Im } Dr(g(\gamma))} = \text{Dr}(g(\gamma))|_{\text{Im } Dr(g(\gamma))},
\]
so
\[
\alpha(g(\gamma))(b) = \upsilon(\text{Dr}(g)) \cdot \upsilon(\text{Dr}(g)(h))
\]
\[
= \upsilon(\text{Dr}(g)) \cdot \upsilon(\text{Dr}(g)(h)) = DA_r(x)(b).
\]
Then
\[
DA_r(x)|_{\text{Im } DA_r(x)} : \text{Im } DA_r(x) \to \text{Im } DA_r(x) \text{ is an automorphism.}
\]
We prove that $A^n = \text{Im } DA_r(x) \oplus \text{Ker } DA_r(x)$ for all $x \in A_W$:
\[
0 = \text{Ker}( DA_r(x)|_{\text{Im } DA_r(x)}) = \text{Im } DA_r(x) \cap \text{Ker } DA_r(x).
\]
If $c \in A^n$, there exists $b \in \text{Im } DA_r(x)$ such that $DA_r(x)(b) = DA_r(x)(c)$. Then $c = b + (c - b)$, with $b \in \text{Im } DA_r(x)$ and $c - b \in \text{Ker } DA_r(x)$. $\text{Ker } DA_r(x)$ is closed, so the direct sum is topological.
We now know that $\text{Im } DA_r(x)$ is a projective $A$-module.
We shall see that its rank is $k$.
First we must prove that for all $x \in A_w$ and $\phi \in X$,
\[
\phi^n(\text{Im } DA_r(x)) = \text{Im } Dr(\phi^n(x))
\]
and
\[
\phi^n(\text{Ker } DA_r(x)) = \text{Ker } Dr(\phi^n(x)).
\]
Take
\[
DA_r(x)(b) \in \text{Im } DA_r(x) \cdot \phi^n(DA_r(x)(b)) = \nu(\text{Dr}(\hat{x})(\hat{b}))\phi
\]
\[
= (\text{Dr}(\hat{x})(\hat{b}))(\phi) = \text{Dr}(\phi^n(x))(\phi^n(b)) \in \text{Im } Dr(\phi^n(x)).
\]
Now take $b \in \text{Ker } DA_r(x)$.
\[
\text{Dr}(\phi^n(x))(\phi^n(b)) = \phi^n(\text{DA}_r(x)(b)) = \phi^n(0) = 0,
\]
so $\phi^n(b) \in \text{Ker } \text{Dr}(\phi^n(x))$, and we have proven both left-to-right inclusions. We have $A^n = \text{Im } DA_r(x) \oplus \text{Ker } DA_r(x)$, and $\phi^n$ is surjective, so
\[
C^n = \phi^n(\text{Im } DA_r(x)) + \phi^n(\text{Ker } DA_r(x)),
\]
but because of the inclusions we have just proven, this sum is direct. Then
\[
C^n = \phi^n(\text{Im } DA_r(x)) \oplus \phi^n(\text{Ker } DA_r(x))
\]
\[
= \text{Im } Dr(\phi^n(x)) \oplus \text{Ker } Dr(\phi^n(x)),
\]
so the inclusions are actually equalities.
Now let $x \in A_w$, $P = \text{Im } DA_r(x)$, $Q = \text{Ker } DA_r(x)$, and $\phi \in X$.
Then $\text{rk}_\phi P = \text{rk}_{A_\phi} P_\phi = \text{rk}_{A_\phi}(A_\phi \otimes_A P)$ is, by Nakayama's Lemma the same as $\dim_C[(A_\phi \otimes_A P) \otimes_{A_\phi} C]$, when $C$ (and also $\phi^n(P)$) has the $A_\phi$-module structure induced by $\phi$. We then have the $A_\phi$-module morphism
\[
q: (A_\phi \otimes_A P) \otimes_{A_\phi} C \to \phi^n(P);
\]
\[
q\left(\sum_j \left(\sum_i a_{ij} \otimes p_{ij}\right) \otimes j\right) = \sum_j \sum_i \lambda_j \phi(a_{ij}) \phi^n(p_{ij}).
\]
Let $v_1, \ldots, v_k$ has a basis for $\phi^n(P) = \text{Im } Dr(\phi^n(x))$, and let $b_1, \ldots, b_k \in P$ such that $\phi^n(b_i) = v_i$ for $i = 1, \ldots, k$. Then $(1/1 \otimes b_i) \otimes 1$, $i = 1, \ldots, k$, are $C$-linearly independent: if $0 = \sum_{i=1}^k \lambda_i(1/1 \otimes b_i) \otimes 1$, then
\[
0 = q(0) = \sum_{i=1}^k \lambda_i \phi^n(b_i) = \sum_{i=1}^k \lambda_i v_i
\]
and $\lambda_i = 0$ for $i = 1, \ldots, k$.
Therefore $\text{rk}_\phi P = \dim_C[(A_\phi \otimes_A P) \otimes_{A_\phi} C] \geq k$.
In a similar manner, and since $\phi^n(Q) = \text{Ker } Dr(\phi^n(x))$, $\text{rk}_\phi Q \geq n - k$. But $\text{rk}_\phi P + \text{rk}_\phi Q = n$, so $\text{rk}_\phi P = k \:\forall \phi \in X$. Then $\text{rk } P = k$. 
To complete our proof, let \( a \in A_M \) and write:
\[
DA_r(x) = \begin{bmatrix} P(x) & Q(x) \\ R(x) & S(x) \end{bmatrix} \begin{bmatrix} \text{Im } DA_r(a) \\ \text{Ker } DA_r(a) \end{bmatrix} \to \begin{bmatrix} \text{Im } DA_r(a) \\ \text{Ker } DA_r(a) \end{bmatrix}.
\]

