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QUASI-NORMAL STRUCTURES FOR CERTAIN SPACES OF OPERATORS ON A HILBERT SPACE

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Let E be a dual Banach space. E is said to have quasi-weak*-normal structure if for each weak* compact convex subset K of E there exists $x \in K$ such that $\|x - y\| < \operatorname{diam}(K)$ for all $y \in K$. E is said to satisfy Lim's condition if whenever $\{x_\alpha\}$ is a bounded net in E converging to 0 in the weak* topology and $\lim \|x_\alpha\| = s$ then $\lim_\alpha \|x_\alpha + y\| = s + \|y\|$ for any $y \in E$. Lim's condition implies (quasi) weak*-normal structure. Let H be a Hilbert space. In this paper, we prove that $\mathcal{T}(H)$, the space of trace class operators on H, always has quasi-weak*-normal structure for any H; $\mathcal{T}(H)$ satisfies Lim's condition if and only if H is finite dimensional. We also prove that the space of bounded linear operator on H has quasi-weak*-normal structure if and only if H is finite dimensional; the space of compact operators on H has quasi-weak-normal structure if and only if H is separable. Finally we prove that if X is a locally compact Hausdorff space, then $C_0(X)^*$ satisfies Lim's condition if and only if $C_0(X)^*$ is isometrically isomorphic to $l_1(\Gamma)$ for some non-empty set Γ.

1. Introduction. Let E be a Banach space. A bounded convex subset K of E has *normal structure* if every non-trivial convex subset H of K contains a point x_0 such that

$$\sup\{\|x_0 - y\| : y \in H\} < \operatorname{diam}(H).$$

Here $\operatorname{diam}(H) = \sup\{\|x - y\|: x, y \in H\}$ denotes the diameter of H. The Banach space E is said to have normal structure if every bounded closed convex subset of E has normal structure. If E is a dual space then E is said to have weak* normal structure if every weak* compact convex subset of E has normal structure. In **[6]** Lim introduced the notion of weak* normal structure and proved that l_1 has this property. It also follows from the proof of Theorem 3 in **[4]** that $l_1(\Gamma)$ has the same property for any non-empty set Γ. Furthermore, an application of Proposition 2 in **[9]** shows that $l_\infty(\Gamma)$ has weak* normal structure if and only if Γ is a finite set.

Let H be a Hilbert space. Let $\mathcal{B}(H)$ be the space of bounded linear operators from H into itself with the operator norm. Let $\mathcal{C}(H)$ be the closed ideal of compact operators in $\mathcal{B}(H)$. Then, as is well known,

$\mathscr{C}(H)^{**} = \mathscr{B}(H)$ and $\mathscr{C}(H)^*$ can be identified with $\mathscr{T}(H)$, the trace-class operators on H with the trace norm (see [**12**, pp. 63–64]). When H is infinite dimensional, it is known that $\mathscr{C}(H)$ and $\mathscr{T}(H)$ contain isometric copies of c_0 and l_1, respectively [**12**, Proposition 1.4 and Theorem 1.6 p. 62]. It follows, then, that the Banach spaces $\mathscr{C}(H)$, $\mathscr{T}(H)$ and $\mathscr{B}(H)$ do not have normal structure unless H is finite dimensional [**1**].

A concept weaker than that of normal structure was introduced by Soardi in [**11**]. A bounded convex subset K of a Banach space has quasi-normal structure (or close-to-normal structure [**13**]) if for every non-trivial closed convex subset H of K, there exists $x \in H$ such that $\|x - y\| < \operatorname{diam}(H)$ for all $y \in H$. A Banach space has quasi-normal structure (quasi-weak-normal structure) if every bounded (weakly compact) closed convex subset has quasi-normal structure. If, in addition , it is a dual Banach space then it has quasi-weak*-normal structure if every weak* compact convex subset has quasi-normal structure.

