ON POLYNOMIAL GENERATORS IN THE ALGEBRA OF COMPLEX FUNCTIONS ON A COMPACT SPACE

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In this paper we prove that in the space of all continuous mappings of a $k$-dimensional compact space $X$ into complex linear space $C^n$ the imbeddings $F: X \rightarrow C^n$ with the property "any complex continuous function on $F(X)$ can be uniformly approximated by complex polynomials on $C^n$" form a dense subset of type $G_δ$, provided that $k < \frac{2}{3}n$.

If it is known [2] that if the algebra of continuous complex functions $C(X)$ for a topological space $X$ has $k$ multiplicative generators then $X$ has to be acyclic (with complex coefficients) in dimensions $\geq k$. In particular, $C(M^k)$ has at least $k + 1$ generators for any closed orientable $k$-manifold $M$. On the other hand, it was proved in [6] that there exist $k + 1$ polynomial generators in the algebra $C(X^k)$ for a finite $k$-dimensional simplicial polyhedron $X^k$. This means that any such function on $X^k$ may be uniformly approximated by complex polynomials in certain specially constructed functions $f_0^*, \ldots, f_k^* \in C(X^k)$. In other words, there exists a continuous embedding $F^* : X^k \rightarrow C^{k+1}$ of the polyhedron $X^k$ into complex vector space $C^{k+1}$ such that any continuous complex valued function on the image $F^*(X^k)$ may be approximated by complex polynomials in the coordinate functions $z_i : C^{k+1} \rightarrow C$, $0 \leq i \leq k$.

It seems that analogous results follow for any compact space $X^k$ (not only for polyhedra). Moreover, it is quite natural to conjecture that for $X^k$ compact the existence of polynomial approximation on $F(X^k) \subset C^{k+1}$ is a "general position" phenomenon with respect to perturbations of $F : X^k \rightarrow C^{k+1}$. Note, that this would be a complete complex analog of the classical Whitney theorems [9] (see also [4]).

In this paper we prove similar propositions for imbeddings $F: X^k \rightarrow C^n$ satisfying the dimensional condition $k \leq \frac{2}{3}n$. In particular, for 2-dimensional compact spaces $X^2$ one has the following result ("complex Whitney theorem"): there are 3 multiplicative generators in the algebra $C(X^2)$, in fact, starting with any $f_1, f_2, f_3 \in C(X^2)$ one can perturb them by an arbitrarily small amount to get a set of multiplicative generators for $C(X^2)$. Note, that this is the best possible general result for $k = 2$. 
Our main result is

**THEOREM A.** Let $3k \leq 2n$. In the space $\text{Map}(X^k, \mathbb{C}^n)$ of all continuous mappings of a $k$-dimensional compact space $X^k$ into complex linear space $\mathbb{C}^n$ consider the mappings $F: X^k \to \mathbb{C}^n$ satisfying the following properties:

1. $F$ is an imbedding;
2. any continuous function on $X^k$ may be approximated by complex polynomials in the multiplicative generators $f_1 = z_1 \circ F, \ldots, f_n = z_n \circ F$, where $z_1, \ldots, z_n$ are complex coordinate functions on $\mathbb{C}^n$;
3. in particular, $F(X^k)$ is polynomially convex in $\mathbb{C}^n$.

These mappings form a dense subset of type $G_δ$ in $\text{Map}(X^k, \mathbb{C}^n)$.

The proof of this theorem is based on the following proposition.

**THEOREM B.** Let $3k \leq 2n$. In the space $\text{SL Map}(Y^k, \mathbb{C}^n)$ of simplicially linear mappings of a finite $k$-dimensional simplicial polyhedron $Y^k$ into $\mathbb{C}^n$ there exists an open and everywhere dense subset of imbeddings $F: Y^k \to \mathbb{C}^n$ such that any continuous function on the image $F(Y^k)$ may be approximated by complex polynomials over $\mathbb{C}^n$ and, consequently, $F(Y^k)$ is polynomially convex in $\mathbb{C}^n$.

We don’t know if Theorems A and B have immediate analogs for smooth regular imbeddings. For example, it is easy to show that there is no smooth regular imbedding $F: \mathbb{C}P^2 \to \mathbb{C}^6$ of complex projective space $\mathbb{C}P^2$ with the tangent bundle of $F(\mathbb{C}P^2)$ being a totally real subbundle of a trivial complex 6-dimensional bundle. On the other hand, $3 \cdot \dim \mathbb{C}P^2 \leq 2 \cdot 6$, which is perfectly consistent with the dimensional assumptions of Theorems A and B.

