QUASINORMAL STRUCTURES FOR CERTAIN SPACES OF OPERATORS ON A HILBERT SPACE

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Let $E$ be a dual Banach space. $E$ is said to have quasi-weak*-*normal structure if for each weak* compact convex subset $K$ of $E$ there exists $x \in K$ such that $\|x - y\| < \text{diam}(K)$ for all $y \in K$. $E$ is said to satisfy Lim's condition if whenever \{ $x_\alpha$ \} is a bounded net in $E$ converging to 0 in the weak* topology and $\lim \|x_\alpha\| = s$ then $\lim \|x_\alpha + y\| = s + \|y\|$ for any $y \in E$. Lim's condition implies (quasi) weak*-*normal structure.

Let $H$ be a Hilbert space. In this paper, we prove that $\mathcal{B}(H)$, the space of trace class operators on $H$, always has quasi-weak*-normal structure for any $H$; $\mathcal{T}(H)$ satisfies Lim's condition if and only if $H$ is finite dimensional. We also prove that the space of bounded linear operator on $H$ has quasi-weak*-normal structure if and only if $H$ is finite dimensional; the space of compact operators on $H$ has quasi-weak-normal structure if and only if $H$ is separable. Finally we prove that if $X$ is a locally compact Hausdorff space, then $C_0(X)^*$ satisfies Lim's condition if and only if $C_0(X)^*$ is isometrically isomorphic to $l_1(\Gamma)$ for some non-empty set $\Gamma$.

1. Introduction. Let $E$ be a Banach space. A bounded convex subset $K$ of $E$ has normal structure if every non-trivial convex subset $H$ of $K$ contains a point $x_0$ such that

$$\sup\{\|x_0 - y\|: y \in H\} < \text{diam}(H).$$

Here $\text{diam}(H) = \sup\{\|x - y\|: x, y \in H\}$ denotes the diameter of $H$. The Banach space $E$ is said to have normal structure if every bounded closed convex subset of $E$ has normal structure. If $E$ is a dual space then $E$ is said to have weak* normal structure if every weak* compact convex subset of $E$ has normal structure. In [6] Lim introduced the notion of weak* normal structure and proved that $l_1$ has this property. It also follows from the proof of Theorem 3 in [4] that $l_1(\Gamma)$ has the same property for any non-empty set $\Gamma$. Furthermore, an application of Proposition 2 in [9] shows that $l_\infty(\Gamma)$ has weak* normal structure if and only if $\Gamma$ is a finite set.

Let $H$ be a Hilbert space. Let $\mathcal{B}(H)$ be the space of bounded linear operators from $H$ into itself with the operator norm. Let $\mathcal{C}(H)$ be the closed ideal of compact operators in $\mathcal{B}(H)$. Then, as is well known,
\( \mathfrak{C}(H)^{**} = \mathfrak{B}(H) \) and \( \mathfrak{C}(H)^* \) can be identified with \( \mathcal{T}(H) \), the trace-class operators on \( H \) with the trace norm (see [12, pp. 63–64]). When \( H \) is infinite dimensional, it is known that \( \mathfrak{C}(H) \) and \( \mathcal{T}(H) \) contain isometric copies of \( c_0 \) and \( l_1 \), respectively [12, Proposition 1.4 and Theorem 1.6 p. 62]. It follows, then, that the Banach spaces \( \mathfrak{C}(H) \), \( \mathcal{T}(H) \) and \( \mathfrak{B}(H) \) do not have normal structure unless \( H \) is finite dimensional [1].

A concept weaker than that of normal structure was introduced by Soardi in [11]. A bounded convex subset \( K \) of a Banach space has quasi-normal structure (or close-to-normal structure [13]) if for every non-trivial closed convex subset \( H \) of \( K \), there exists \( x \in H \) such that \( \|x - y\| < \text{diam}(H) \) for all \( y \in H \). A Banach space has quasi-normal structure (quasi-weak-normal structure) if every bounded (weakly compact) closed convex subset has quasi-normal structure. If, in addition, it is a dual Banach space then it has quasi-weak*-normal structure if every weak* compact convex subset has quasi-normal structure.