In §2 of this paper, we prove, among other things, three theorems on quasi-normal structure and its generalizations for certain spaces of operators on a Hilbert space H. First, we prove that $\mathscr{B}(H)$ has quasi-weak*-normal structure if and only if H is finite dimensional (Theorem 1). Secondly, we prove that $\mathscr{T}(H)$ has quasi-weak*-normal structure for any H (Theorem 2). Finally, we prove that $\mathscr{C}(H)$ has quasi-weak-normal structure if and only if H is separable (Theorem 3). A table summarizing our results is provided at the end.

Let E be a Banach space. Then E^* is said to satisfy *Lim's condition* if whenever $\{\phi_\alpha\}$ is a bounded net in E^*, ϕ_α converges to 0 in the weak* topology and $\lim_\alpha\|\phi_\alpha\| = s$, then $\lim_\alpha\|\phi_\alpha + \psi\| = s + \|\psi\|$ for any $\psi \in E^*$. In [**6**], Lim showed that l_1 satisfies this condition for sequences. Also a simple modification of the proof of Theorem 3 [**4**] shows that Lim's condition implies weak* normal structure (see Lemma 4). We prove in section 4 that (Theorem 4) if X is a locally compact Hausdorff space, then the dual Banach space $C_0(X)^*$ satisfy Lim's condition if and only if $C_0(X)^*$ is isometric isomorphic to $l_1(\Gamma)$ for some non-empty set Γ. We also prove that (Theorem 5) if H is a Hilbert space, then $\mathscr{T}(H)$ satisfy Lim's condition if and only if H is finite dimensional.

As known [**6**, Theorem 1], if E is dual Banach space with weak* normal structure, then every nonexpansive mapping T of a non-empty weak* compact convex subset K of E (i.e. $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in K$) into itself has a fixed point. Also [**13**, Theorem 1] if E is a Banach space with quasi-weak-normal structure if and only if every

Kannan map T of a non-empty weakly compact convex subset K of E (i.e. $\|Tx - Ty\| \le (\|x - Tx\| + \|y - Ty\|)/2$, for any $x, y \in K$) into itself has a fixed point.

2. Quasi-normal structures.

THEOREM 1. *Let H be a Hilbert space. Then $\mathscr{B}(H)$ has quasi-weak*-normal structure if and only if H is finite dimensional.*

Proof. If H is finite dimensional, then $\mathscr{B}(H)$ is finite dimensional. Hence $\mathscr{B}(H)$ has normal structure.

Conversely if H is infinite dimensional, write $H = l_2(\Gamma)$ where Γ is a complete orthonormal basis of H. Consider the map ρ: $l_\infty(\Gamma) \to \mathscr{B}(l_2(\Gamma))$ defined by

$$\rho(f)(h)(t) = f(t)h(t), \qquad t \in \Gamma.$$

Then ρ is an isometry and algebra isomorphism of $l_\infty(\Gamma)$ into $\mathscr{B}(l_2(\Gamma))$ which is continuous when $l_\infty(\Gamma)$ has the weak* topology and $\mathscr{B}(H)$ has the weak operator topology. By Proposition 2 in [**9**], there exists a weak* compact convex subset K of $l_\infty(\Gamma)$ such that for each $f \in K$, there exists $g \in K$ with $\|f - g\|_\infty = \operatorname{diam}(K) > 0$. Since weak* topology and the weak operator topology agree on bounded subset of $\mathscr{B}(H)$, $\rho(K)$ is also a weak* compact convex subset of $\mathscr{B}(H)$ with positive diameter. In particular $\mathscr{B}(H)$ does not have the quasi-weak*-normal structure.