Prior to the proof of Theorem B we need to introduce some terminology and to prove some auxiliary propositions.

Let $\mathcal{L}$ be any finite family of real affine subspaces $\{V_\alpha\}_{\alpha \in \mathcal{L}}$ of $\mathbb{C}^n$ with the property $V_\alpha \not\subseteq V_\beta$ for any pair $\alpha, \beta \in \mathcal{L}$, $\alpha \neq \beta$. Consider the subspace $|\mathcal{L}| = \bigcup_{\alpha \in \mathcal{L}} V_\alpha \subset \mathbb{C}^n$. In fact, it is a stratified set with the stratification induced by the multiple intersections of different spaces $V_\alpha$ parameterized by $\mathcal{L}$.

We say that the family $\mathcal{L}$ is *totally real* if any $V_\alpha \subset |\mathcal{L}|$, $\alpha \in \mathcal{L}$, is a totally real affine subspace of $\mathbb{C}^n$, i.e. it does not contain any complex line. Of course, if $\mathcal{L}$ is totally real, then its dimension $\dim \mathcal{L} = \max_{\alpha \in \mathcal{L}} (\dim \mathbb{R} V_\alpha)$ is not greater than $n$. 
We denote by $V^C_a$ the complexification of $V_a \subset \mathbb{C}^n$ (which for totally real $V_a$ is an affine subspace of real dimension $2 \dim V_a$). We call a totally real family $L$ weakly generic if the following holds: $V_\beta \not\supseteq V_\alpha$ implies $V^C_\beta \not\supseteq V_\alpha$ for any $\alpha, \beta \in L$.

One can associate a new family $D L$ with any (totally real) family $L$. This derived family $D L$ is formed by all the spaces $V_{\alpha,\beta} = V_\alpha \cap V^C_\beta$, $V_\beta \not\supseteq V_\alpha$, and which are maximal with respect to inclusion relations. In fact, $|D L|$ contains $V_\alpha \cap V_\beta$ for any pair $\alpha, \beta \in L$. If $L$ is weakly generic then $\dim L > \dim D L$. Moreover, if $L$ is totally real then $D L$ also has this property.

We call a totally real family $L$ perfectly generic if $L$ and all its derived families $D L$, $D(D L)$, \ldots, are weakly generic. Note, that if $L$ is perfectly generic then its $(k+1)$-derivative $D^{(k+1)} L = \emptyset$, where $k = \dim L$.

The following Lemma is the main step to prove Theorem B.

**Lemma 1.** Given a totally real and perfectly generic family $L$ of real affine subspace of $\mathbb{C}^n$, $\dim L < n$, and any compact subset $K \subset |L|$, then any continuous complex function on $K$ may be uniformly approximated by complex polynomials in coordinate functions $z_1, \ldots, z_n$ on $\mathbb{C}^n$. In particular, $K$ is polynomially convex in $\mathbb{C}^n$.

Let $C(K)$ be the algebra of all continuous functions on $K$. Let $\mathcal{P}(K)$ denote the uniform closure in $C(K)$ of the subalgebra multiplicatively generated by the functions $\text{Res}_K(z_i)$, $1 \leq i \leq n$. By Bishop’s theorem on maximal antisymmetric subdivisions to prove that $\mathcal{P}(K) = C(K)$ it is sufficient to show that any antisymmetry set $\Omega$ for $\mathcal{P}(K)$ is a singleton [3]. Recall, that a subset $\Omega \subset K$ is called an antisymmetry set for $\mathcal{P}(K)$ if any function $f \in \mathcal{P}(K)$ which is real valued on $\Omega$, in fact, is constant.

As a first step we prove that any antisymmetry set $\Omega$ is a singleton or is contained in the intersection of $K$ with the derived family $|D L|$ (providing that $L$ is totally real and weakly generic). Denote by $\hat{\Omega}_\alpha$ the intersection $V_\alpha \cap \Omega$ and by $\hat{\Omega}_\alpha$ the intersection $\hat{V}_\alpha \cap \Omega$, where $\hat{V}_\alpha = V_\alpha \setminus (V_\alpha \cap |D L|) = V_\alpha \setminus \bigcup_{\beta \neq \alpha} (V_\alpha \cap V^C_\beta)$. Note that $L$ weakly generic implies that $\hat{V}_\alpha$ is open and everywhere dense in $V_\alpha$.