In §2 of this paper, we prove, among other things, three theorems on quasi-normal structure and its generalizations for certain spaces of operators on a Hilbert space \( H \). First, we prove that \( \mathfrak{B}(H) \) has quasi-weak*-normal structure if and only if \( H \) is finite dimensional (Theorem 1). Secondly, we prove that \( \mathcal{T}(H) \) has quasi-weak*-normal structure for any \( H \) (Theorem 2). Finally, we prove that \( \mathfrak{C}(H) \) has quasi-weak-normal structure if and only if \( H \) is separable (Theorem 3). A table summarizing our results is provided at the end.

Let \( E \) be a Banach space. Then \( E^* \) is said to satisfy Lim's condition if whenever \( \{ \phi_\alpha \} \) is a bounded net in \( E^* \), \( \phi_\alpha \) converges to 0 in the weak* topology and \( \lim \alpha \| \phi_\alpha \| = s \), then \( \lim \alpha \| \phi_\alpha + \psi \| = s + \| \psi \| \) for any \( \psi \in E^* \). In [6], Lim showed that \( l_1 \) satisfies this condition for sequences. Also a simple modification of the proof of Theorem 3 [4] shows that Lim's condition implies weak* normal structure (see Lemma 4). We prove in section 4 that (Theorem 4) if \( X \) is a locally compact Hausdorff space, then the dual Banach space \( C_0(X)^* \) satisfy Lim's condition if and only if \( C_0(X)^* \) is isometric isomorphic to \( l_1(\Gamma) \) for some non-empty set \( \Gamma \). We also prove that (Theorem 5) if \( H \) is a Hilbert space, then \( \mathcal{T}(H) \) satisfy Lim's condition if and only if \( H \) is finite dimensional.

As known [6, Theorem 1], if \( E \) is dual Banach space with weak* normal structure, then every nonexpansive mapping \( T \) of a non-empty weak* compact convex subset \( K \) of \( E \) (i.e. \( \| Tx - Ty \| \leq \| x - y \| \) for any \( x, y \in K \)) into itself has a fixed point. Also [13, Theorem 1] if \( E \) is a Banach space with quasi-weak-normal structure if and only if every
Kannan map $T$ of a non-empty weakly compact convex subset $K$ of $E$ (i.e. $||Tx - Ty|| \leq (||x -Tx|| + ||y - Ty||)/2$, for any $x, y \in K$) into itself has a fixed point.

2. Quasi-normal structures.

**Theorem 1.** Let $H$ be a Hilbert space. Then $\mathcal{B}(H)$ has quasi-weak*-normal structure if and only if $H$ is finite dimensional.

**Proof.** If $H$ is finite dimensional, then $\mathcal{B}(H)$ is finite dimensional. Hence $\mathcal{B}(H)$ has normal structure.

Conversely if $H$ is infinite dimensional, write $H = l_2(\Gamma)$ where $\Gamma$ is a complete orthonormal basis of $H$. Consider the map $\rho: l_\infty(\Gamma) \to \mathcal{B}(l_2(\Gamma))$ defined by

$$\rho(f)(h)(t) = f(t)h(t), \quad t \in \Gamma.$$ 

Then $\rho$ is an isometry and algebra isomorphism of $l_\infty(\Gamma)$ into $\mathcal{B}(l_2(\Gamma))$ which is continuous when $l_\infty(\Gamma)$ has the weak* topology and $\mathcal{B}(H)$ has the weak operator topology. By Proposition 2 in [9], there exists a weak* compact convex subset $K$ of $l_\infty(\Gamma)$ such that for each $f \in K$, there exists $g \in K$ with $||f - g||_\infty = \text{diam}(K) > 0$. Since weak* topology and the weak operator topology agree on bounded subset of $\mathcal{B}(H)$, $\rho(K)$ is also a weak* compact convex subset of $\mathcal{B}(H)$ with positive diameter. In particular $\mathcal{B}(H)$ does not have the quasi-weak*-normal structure.

**Lemma 1.** Let $E$ be a dual Banach space. Then $E$ has quasi-weak*-normal structure if it satisfies

whenever $\{x_\alpha\}$ is a net in $E$, $x_\alpha$ converges to $x$ in the weak* topology and $||x_\alpha||$ converges to $||x||$, then $x_\alpha$ converges to $x$ in norm.