LEMMA 1. *Let E be a dual Banach space. Then E has quasi-weak*-normal structure if it satisfies*

(**) *whenever $\{x_\alpha\}$ is a net in E, x_α converges to x in the weak* topology and $\|x_\alpha\|$ converges to $\|x\|$, then x_α converges to x in norm.*

Proof. Suppose there exists a weak* compact convex subset K of E, $\operatorname{diam}(K) > 0$, such that for each $x \in K$, there exists $T(x) \in K$ with $\|x - T(x)\| = \operatorname{diam}(K)$. Following an idea of Wong [**13**, Theorem 2], let $W(K)$ denote the supremum of $\{|H|; H \text{ is a diametral subset of } K\}$ (H is diametral if $\|x_1 - x_2\| = \operatorname{diam}(K)$ whenever $x_1, x_2 \in H$, $x_1 \neq x_2$). As shown in the proof of Theorem 2 in [**13**], $W(K)$ is infinite. Let $\{x_n\}$ be a sequence in K such that $\|x_n - x_m\| = \operatorname{diam}(K)$, $n \neq m$. Since K is weak* compact, there exists a subnet $\{x_{n_\alpha}\}$ of $\{x_n\}$ such that x_{n_α} converges to some $z \in K$ in the weak*-topology. Passing to a subnet if necessary, we may assume that the net $\{\|x_{n_\alpha} - T(z)\|\}$ also converges. Then

$$\operatorname{diam}(K) = \|z - T(z)\| \le \lim_\alpha \|x_{n_\alpha} - T(z)\| \le \operatorname{diam}(K).$$

So $\lim_\alpha \|x_{n_\alpha} - T(z)\| = \operatorname{diam}(K)$. Since $\{x_{n_\alpha} - T(z)\}$ converges in the weak* topology to $z - T(z)$, and $\lim_\alpha \|x_{n_\alpha} - T(z)\| = \|z - T(z)\|$, it follows that $\{x_{n_\alpha} - T(z)\}$ converges in norm to $z - T(z)$. In particular, the net $\{x_{n_\alpha}\}$ converges in norm to z also. This contradicts the choice of the sequence $\{x_n\}$.

The next lemma is due to K. McKennon [7, Lemma, 3.2]. For the sake of completeness, we give a short proof.

LEMMA 2 (*McKennon* [7]). *Let A be a C^*-algebra and $\{e_\alpha\}$ be an approximate identity of A, $e_\alpha \ge 0$ and $\|e_\alpha\| \le 1$. Let $\{\phi_\beta\}$ be a net in A^* such that $\phi_\beta \to \phi$ in the weak* topology and $\|\phi_\beta\| \to \|\phi\|$. Then for any $\varepsilon > 0$, there exists α_0, β_0 such that*

(1) $$\|R_{e_{\alpha_0}}\phi - \phi\| < \varepsilon$$

and

(2) $$\|R_{e_{\alpha_0}}\phi_\beta - \phi_\beta\| < \varepsilon$$

for all $\beta \ge \beta_0$, where $R_e\phi(x) = \phi(xe)$.

Proof. Let $x \in A$, $\|x\| \le 1$. Then using [5, Lemma 3.3] and some properties of positive linear functionals, we obtain the following estimate

$$\begin{aligned} |\langle R_{e_\alpha}\phi - \phi, x\rangle|^2 &= |\langle \phi, x \cdot e_\alpha - x\rangle|^2 = |\langle \phi, x \cdot (1 - e_\alpha)\rangle|^2 \\ &\le \|\phi\|\, |\phi|\big[(x \cdot (1 - e_\alpha))^*(x \cdot (1 - e_\alpha))\big] \\ &= \|\phi\|\, |\phi|\, [(1 - e_\alpha)x^*x(1 - e_\alpha)] \\ &\le \|\phi\|\, |\phi|\, [(1 - e_\alpha)(1 - e_\alpha)] \le \|\phi\|\, |\phi|(1 - e_\alpha), \end{aligned}$$

where 1 is the identity in the enveloping von Neumann algebra A^{**} of A and $|\phi|$ is the absolute value of ϕ. Since the net $\{e_\alpha\}$ converges to 1 in the weak* topology of A^{**}, (1) follows from the above estimate.

A similar estimate as above shows that

$$\|R_{e_{\alpha_0}}\phi_\beta - \phi_\beta\| \le \|\phi_\beta\|\, |\phi_\beta|(1 - e_{\alpha_0}).$$

Using [5, Lemma 3.5] and the fact that for each positive form $|\phi_\beta|$, $\|\,|\phi_\beta|\,\| = \|\phi_\beta\| = |\phi_\beta|(1)$, the right side of the above estimate converges to $\|\phi\|\, |\phi|(1 - e_{\alpha_0})$. Hence (2) follows.