For any two points $a, b \in \hat{\Omega}_\alpha$, $\alpha \in L$, we construct a polynomial $P_\alpha = P_\alpha(z_1, \ldots, z_n)$ which is real-valued on $|L|$ and separates $a$ and $b$. Note, that for any two points $a, b \notin V^C_\beta$ one can find a linear polynomial $L_\beta: \mathbb{C}^n \to \mathbb{C}$ which is zero on $V^C_\beta$ and such that $L_\beta(a) \neq 0 \neq L_\beta(b)$. Now take the product $Q_\alpha = \Pi_{\beta \neq \alpha} L_\beta$. The polynomial $Q_\alpha$ is zero on each $V^C_\beta$, $\beta \neq \alpha$, and $Q_\alpha(a) \neq 0 \neq Q_\alpha(b)$. Over $V_\alpha$ one can represent $Q_\alpha$ in the
form \( S_\alpha + i T_\alpha \) where the polynomials \( S_\alpha: V_\alpha \to \mathbb{C}, T_\alpha: V_\alpha \to \mathbb{C} \) are real valued. Denote by \( \hat{Q}_\alpha^*: V_\alpha \to \mathbb{C} \) the polynomial \( S_\alpha - i T_\alpha \). Using that \( V_\alpha \) is totally real, one can extend \( \hat{Q}_\alpha^* \) to a polynomial \( Q_\alpha^*: \mathbb{C}^n \to \mathbb{C} \) (first take the analytic extension of \( \hat{Q}_\alpha^* \) from \( V_\alpha \) to \( V_\alpha^C \) and then use a complex linear projection \( \mathbb{C}^n \to V_\alpha^C \)).

Consider the product \( Q_\alpha Q_\alpha^* \) of the polynomials \( Q_\alpha \) and \( Q_\alpha^* \). This complex polynomial has the following remarkable properties: (1) \( Q_\alpha Q_\alpha^* \) is zero for any \( \beta \neq \alpha \); (2) \( Q_\alpha Q_\alpha^* \) is real valued on \( V_\alpha \); (3) \( Q_\alpha Q_\alpha^*(a) \neq 0 \neq Q_\alpha Q_\alpha^*(b) \).

Again, using that \( V_\alpha \) is totally real, one can construct some polynomial \( G_\alpha: \mathbb{C}^n \to \mathbb{C} \) which is real-valued on \( V_\alpha \) and such that \( G_\alpha Q_\alpha Q_\alpha^*(a) \neq G_\alpha Q_\alpha Q_\alpha^*(b) \) (recall, that \( Q_\alpha Q_\alpha^* \) cannot simultaneously vanish at \( a \) and \( b \)). Hence, the polynomial \( P_\alpha = G_\alpha \cdot Q_\alpha \cdot Q_\alpha^* \) separates \( a \) and \( b \). Moreover, it is real-valued on \( V_\alpha \) and vanishes on any \( V_\beta, \beta \neq \alpha \). Consequently, \( \Omega_\alpha \) is a singleton or \( \Omega_\alpha \subseteq |D \mathcal{L}| \). In fact, if \( \Omega_\alpha = \Omega_\beta \) is a singleton \( a \), then \( \Omega = \Omega_\alpha \) (note that \( Q_\alpha Q_\alpha^*(a) \neq 0 \) and, hence, it separates \( a \) from \( |D \mathcal{L}| \) or \( \cup_{\beta \neq a} V_\beta \)).

To complete the proof of Lemma 1 we apply inductively the same argument to the derived families \( D \mathcal{L}, D^2 \mathcal{L}, \ldots \) and use that \( \mathcal{L} \) is perfectly generic (i.e. each \( D^s \mathcal{L}, s = 1, 2, \ldots \) is weakly generic and totally real). \( \square \)