Since \( DA_r(a) \) is an indempotent, \( DA_r(a)|_{\text{Im } DA_r(a)} \) is the identity, and \( P(a) = I \). But \( \text{Im } DA_r(a) \) is a Banach space, so by the continuity of \( P \), \( P(x) \) is an automorphism of \( \text{Im } DA_r(a) \) for all \( x \) in a neighborhood \( U \) of \( a \).

We have then verified conditions (iv) of Theorem 1 for all \( x \in U \). Therefore, \( A_r \) is \( A \)-direct at \( x \) for all \( x \) in a neighborhood of \( A_M \).

Observe that the tangent space \( T_a(A_M) \) at \( a \) is \( \text{Im } DA_r(a) \). These are of course projective \( A \)-modules of rank \( k \), but they need not be isomorphic on different connected components of \( A_M \). In fact, some of these modules may be free while others may not.

Now consider for any Banach algebra \( A \), the category \( M(A) \) whose objects are analytic manifolds modeled on projective \( A \)-modules, with morphisms holomorphic functions whose differentials are \( A \)-module morphism, and the ordinary composition. Let \( M \) be the category of closed analytic submanifolds of open subsets of finite products of \( C \). Then we have:

**Proposition 3.3.** \( A(\cdot) \) is a covariant functor from \( M \) to \( M(A) \).

**Proof.** \( A_M \) is defined for every object in \( M \) and is an object of \( M(A) \), by Theorem 3. Now let \( M \) and \( N \) be two objects of \( M \) and \( h: M \to N \) an holomorphic function. \( h \) can be extended to an open neighborhood \( W \) of \( M \) for example by \( h \circ r \). If \( \bar{h} \) is such an extension, then we can define \( A_{\bar{h}} \) as before Lemma 3.2. Now define \( A_h \) to be the restriction of \( A_{\bar{h}} \) to \( A_M \), for any extension \( \bar{h} \) of \( h \). Obviously, \( \text{Im } A_{\bar{h}} = A_{\bar{h}}(A_M) \subseteq A_N \), and if \( h_1 \) and \( h_2 \) are two extensions of \( h \), and \( a \in A_M \), \( a = v(f) \) with \( f \in \emptyset(X, M) \), then
\[
A_{h_1}(a) = v(h_1 \circ f) = v(h \circ f) = v(h_2 \circ f) = A_{h_2}(a),
\]
so \( A_h \) is well defined. The rest of the Proposition is easily verified.