THEOREM 2. *Let H be a Hilbert space. Then $\mathcal{T}(H)$ has the quasi-weak*-normal structure.*

Proof. By Lemma 1, it suffices to show that $\mathcal{T}(H) = \mathcal{C}(H)^*$ has property $(**)$. Let $\mathcal{P}$ denote all orthogonal projections of H onto a finite dimensional subspace of H. Order $\mathcal{P}$ by: $P \geq Q$ iff $QP = PQ = Q$. Then $(\mathcal{P}, \leq)$ is an approximate identity for $\mathcal{C}(H)$. Since every $T \in \mathcal{C}(H)$ can be written in the form $T = T_1 + iT_2$, T_i self-adjoint, $i = 1, 2$, it suffices to show that if T is self-adjoint, then $\lim \|TP - T\| = \lim \|PT - T\| = 0$. Indeed, if $T \in \mathcal{C}(H)$ and T is self-adjoint, then by the spectral theorem $T = \sum_{i=1}^{\infty} \lambda_i P_i$, where $\lambda_i \to 0$ as $i \to \infty$ and $P_i \in \mathcal{P}$. Given $\varepsilon > 0$ choose n such that $\|T - \sum_{i=1}^{n} \lambda_i P_i\| < \varepsilon$. Let $Q \in \mathcal{P}$ be such that $Q \geq P_i$, $i = 1, 2, \ldots, n$. Then for all $P \geq Q$,

$$\|TP - T\| \leq \|TP - S_n P\| + \|S_n P - S_n\| + \|S_n - T\| < 2\varepsilon,$$

where $S_n = \sum_{i=1}^{n} \lambda_i P_i$. Similarly, we can show $\lim \|PT - T\| = 0$. We also note that each $P \in \mathcal{P}$ is positive and has norm one.

Let $\{\phi_\beta\}$ be a net in $\mathcal{C}(H)^*$ converging to some $\phi \in \mathcal{C}(H)^*$ in the weak* topology and $\|\phi_\beta\| \to \|\phi\|$. By Lemma 2, there exists $P_0 \in \mathcal{P}$ and β_0 such that

$$\|R_{P_0}\phi - \phi\| < \varepsilon/2 \quad \text{and} \quad \|R_{P_0}\phi_\beta - \phi_\beta\| < \varepsilon/2 \tag{3}$$

for all $\beta \geq \beta_0$. By considering the reversed C^*-algebra, we may also assume that

$$\|L_{P_0}\phi - \phi\| < \varepsilon/2 \quad \text{and} \quad \|L_{P_0}\phi_\beta - \phi_\beta\| < \varepsilon/2. \tag{4}$$

where $(L_{P_0}\phi)(T) = \phi(P_0 T)$, $T \in \mathcal{C}(H)$. Consequently, if $\beta \geq \beta_0$,

$$\|R_{P_0}L_{P_0}\phi - \phi\| \leq \|R_{P_0}L_{P_0}\phi - R_{P_0}\phi\| + \|R_{P_0}\phi - \phi\| < \varepsilon \tag{5}$$

since $\|R_{P_0}\| \leq \|P_0\| = 1$ by (3) and (4). Similarly

$$\|R_{P_0}L_{P_0}\phi_\beta - \phi_\beta\| < \varepsilon. \tag{6}$$

Also, $P_0\mathcal{C}(H)P_0$ is a finite dimensional algebra over C. Hence, $\{\phi_\beta\}$, restricted to $P_0\mathcal{C}(H)P_0$ converges to ϕ in norm. Consequently, there exists $\beta_1 \geq \beta_0$ such that

$$\|T_{P_0}L_{P_0}\phi_\beta - R_{P_0}L_{P_0}\phi\| < \varepsilon \tag{7}$$

if $\beta \geq \beta_1$. Now if $\beta \geq \beta_1$, we have

$$\|\phi_\beta - \phi\| \leq \|\phi_\beta - R_{P_0}L_{P_0}\phi_\beta\| + \|R_{P_0}L_{P_0}\phi_\beta - R_{P_0}L_{P_0}\phi\| + \|R_{P_0}L_{P_0}\phi - \phi\| < 3\varepsilon.$$

by (5), (6) and (7).