As we mentioned before, \(|\mathcal{L}|\) is a stratified space with the stratification induced by different intersections \( V_\alpha = V_{\alpha_1} \cap V_{\alpha_2} \cap \cdots \cap V_{\alpha_s}, \alpha_1, \alpha_2, \ldots, \alpha_s \in \mathcal{L} \). In this way, starting with \( \mathcal{L} \) one can produce a new family \( \mathcal{L} \supset \mathcal{L} \) of real affine subspaces parameterizing different multiple intersections. Let us say that \( \mathcal{L} \) is a generic family if any two spaces \( V_\alpha \) and \( V_\beta \) are in “general position” in \( \mathbb{C}^n \) for each pair \( \alpha, \beta \in \mathcal{L} \), i.e. \( V_\alpha \cap V_\beta \) is of the smallest possible dimension, provided that \( V_\alpha \cap V_\beta \) is fixed. More precisely, the spaces \( W_{\alpha, \beta} \subseteq V_\alpha \) and \( V_\beta \) should be in general position as real subspaces of \( \mathbb{C}^n \), where \( W_{\alpha, \beta} \) denotes a subspace of \( V_\alpha \) which does not intersect \( V_\alpha \cap V_\beta \) and which is of a maximal possible dimension.

Denote by \( L \text{Imb}(|\mathcal{L}|, \mathbb{C}^n) \) the space of all linear imbeddings of the space \(|\mathcal{L}|\) into \( \mathbb{C}^n \). Here \(|\mathcal{L}|\) is considered without the ambient space \( \mathbb{C}^n \), but with the fixed real linear structure for each \( V_\alpha \subset |\mathcal{L}|, \alpha \in \mathcal{L} \). Let \( A(2n, \mathbb{R}) \) be the Lie group of all real affine transformations of \( \mathbb{R}^{2n} \approx \mathbb{C}^n \). This group acts naturally on \( L \text{Imb}(|\mathcal{L}|, \mathbb{C}^n) \). For any \( F \in L \text{Imb}(|\mathcal{L}|, \mathbb{C}^n) \) and \( g \in A(2n, \mathbb{R}) \) we denote by \( g(F) \) the imbedding

\[
|\mathcal{L}| \to F(|\mathcal{L}|) \xrightarrow{g} g(F(|\mathcal{L}|)) \subset \mathbb{C}^n.
\]
Lemma 2. Let \( \dim \mathcal{L} \leq n \). Then the linear imbeddings \( F: |\mathcal{L}| \to C^n \) with the property \( "F(|\mathcal{L}|)" \) is totally real and generic” form an open and everywhere dense set \( \mathcal{G} \) in the space \( L_{\text{Imb}}(|\mathcal{L}|, C^n) \). Moreover, for any \( F_0 \in L_{\text{Imb}}(|\mathcal{L}|, C^n) \) the set \( A_{F_0} \) of affine transformations \( g \) with the property \( g(F_0) \) \( \in \mathcal{G} \) form an open and everywhere dense subset of \( A(2n, \mathbb{R}) \).

The properties of \( F(|\mathcal{L}|) \) being totally real and generic are both general position properties. Hence, the openness of \( \mathcal{G} \) in \( L_{\text{Imb}}(|\mathcal{L}|, C^n) \) or of \( A_{F_0} \) in \( A(2n, \mathbb{R}) \) is obvious. So, we have to prove that \( \mathcal{G} \) and \( A_{F_0} \) are everywhere dense in the corresponding spaces.