**Proof.** Suppose there exists a weak* compact convex subset $K$ of $E$, $\text{diam}(K) > 0$, such that for each $x \in K$, there exists $T(x) \in K$ with $||x - T(x)|| = \text{diam}(K)$. Following an idea of Wong [13, Theorem 2], let $W(K)$ denote the supremum of $\{|H|; H$ is a diametral subset of $K\}$ ($H$ is diametral if $||x_1 - x_2|| = \text{diam}(K)$ whenever $x_1, x_2 \in H, x_1 \neq x_2$). As shown in the proof of Theorem 2 in [13], $W(K)$ is infinite. Let $\{x_n\}$ be a sequence in $K$ such that $||x_n - x_m|| = \text{diam}(K), n \neq m$. Since $K$ is weak* compact, there exists a subnet $\{x_{n_\alpha}\}$ of $\{x_n\}$ such that $x_{n_\alpha}$ converges to some $z \in K$ in the weak* topology. Passing to a subnet if necessary, we
may assume that the net \( \{ \| x_n - T(z) \| \} \) also converges. Then
\[
\text{diam}(K) = \| z - T(z) \| \leq \lim_\alpha \| x_n - T(z) \| \leq \text{diam}(K).
\]
So \( \lim_\alpha \| x_n - T(z) \| = \text{diam}(K) \). Since \( \{ x_n - T(z) \} \) converges in the weak* topology to \( z - T(z) \), and \( \lim_\alpha \| x_n - T(z) \| = \| z - T(z) \| \), it follows that \( \{ x_n - T(z) \} \) converges in norm to \( z - T(z) \). In particular, the net \( \{ x_n \} \) converges in norm to \( z \) also. This contradicts the choice of the sequence \( \{ x_n \} \).

The next lemma is due to K. McKennon [7, Lemma, 3.2]. For the sake of completeness, we give a short proof.

**Lemma 2 (McKennon [7]).** Let \( A \) be a C*-algebra and \( \{ e_\alpha \} \) be an approximate identity of \( A \), \( e_\alpha \geq 0 \) and \( \| e_\alpha \| \leq 1 \). Let \( \{ \phi_\beta \} \) be a net in \( A^* \) such that \( \phi_\beta \to \phi \) in the weak* topology and \( \| \phi_\beta \| \to \| \phi \| \). Then for any \( \epsilon > 0 \), there exists \( \alpha_0, \beta_0 \) such that
\begin{align*}
(1) & \quad \| R_{e_{\alpha_0}} \phi - \phi \| < \epsilon \\
(2) & \quad \| R_{e_{\alpha_0}} \phi_\beta - \phi_\beta \| < \epsilon
\end{align*}
for all \( \beta \geq \beta_0 \), where \( R_{e\phi}(x) = \phi(xe) \).

**Proof.** Let \( x \in A \), \( \| x \| \leq 1 \). Then using [5, Lemma 3.3] and some properties of positive linear functionals, we obtain the following estimate
\[
\left| \langle R_{e_\alpha} \phi - \phi, x \rangle \right|^2 = \left| \langle \phi, x \cdot e_\alpha - x \rangle \right|^2 = \left| \langle \phi, x \cdot (1 - e_\alpha) \rangle \right|^2 \\
\leq \| \phi \| \| \phi \| \left[ (x \cdot (1 - e_\alpha))^* (x \cdot (1 - e_\alpha)) \right] \\
= \| \phi \| \| \phi \| \left[ (1 - e_\alpha)^* x^* x (1 - e_\alpha) \right] \\
\leq \| \phi \| \| \phi \| \left[ (1 - e_\alpha)(1 - e_\alpha) \right] \leq \| \phi \| \| \phi \| (1 - e_\alpha),
\]
where 1 is the identity in the enveloping von Neumann algebra \( A^{**} \) of \( A \) and \( | \phi \| \) is the absolute value of \( \phi \). Since the net \( \{ e_\alpha \} \) converges to 1 in the weak* topology of \( A^{**} \), (1) follows from the above estimate.