REMARK. Clearly if a dual Banach space E has the weak* normal structure then E has the quasi-weak*-normal structure. But the converse is false. Indeed, let E the space of absolutely summable real sequences with norm

$$\|x\| = \max\{\|x^+\|_1, \|x^-\|_1\}$$

where x^+, x^- denote the positive and negative part of x respectively and $\|x\|_1 = \Sigma_{i=1}^{\infty}|x_i|$. Then, as shown by Lim [**6**] (Lemma 1 and Example 1), E is a dual Banach space which does not have weak* normal structure. However, since E is separable, an argument similar to that of Wong [**13**, Theorem 2] shows that E has quasi-weak*-normal structure.

Problem 1. Does the trace class operator $\mathscr{T}(H) = \mathscr{C}(H)^*$ with dual norm have the weak* normal structure or the weak normal structure?

LEMMA 3. *Let Γ be a non-empty set. Then $c_0(\Gamma)$ has the quasi-weak-normal structure if and only if Γ is countable.*

Proof. If Γ is countable then $c_0(\Gamma)$ is norm separable. Hence each weakly compact convex subset of $c_0(\Gamma)$ has quasi-normal structure by Theorem 2 in [**13**].

Conversely, if Γ is not countable, consider Γ as a group (say the free group on $|\Gamma|$ generators). Pick $a \in \Gamma$. Let $f = \delta_a$ i.e. $f(x) = 1$ if $x = a$ and $f(x) = 0$ if $x \neq a$. Let K denotes the closed convex hull of $\{l_x f;\ x \in \Gamma\}$, where $(l_x f)(t) = f(xt)$, $t \in \Gamma$. Then K is weakly compact ([**2**, Corollary 3.7]) and diam$(K) = 1$. Now if $g \in K$, let $\sigma \subseteq \Gamma$ be a countable set such that $g(t) = 0$ if $t \in \Gamma \sim \sigma$. Pick $z \in \Gamma \sim \sigma$ and let $h = \delta_z$. Then $h \in K$ and $\|g - h\|_\infty = 1$. Hence K does not have quasi-normal structure.

THEOREM 3. *Let H be a Hilbert space. Then H is separable if and only if $\mathscr{C}(H)$ has quasi-weak-normal structure.*

Proof. If H is separable, then $\mathscr{C}(H)^*$ is separable [**10**, Proposition 2.1.10]). Hence $\mathscr{C}(H)$ is separable. Consequently every weakly compact convex subset of $\mathscr{C}(H)$ has quasi-normal structure by [**13**, Theorem 2].

Conversely, if H is not separable, then H is isomorphic to $l_2(\Gamma)$ for an uncountable set Γ. Consider the map ρ: $c_0(\Gamma) \to \mathscr{B}(l_2(\Gamma))$ defined by

$$\rho(f)(h)(t) = f(t)h(t), \qquad t \in \Gamma,$$

then ρ is an isometry and an algebra isomorphism of $c_0(\Gamma)$ into $\mathscr{B}(l_2(\Gamma))$. Furthermore, $\rho(f)$ is compact for each $f \in c_0(\Gamma)$. By Lemma 3, there exists a weakly compact convex subset K in $c_0(\Gamma)$ which does not have quasi-normal structure. In particular, $\rho(K)$ is a weakly compact convex subset of $\mathscr{C}(H)$ which does not have quasi-normal structure also.

Summary. In the Table we shall abbreviate normal structure by n.s., quasi-normal structure by q.n.s., etc. We assume Γ is not finite and H is not finite dimensional.

$c_0(\Gamma)$	$l_1(\Gamma)$	$l_\infty(\Gamma)$
No n.s.	No n.s.	No n.s.
q.w.n.s.	w*.n.s.	No q.w*.n.s
⇕		
Γ is countable	w.n.s.	