For any \( V_\alpha \subset F_0(|\mathcal{L}|) \), \( \alpha \in \mathcal{L} \), \( F_0 \in L_{\text{Imb}}(|\mathcal{L}|, C^n) \) consider the subset \( \rho_\alpha \subset A(2n, \mathbb{R}) \) such that \( g \in \rho_\alpha \) iff \( g(V_\alpha) \) is totally real. If \( \dim V_\alpha \leq n \) then one can check that \( \rho_\alpha \) is open and everywhere dense in \( A(2n, \mathbb{R}) \). Consequently, \( \rho_\mathcal{L} = \bigcap_{\alpha \in \mathcal{L}} \rho_\alpha \) is open and everywhere dense as well. Picking some \( \tilde{g} \in \rho_\mathcal{L} \) sufficiently close to the identity one can approximate \( F_0 \) by a totally real imbedding \( \tilde{F}_0 = \tilde{g}(F_0) \). Hence, for \( \dim \mathcal{L} \leq n \) totally real imbeddings are everywhere dense in \( L_{\text{Imb}}(|\mathcal{L}|, C^n) \). Now take any pair of affine subspaces \( V_\alpha, V_\beta \subset F_0(|\mathcal{L}|) \), \( \alpha, \beta \in \mathcal{L} \), such that \( V_\alpha \subsetneq V_\beta \). Recall, that \( W_{\alpha,\beta} \) is a subspace of \( V_\alpha \) of a maximal dimension such that \( W_{\alpha,\beta} \cap (V_\alpha \cap V_\beta) = \emptyset \). Consider the following subset \( \Sigma_{\alpha,\beta} \subset A(2n, \mathbb{R}) \). An element \( g \in \Sigma_{\alpha,\beta} \) iff \( g(W_{\alpha,\beta}) \) is in general position with the complex subspace \( [g(V_\beta)]^C \). Again, the openness of \( \Sigma_{\alpha,\beta} \) is obvious. To prove that \( \Sigma_{\alpha,\beta} \) is dense in \( A(2n, \mathbb{R}) \) we show that the identity transformation \( e \in A(2n, \mathbb{R}) \) can be approximated by some \( g \in A(2n, \mathbb{R}) \) with the property \( g(V_\beta) = V_\beta \) and \( g(W_{\alpha,\beta}) \) being transversal to \( V_\beta^C \). Note, that by the construction, \( W_{\alpha,\beta} \) and \( V_\beta \) are in general position in \( C^n \). Take \( \tilde{W}_{\alpha,\beta} \subset C^n \) sufficiently close to \( W_{\alpha,\beta} \) (so it still will be in general position with \( V_\beta \)) and transverse to \( V_\beta^C \). Now it is easy to construct a real affine transformation \( g \) mapping \( W_{\alpha,\beta} \) onto \( \tilde{W}_{\alpha,\beta} \) and identical on \( V_\beta \). Moreover, this \( g \) can be taken close to \( e \). So, the subset \( \Sigma_{\mathcal{L}} = \rho_\mathcal{L} \cap (\bigcap_{\{V_\alpha \subsetneq \mathcal{L} \}} \Sigma_{\alpha,\beta}) \) of \( A(2n, \mathbb{R}) \) is open and everywhere dense. This implies that totally real and generic imbeddings are open and everywhere dense in \( L_{\text{Imb}}(|\mathcal{L}|, C^n) \), provided that \( \dim \mathcal{L} \leq n. \square \)

Lemma 3. If \( \dim \mathcal{L} \leq \frac{3}{2} n \) then \( \mathcal{L} \) totally real and generic implies that \( \mathcal{L} \) is perfectly generic. Consequently, the set of imbeddings \( F \in L_{\text{Imb}}(|\mathcal{L}|, C^n) \) with the property \( "\mathcal{P}(K) = C(K)" \) for any compact \( K \subset F(|\mathcal{L}|) \) contains an open and everywhere dense subset of \( L_{\text{Imb}}(|\mathcal{L}|, C^n) \).

If \( \dim V_\alpha + 2 \dim V_\beta \leq 2n; \alpha, \beta \in \mathcal{L}, \) and \( \mathcal{L} \) is generic then \( V_\alpha \cap V_\beta^C = V_\alpha \cap V_\beta \) (when \( V_\alpha \cap V_\beta \neq \emptyset \)) or \( V_\alpha \cap V_\beta^C \) is at most a singleton (when \( V_\alpha \cap V_\beta = \emptyset \) and \( \dim V_\alpha + 2 \dim V_\beta = 2n \)) (see Fig. 1). Hence, under
these dimensional assumptions $|D\mathcal{L}| = |\mathcal{L}'| \cup M$, where $|\mathcal{L}'|$ is formed by $V_\alpha$, $\alpha \in \mathcal{L}' \setminus \mathcal{L}$ (i.e. $\alpha$ is not a maximal element of $\mathcal{L}'$) and $M$ is a finite set of points (0-dimensional subspaces) in $C^n$. Note, that $\mathcal{L}$ generic implies that $\mathcal{L}' \cup M$ is a generic family too. In fact, any subfamily of a generic family is generic. So, $\mathcal{L}'$ is generic. By the construction $V_\beta^C \cap M = \emptyset$ for any $\beta \in \mathcal{L}'$. All the higher derivatives $D^s\mathcal{L}$, $s > 1$, will be just subfamilies of $\mathcal{L}$ and, hence, are generic (weakly generic). So, $\mathcal{L}$ is perfectly generic and Lemmas 1 and 2 imply Lemma 3. □

![Figure 1](image_url)

**Remark.** Lemma 3 is the only place where we are using the dimensional restriction $\dim X \leq \frac{3}{2}n$. We conjecture that this lemma holds just if $\dim \mathcal{L} < n$, which would imply Theorems A and B for compact spaces or for finite polyhedra of dimensions less than $n$.