There are many holomorphic functions in \( A^m \) whose differentials are \( A \)-module morphisms, but which are not of the form \( A_h \) for any \( h \). As an example, take \( a \in A \) such that there are \( x \in A \), and \( \phi, \psi \in X \) with \( \phi(x) = \psi(x) \neq 0 \) and \( \phi(a) \neq \psi(a) \); and consider \( L_a: A \to A \) defined by \( L_a(y) = ay \). \( L_a \) is \( A \)-linear, but \( L_a \neq A_h \) for all \( h \): if \( L_a \) were \( A_h \), \( ax = L_a(x) = A_h(x) = v(h \circ \hat{x}) \), so over \( X, \hat{a} \hat{x} = h \circ \hat{x} \), and then
\[
\phi(a) \cdot \phi(x) = h(\phi(x)) = h(\psi(x)) = \psi(a) \psi(x).
\]
Hence, \( \phi(a) = \psi(a) \), contrary to our assumptions.

Finally, we wish to compare \( A_M \) and \( A^M \).
Proposition 3.4. $A^M = A_M + \text{Rad}(A)^n$.

Proof. Let $\mathcal{N} = \{ f \in \mathcal{O}(X, C): f|_X = 0 \}$. Then $\nu(\mathcal{N}) = \text{Rad}(A)$: if $f \in \mathcal{N}$, $\nu(f)|_X = f|_X = 0$, so $\nu(\mathcal{N}) \subseteq \text{Rad}(A)$; on the other hand, if $a \in \text{Rad}(A)$, $a = \nu(\hat{a})$ with $\hat{a}|X = 0$. We identify also $\text{Rad}(A)^n$ with $\nu(\mathcal{N}^n)$. Note that $A^M \subseteq A_W$, for if $\hat{a}(X) = sp(a) \subseteq M$, then $\hat{a} \in \mathcal{O}(X, W)$. Now take $a \in A^M$, and put $a = A_r(a) + (a - A_r(a))$. $A_r(a) \in A_M$, and

$$a - A_r(a) = \nu(\hat{a}) - \nu(r \circ \hat{a}) = \nu(\hat{a} - r \circ \hat{a}) \in \text{Rad}(A)^n,$$

because $\hat{a} - r \circ \hat{a} \in \mathcal{N}^n$. For the other inclusion, let $b \in A_M$ and $c \in \text{Rad}(A)^n$. $c = \nu(g)$, with $g \in \mathcal{N}^n$. Then

$$\text{sp}(b + c) = \overline{b + c}(X) = (\hat{b} + \nu(g))(X)$$

$$= (\hat{b} + g)(X) = \hat{b}(X) = \text{sp}(b) \subseteq M.$$

Corollary 3.5. $A^M$ and $A_M$ have the same homotopy type. If $A$ is semisimple, then $A^M = A_M$. (See also [7; 2.8].)

Proof. Let $:\iota: A_M \to A^M$ denote the inclusion. $A_r \circ \iota$ is the identity on $A_M$ and it is easily verified that $\iota \circ A_r$ is homotopic to the identity on $A^M$.

4. An example. We wish to consider briefly an example of a spectral set. Suppose $A$ is semisimple, and the manifold $M$ is given as the zero set of a holomorphic function

$$W \xrightarrow{F} C^k.$$

It has been established in the last paragraph that $A_M$ is a Banach manifold. This would have been a much simpler matter in this particular case, but a bit more can be said. Lift $F$ to an analytic function

$$A_W \xrightarrow{A_F} A^k$$

and the zero set of $A_F$ is exactly $A_M$. To see this, let $a \in A_M$; then $a = \nu(f)$ with $f \in \mathcal{O}(X, M)$, and $A_F(a) = \nu(F \circ f) = \nu(0) = 0$, so $a \in A_F^{-1}(0)$. Now if $A_F(a) = 0$, $\nu(F \circ \hat{a}) = 0$ and $F \circ \hat{a} = 0$ over $X$. Hence $F(sp(a)) = \{0\}$, and $sp(a) \subseteq M$. We then have $A_M \subseteq A_F^{-1}(0) \subseteq A^M$, but since $A$ is semisimple, all three are the same.

Now take $W = GL_n(C)$, and $G$ a Lie subgroup of $W$ which is the zero set of analytic functions, for instance an algebraic group. Then the corresponding zero set of the same functions in $GL_n(A)$ is a Lie subgroup of $GL_n(A)$.

It can in fact be shown that all Lie groups give rise to Banach Lie groups, and that these have tangent spaces which are free $A$-modules.
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PABELLON I - CIUDAD UNIVERSITARIA
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