$\mathscr{C}(H)$	$\mathscr{T}(H)$	$\mathscr{B}(H)$
No n.s.	No n.s.	No n.s.
q.w.n.s.	w*.n.s.(?)	No q.w*.n.s.
⇕		
H is separable	w.n.s.(?)	
	q.w*.n.s.	

3. On Lim's condition. Let E be a Banach space. Then E^* is said to satisfy *Lim's condition* if whether $\{\phi_\alpha\}$ is a bounded net in E^*, ϕ_α converges to 0 in the weak* topology and $\lim_\alpha \|\phi_\alpha\| = s$, then $\lim_\alpha \|\phi_\alpha + \psi\| = s + \|\psi\|$ for any $\psi \in E^*$.

In [**6**], Lim showed that l_1 satisfies this condition for sequences.

LEMMA 4. *Let E be a Banach space. If E^* satisfies Lim's condition, then E^* has the following properties*:

(a) *Norm and weak* topology agree on* $S = \{\phi \in E^*; \|\phi\| = 1\}$

(b) *For any* $0 < \varepsilon < 2$, *if* $\{\phi_\alpha\}$ *is a net in* E^*, $\|\phi_\alpha\| \le 1$, $\phi_\alpha \to \phi$ *in the weak*-topology and* $\|x_\alpha - x_\beta\| \ge \varepsilon$ *for all* $\alpha \ne \beta$, *then* $\|\phi\| \le 1 - \varepsilon/2$.

In particular, E^* has the Radon Nikodym Property and weak normal structure.*

Proof. (a) Let $\{\phi_\alpha\}$ be a net in S, $\phi \in S$ such that $\phi_\alpha \to \phi$ in the weak*-topology. Suppose $\|\phi_\alpha - \phi\| \to 0$. Then we may assume, by passing to a subnet if necessary, that $\|\phi_\alpha - \phi\| \ge \varepsilon$ for each α. Since $\{\|\phi_\alpha - \phi\|\}$ is bounded by 2, we may further assume that $\lim_\alpha \|\phi_\alpha - \phi\| = s \ge \varepsilon > 0$.

Let $\psi_\alpha = \phi_\alpha - \phi$. Then $\psi_\alpha \to 0$ in the weak*-topology but

$$1 = \lim_\alpha \|\phi + (\phi_\alpha - \phi)\| = \|\phi\| + s > 1$$

which is impossible.

(b) We may assume that $\|\phi_\alpha - \phi\| \geq \varepsilon/2$ for each α, and

$$\lim_\alpha \|\phi_\alpha - \phi\| = s.$$

Then by Lim's condition,

$$\lim_\alpha \|\phi_\alpha\| = \lim_\alpha \|(\phi_\alpha - \phi) + \phi\| = s + \|\phi\|$$

i.e. $s + \|\phi\| \leq 1$ or $\|\phi\| \leq 1 - s \leq 1 - \varepsilon/2$.

The last statement follows from Corollary 8 and Proposition 9 in [**8**], and the proof of Theorem 3 [4] (That E^* has weak* normal structure also follows simple modification of Lim's proof of Theorem 3 in [**6**]).

Given a locally compact Hausdorff space X, let $C_0(X)$ denote the C^*-algebra of complex-valued continuous functions f on X such that for any $\varepsilon > 0$ there exists a compact subset σ of X such that $|f(x)| \leq \varepsilon$ for $x \in X \sim \sigma$ with the supremum norm.

THEOREM 4. *Let X be a locally compact Hausdorff space. The dual Banach space $C_0(X)^*$ satisfies Lim's condition if and only if $C_0(X)^*$ is isometric isomorphic to $l_1(\Gamma)$ for some non-empty set Γ.*

Proof. If $C_0(X)^*$ satisfies Lim's condition, then, by Lemma 4, $C_0(X)^*$ has the Radon Nikodym Property. Since $C_0(X)^{**} = M$ is the enveloping von Neumann algebra of the C^*-algebra $C_0(X)$, it follows from Theorem 4 in [3] that M is the direct sum of Type I factors i.e. M is isomorphic to $\Sigma_{\alpha\in\Gamma} \oplus \mathcal{B}(H_\alpha)$. Since M is commutative, $H_\alpha = C$ for each $\alpha \in \Gamma$. In particular, $C_0(X)^* \approx l_1(\Gamma)$.