Now we are able to prove Theorem B. Any simplicially linear mapping $F: Y^k \to C^n$ is uniquely determined by the images $\{F(y_j)\}_j$ of the vertices $\{y_j\}_j$ of the simplicial polyhedron $Y^k$. If the points $\{F(y_j)\}$ are in general position over the field $R$ in $C^n \approx R^{2n}$, it follows from standard dimensional considerations that $F$ is an imbedding for $k < n$. Actually, if they are in general position in $C^n$ over $C$, then any real affine subspace passing through arbitrary $s$ points $\{F(y_{j_s})\}_s$, $s \leq n$, is totally real.

Let $\Delta^s_\alpha \subset Y^k$ denote an $s$-dimensional simplex of $Y^k$, where index $\alpha$ enumerates such simplices. For any $\Delta^s_\alpha \subset Y^k$ and $F \in \text{SL Imb}(Y^k, C^n)$ consider the real $s$-dimensional affine subspace $V_{\alpha,F}$ in $C^n$, containing $F(\Delta^s_\alpha)$. This correspondence $\Delta^s_\alpha \mapsto V_{\alpha,F}$ defines a family of subspaces $\mathcal{L}_F$ (the corresponding family $\mathcal{L}_F$ consists of $V_{\alpha,F}$, where $\Delta^s_\alpha$ is not a subsimplex of any other simplex of $Y^k$).
Starting with any mapping $F \in \text{SL Map}(Y^k, C^n)$ one can approximate $F$ by an imbedding $\tilde{F}(k < n)$. Note that the group $A(2n, \mathbb{R})$ acts naturally on $\text{SL Map}(Y^k, C^n)$, moreover, the subspace $\text{SL Imb}(Y^k, C^n) \subset \text{SL Map}(Y^k, C^n)$ obviously is invariant under this action. By Lemma 2 and using the continuity of the correspondence $F \sim |\mathcal{L}_F|$ one can approximate $\tilde{F} \in \text{SL Imb}(Y^k, C^n)$ by some imbedding $g(\tilde{F})$, $g \in A(2n, \mathbb{R})$ with the property $g(|\mathcal{L}_F|)$ is totally real and generic. By Lemma 3 such a family will be perfectly generic, provided that $3k \leq 2n$. Hence, by Lemma 1 $g(\tilde{F}(Y^k)) \subset g(|\mathcal{L}_F|)$ admits polynomial approximation.

The properties "$\mathcal{L}_F$ totally real, generic, perfectly generic" obviously are stable with respect to small perturbations of $F \in \text{SL Imb}(Y^k, C^n)$. Hence, for $k \leq \frac{3}{2}n$ the subset $\{ F \in \text{SL Imb}(Y^k, C^n) | \mathcal{L}_F \text{ is totally real and perfectly generic} \}$ is open and everywhere dense in $\text{SL Map}(Y^k, C)$, which completes the proof of Theorem B.

Now we derive Theorem A from Theorem B.

Let $X^k$ be any compact space. Let $\Theta_{\epsilon, \delta}$ be the subset of $\text{Map}(X^k, C^n)$ defined by the following two properties: (1) the diameter of the inverse-image $F^{-1}(y)$ of any point $y \in C^n$ is less than $\delta$; (2) the functions $\bar{z}_1, \ldots, \bar{z}_n$ on $F(X^k)$, where $\bar{\cdot}$ denotes the complex conjugation, may be approximated to within $\epsilon$ by complex polynomials in $z_1, \ldots, z_n$. It is readily verified that $\Theta_{\epsilon, \delta}$ is an open set of $\text{Map}(X^k, C^n)$.

Now choose some countable monotone sequence $\{ \epsilon_i \} \to 0$, $\{ \delta_i \} \to 0$. It is easy to verify that $\bigcap_i \Theta_{\epsilon_i, \delta_i}$ is the set $\Theta$ of all imbeddings $F$ admitting polynomial approximation on $F(X^k)$. Indeed, if we let $\delta_i \to 0$ property (1) of the sets $\Theta_{\epsilon_i, \delta_i}$ guarantees that the limiting mapping is an imbedding. Property (2) of the sets implies that if $F \in \bigcap_i \Theta_{\epsilon_i, \delta_i}$ then the functions $\{ \bar{z}_j \}$ on the image $F(X^k)$ may be approximated to within arbitrary accuracy by polynomials in $\{ z_j \}$. On the other hand, by the Weierstrass-Stone theorem any continuous function on $F(X^k)$ may be approximated by polynomials in $\{ z_j \}$; hence it may be approximated by polynomials in the variables $\{ z_j \}$ alone.