A similar estimate as above shows that
\[
\| R_{e_{\alpha_0}} \phi_\beta - \phi_\beta \| \leq \| \phi_\beta \| \| \phi_\beta \| (1 - e_{\alpha_0}).
\]
Using [5, Lemma 3.5] and the fact that for each positive form \( | \phi_\beta \| \), \( \| \phi_\beta \| = | \phi_\beta | = | \phi_\beta | (1) \), the right side of the above estimate converges to \( \| \phi \| \| \phi \| (1 - e_{\alpha_0}) \). Hence (2) follows.
THEOREM 2. Let $H$ be a Hilbert space. Then $\mathcal{T}(H)$ has the quasi-weak*-normal structure.

*Proof.* By Lemma 1, it suffices to show that $\mathcal{T}(H) = \mathcal{C}(H)^*$ has property (**) Let $\mathcal{P}$ denote all orthogonal projections of $H$ onto a finite dimensional subspace of $H$. Order $\mathcal{P}$ by: $P \geq Q$ iff $QP = PQ = Q$. Then $(\mathcal{P}, \leq)$ is an approximate identity for $\mathcal{C}(H)$. Since every $T \in \mathcal{C}(H)$ can be written in the form $T = T_1 + iT_2$, $T_1$ self-adjoint, $i = 1, 2$, it suffices to show that if $T$ is self-adjoint, then $\lim ||TP - T|| = \lim ||PT - T|| = 0$. Indeed, if $T \in \mathcal{C}(H)$ and $T$ is self-adjoint, then by the spectral theorem $T = \sum_{i=1}^{\infty} \lambda_i P_i$, where $\lambda_i \to 0$ as $i \to \infty$ and $P_i \in \mathcal{P}$. Given $\varepsilon > 0$ choose $n$ such that $||T - \sum_{i=1}^{n} \lambda_i P_i|| < \varepsilon$. Let $Q \in \mathcal{P}$ be such that $Q \geq P_i$, $i = 1, 2, \ldots, n$. Then for all $P \geq Q$,

$$||TP - T|| \leq ||TP - S_n P|| + ||S_n P - S_n|| + ||S_n - T|| < 2\varepsilon,$$

where $S_n = \sum_{i=1}^{n} \lambda_i P_i$. Similarly, we can show $\lim ||PT - T|| = 0$. We also note that each $P \in \mathcal{P}$ is positive and has norm one.

Let $\{\phi_\beta\}$ be a net in $\mathcal{C}(H)^*$ converging to some $\phi \in \mathcal{C}(H)^*$ in the weak* topology and $||\phi_\beta|| \to ||\phi||$. By Lemma 2, there exists $P_0 \in \mathcal{P}$ and $\beta_0$ such that

(3) $\|R_{P_0}\phi - \phi\| < \varepsilon/2$ and $\|R_{P_0}\phi_\beta - \phi_\beta\| < \varepsilon/2$

for all $\beta \geq \beta_0$. By considering the reversed $C^*$-algebra, we may also assume that

(4) $\|L_{P_0}\phi - \phi\| < \varepsilon/2$ and $\|L_{P_0}\phi_\beta - \phi_\beta\| < \varepsilon/2$.

where $(L_{P_0}\phi)(T) = \phi(P_0 T)$, $T \in \mathcal{C}(H)$. Consequently, if $\beta \geq \beta_0$,

(5) $\|R_{P_0}L_{P_0}\phi - \phi\| \leq \|R_{P_0}L_{P_0}\phi - R_{P_0}\phi\| + \|R_{P_0}\phi - \phi\| < \varepsilon$

since $\|R_{P_0}\phi\| \leq \|P_0\| = 1$ by (3) and (4). Similarly

(6) $\|R_{P_0}L_{P_0}\phi_\beta - \phi_\beta\| < \varepsilon$.

Also, $P_0\mathcal{C}(H)P_0$ is a finite dimensional algebra over $C$. Hence, $\{\phi_\beta\}$, restricted to $P_0\mathcal{C}(H)P_0$ converges to $\phi$ in norm. Consequently, there exists $\beta_1 \geq \beta_0$ such that