Suppose $C_0(X)^*$ is isometric isomorphic to $l_1(\Gamma)$ for some non empty set Γ. We may assume that Γ is infinite. Let $\{f_\alpha\}$ be a bounded net in $l_1(\Gamma)$ such that $f_\alpha \to 0$ in the weak*-topology and $\lim_\alpha \|f_\alpha\| = s$. Let $g \in l_1(\Gamma)$. Since $\|f_\alpha - g\| \leq \|f_\alpha\| + \|g\|$ for each α, we may assume, by passing to a subnet if necessary, that $\lim_\alpha \|f_\alpha - g\| = t$ exists. Clearly we have $t \leq s + \|g\|$. To see that we actually have equality, let $\varepsilon > 0$. Observe that in $l_1(\Gamma)$,

$$\text{(1)} \qquad \|f_\alpha - g\| \geq \|f_\alpha\| - \|g\| + 2\sum_{s\in\sigma}\left(|g(s)| - |f_\alpha(s)|\right)$$

for any subset σ of Σ. Now let σ be a finite subset such that $\Sigma_{s\in\sigma}|g(s)| > \|g\| - \varepsilon$. For this σ, we can choose α_0, using the weak* convergence of f_α and the convergence of $\|f_\alpha\|$, so that for all $\alpha \geq \alpha_0$ we have $\Sigma_{s\in\sigma}|f_\alpha(s)| < \varepsilon$ and $\|f_\alpha\| > s - \varepsilon$. Then for all $\alpha \geq \alpha_0$ we have from (1)

$$\|f_\alpha - g\| \geq s - \varepsilon - \|g\| + 2\|g\| - 2\varepsilon - 2\varepsilon = s + \|g\| - 5\varepsilon.$$

Thus $t \geq s + \|g\|$.

Problem 2. Let X be a locally compact Hausdorff space. When does $C_0(X)^*$ have the weak* normal structure?

THEOREM 5. *Let H be a Hilbert space. Then $\mathscr{T}(H)$ satisfies Lim's condition if and only if H is finite dimensional.*

Proof. If H is finite dimensional, then $\mathscr{T}(H)$ is finite dimensional. Hence $\mathscr{T}(H)$ satisfies Lim's condition.

If H is infinite dimensional, let $\{\xi_n, n = 1,2,\ldots\}$ be an orthonormal sequence in H. For each $n = 1,2,\ldots,$ define $\phi_n(T) = \langle T\xi_1, \xi_n\rangle$. Then $\phi_n \in \mathscr{T}(H)$, $\|\phi_n\| = 1$ and $\phi_n \to 0$ weakly. Indeed, if $T \in \mathscr{B}(H)$, then

$$\infty > \|T\xi_1\|^2 = \sum_{\alpha\in I}^{\infty} |\langle T\xi_1, \xi_\alpha\rangle|^2 \geq \sum_{n=1}^{\infty} |\langle T\xi_1, \xi_n\rangle|^2 = \sum_{n=1}^{\infty} |\phi_n(T)|^2$$

where $\{\xi_\alpha\}_{\alpha\in I}$ is a complete orthonormal set of H containing $\{\xi_n\}$. So $\phi_n(T) \to 0$. Also $\|\phi_n - \phi_1\| \leq \sqrt{2}$ for each n. Hence $\overline{\lim}_n \|\phi_n - \phi_1\| \leq \sqrt{2}$ i.e. $\lim_n \|\phi_n - \phi_1\| \neq \lim \|\phi_n\| + \|\phi_1\|$. In particular, $\mathscr{T}(H)$ does not satisfy Lim's condition.

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