To complete the proof, it remains to verify that every set $\Theta_{\epsilon_i, \delta_i}$ is dense in $\text{Map}(X^k, C^n)$.

Let $F' \in \text{Map}(X^k, C^n)$ be an arbitrary mapping. In accordance with the classical Alexandroff construction [1], if $m < n$, then for any $\epsilon, \delta > 0$ there is a mapping $F: X^k \to C^n$ such that $F(X^k)$ is contained in a $k$-dimensional simplicial polyhedron $Y^k$ simplicially-linearly imbedded in $C^n$, in such a way that

(a) $\rho(F', F) < \epsilon$, where $\rho$ is the natural distance between mappings;
(b) $\text{diam}(F^{-1}(y)) < \delta$ for any point $y \in Y^k$. 


(A complete proof of this theorem can also be found in [4], Chapter V, §3).

Set $\delta = \delta_j$. By a trivial modification of this construction one can guarantee that, in addition to these two properties (a) and (b), the family of affine subspaces $L_{\text{id}}$ (generated by id: $Y^k \to C^n$) will be totally real and generic (just use the appropriate transformation from $A(2n, \mathbb{R})$). If $3k \leq 2n$ then, by Lemma 3, these properties are a sufficient condition for the existence of polynomial approximation on the polyhedron $Y^k$. The modification is as follows. By Theorem B there exists an imbedding $\kappa: Y^k \to C^n$, arbitrarily close to the original imbedding $\text{id}: Y^k \to C^n$, such that continuous functions admit polynomial approximation on $\kappa(Y^k)$. The imbedding $\kappa \in \text{SL Map}(Y^k, C^n)$ may be chosen in such a way that $\rho(F', \kappa \circ F) < \varepsilon$, while $\text{diam}(F^{-1} \circ \kappa^{-1}(y)) < \delta_j$ for any $y \in C^n$. Moreover, the functions $\{\tilde{z}_j\}$ may be approximated on $\kappa \circ F(X^k)$ to within arbitrary accuracy by polynomials in $\{z_j\}$, i.e., $\kappa \circ F \in \Theta_{\varepsilon, \delta_j}$ and $\kappa \circ F$ is in the $\varepsilon$-neighborhood of the original mapping $F'$. This proves that $\Theta_{\varepsilon, \delta_j}$ is dense in $\text{Map}(X^k, C^n)$.

Recall that for any compact set $K$ in $C^n$ the space of maximal ideals of the algebra $\mathcal{P}(K)$ is precisely the polynomially convex hull of $K$. Therefore, if $\mathcal{P}(K)$ coincides with the algebra of all complex functions, then $K$ is polynomially convex and this property is hereditary with respect to compact subsets of $K$. Thus, if $3k \leq 2n$ the polynomially convex imbeddings of a $k$-dimensional compact space into $C^n$ form a massive set (i.e. of type $G_\delta$). This completes the proof of Theorem A. $\square$

It is obvious that if all continuous functions on a compact subset $K \subset C^n$ admit polynomial approximation, this property is hereditary with respect to closed subsets and therefore, in particular, the intersection $K \cap C^l$ of a compact subset $K$ with any affine complex subspace also admits approximation by polynomials in $z_1, \ldots, z_n$. In particular, in the case $k = l$, it follows from the maximum modulus theorem that the set $K \cap C^1$ is necessarily nowhere dense in $C^1$ and has connected complement.

**Corollary.** Let $X^k$ be a $k$-dimensional compact space. If $3k \leq 2n$, the imbeddings $F \in \text{Map}(X^k, C^n)$ such that the intersection of $F(X^k)$ with any complex straight line $C^1 \subset C^n$ is nowhere dense in $C^1$ and the complement of the intersection is connected in $C^1$ form a dense subset of type $G_\delta$. $\square$

Let $M^k$ be a PL-manifold. Then starting with an arbitrary locally flat PL-imbedding $F_0: M^k \to C^n$ ($k < n$) it is possible to find an element $g \in A(2n, \mathbb{R})$ such that $g(F_0)(M^k)$ will generate a totally real and generic
family of affine subspaces and, hence, for \( k \leq \frac{2}{3}n \) one has polynomial approximation on \( g(F_0)(M^k) \). Moreover, \( g(F_0)(M^k) \) is again locally flat. Thus, by Theorem B for \( k \leq \frac{2}{3}n \) there exists a PL-embedding \( F \) of \( M^k \) in \( \mathbb{C}^n \) with \( F(M^k) \) having a nice normal PL-bundle and admitting polynomial approximation (hence, \( F(M^k) \) is polynomially convex in \( \mathbb{C}^n \)). In particular, the tangent bundle to \( F(M^k) \) is formed by \text{“totally real”} blocks.

Considering smooth or real-analytic manifolds \( M^k \), it would be natural to try to prove \text{“smooth or analytic”} analogs of Theorems A and B. But it seems quite unlikely that such propositions can be established. As a matter of fact, for \( k > \frac{2}{3}n \) there exist profound topological obstacles to the existence of totally real and regular imbedding, i.e., imbeddings \( F: M^k \rightarrow \mathbb{C}^n \) such that \( dF \) is nondegenerate and \( dF(T_x M^k) \) is totally real for any tangent space \( T_x M^k \) of \( M^k \).

One can find a very good discussion of similar and more delicate analytic phenomena in [7] and [8] §§17, 18 basically, for the case \( k \geq n \).

As an example, let us consider regular imbeddings \( F: \mathbb{C}P^k \rightarrow \mathbb{C}^n \) of complex projective space \( \mathbb{C}P^k \). Let \( \tau \) be a tangent bundle of \( F(\mathbb{C}P^k) \) and assume that it is a totally real subbundle of the complex tangent bundle to \( \mathbb{C}^n \). Hence, its complexification \( \tau^C \) is isomorphic to \( \tau \oplus J \tau \), where the infinitesimal operator \( J \) is induced by multiplication of vectors by the imaginary unit \( i \). Let \( \nu \) be the bundle complementary to \( \tau \oplus J(\tau) \), i.e., \( \tau \oplus J(\tau) \oplus \nu = \tau(\mathbb{C}^n) |F(\mathbb{C}P^k) \) is the trivial bundle. Since \( \tau_x \oplus J(\tau_x) \) is a complex subspace of \( \mathbb{C}^n \), we may assume that \( \nu \) is a complex bundle of complex dimension \( n - 2k \). The Chern class \( c(\tau^C) \) of \( \tau^C \) is equal to

\[
\sum_{i=0}^{k} c_i(\tau) \times \sum_{i=0}^{k} (-1)^i c_i(\tau) \quad \text{or} \quad (1 - h^2)^{k+1},
\]

where \( h \in H^2(\mathbb{C}P^k; \mathbb{Z}) \) is a standard generator and \((1 - h^2)^{k+1} \) is considered as an element of the ring \( \mathbb{Z}[h]/\{h^{k+1} = 0\} \) [5]. Since \( \tau^C \oplus \nu \) is trivial, it follows that \( c(\nu) \cdot c(\tau^C) = 1 \). The element \( c(\tau^C) \) is invertible in the ring \( \mathbb{Z}[h]/\{h^{k+1} = 0\} \). As a representative of the inverse element, we take the polynomial \( \sum_{i=0}^{[k/2]} \alpha_i h^{2i} \), where \( \{\alpha_i\} \) are different from zero. Therefore \( c(\nu) = \sum_{i=0}^{[k/2]} \alpha_i h^{2i} \) and, since \( \alpha_i \neq 0 \), the complex dimension \( n - 2k \) of \( \nu \) cannot be less than \( 2 \cdot [k/2] \). Thus, when \( n < 2k + 2[k/2] \), there exist no totally real immersions of \( \mathbb{C}P^k \) into \( \mathbb{C}^n \). In fact, the Euler class of the normal bundle of oriented submanifolds in \( \mathbb{R}^{2n} \) should be trivial [5], which ruins the possibility for regular totally real imbeddings \( \mathbb{C}P^k \rightarrow \mathbb{C}^{3k} \), \( k \) even. Since \( \dim_{\mathbb{R}} \mathbb{C}P^k = 2k \), it follows that the \text{“allowed”} dimensions \( n \) satisfy
the conditions $3 \dim_R(\mathbb{C}P^k) < 2n$, which should be compared with the dimensional condition that figures in Theorems A and B.

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