(7) $\|T_{P_0}L_{P_0}\phi_\beta - R_{P_0}L_{P_0}\phi\| < \varepsilon$

if $\beta \geq \beta_1$. Now if $\beta \geq \beta_1$, we have

$$\|\phi_\beta - \phi\| \leq \|\phi_\beta - R_{P_0}L_{P_0}\phi_\beta\| + \|R_{P_0}L_{P_0}\phi_\beta - R_{P_0}L_{P_0}\phi\| + \|R_{P_0}L_{P_0}\phi - \phi\| < 3\varepsilon.$$

by (5), (6) and (7).
REMARK. Clearly if a dual Banach space $E$ has the weak* normal structure then $E$ has the quasi-weak*-normal structure. But the converse is false. Indeed, let $E$ the space of absolutely summable real sequences with norm

$$\|x\| = \max\{\|x^+\|_1, \|x^-\|_1\}$$

where $x^+$, $x^-$ denote the positive and negative part of $x$ respectively and $\|x\|_1 = \sum_{i=1}^{\infty}|x_i|$. Then, as shown by Lim [6] (Lemma 1 and Example 1), $E$ is a dual Banach space which does not have weak* normal structure. However, since $E$ is separable, an argument similar to that of Wong [13, Theorem 2] shows that $E$ has quasi-weak*-normal structure.

**Problem 1.** Does the trace class operator $\mathcal{T}(H) = \mathcal{C}(H)^*$ with dual norm have the weak* normal structure or the weak normal structure?

**Lemma 3.** Let $\Gamma$ be a non-empty set. Then $c_0(\Gamma)$ has the quasi-weak-normal structure if and only if $\Gamma$ is countable.

**Proof.** If $\Gamma$ is countable then $c_0(\Gamma)$ is norm separable. Hence each weakly compact convex subset of $c_0(\Gamma)$ has quasi-normal structure by Theorem 2 in [13].

Conversely, if $\Gamma$ is not countable, consider $\Gamma$ as a group (say the free group on $|\Gamma|$ generators). Pick $a \in \Gamma$. Let $f = \delta_a$ i.e. $f(x) = 1$ if $x = a$ and $f(x) = 0$ if $x \neq a$. Let $K$ denotes the closed convex hull of \{ $l_x f; x \in \Gamma$ \}, where $(l_x f)(t) = f(xt), t \in \Gamma$. Then $K$ is weakly compact ([2, Corollary 3.7]) and $\text{diam}(K) = 1$. Now if $g \in K$, let $\sigma \subseteq \Gamma$ be a countable set such that $g(t) = 0$ if $t \in \Gamma - \sigma$. Pick $z \in \Gamma - \sigma$ and let $h = \delta_z$. Then $h \in K$ and $\|g - h\|_\infty = 1$. Hence $K$ does not have quasi-normal structure.

**Theorem 3.** Let $H$ be a Hilbert space. Then $H$ is separable if and only if $\mathcal{C}(H)$ has quasi-weak-normal structure.

**Proof.** If $H$ is separable, then $\mathcal{C}(H)^*$ is separable [10, Proposition 2.1.10]. Hence $\mathcal{C}(H)$ is separable. Consequently every weakly compact convex subset of $\mathcal{C}(H)$ has quasi-normal structure by [13, Theorem 2].

Conversely, if $H$ is not separable, then $H$ is isomorphic to $l_2(\Gamma)$ for an uncountable set $\Gamma$. Consider the map $\rho: c_0(\Gamma) \to \mathcal{B}(l_2(\Gamma))$ defined by

$$\rho(f)(h)(t) = f(t)h(t), \quad t \in \Gamma,$$

then $\rho$ is an isometry and an algebra isomorphism of $c_0(\Gamma)$ into $\mathcal{B}(l_2(\Gamma))$. Furthermore, $\rho(f)$ is compact for each $f \in c_0(\Gamma)$. By Lemma 3, there exists a weakly compact convex subset $K$ in $c_0(\Gamma)$ which does not have
quasi-normal structure. In particular, \( \rho(K) \) is a weakly compact convex subset of \( C(H) \) which does not have quasi-normal structure also.

**Summary.** In the Table we shall abbreviate normal structure by n.s., quasi-normal structure by q.n.s., etc. We assume \( \Gamma \) is not finite and \( H \) is not finite dimensional.

<table>
<thead>
<tr>
<th>( c_0(\Gamma) )</th>
<th>( l_1(\Gamma) )</th>
<th>( l_\infty(\Gamma) )</th>
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<td>No n.s.</td>
<td>No n.s.</td>
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<tr>
<td>q.w.n.s.</td>
<td>w*.n.s.</td>
<td>No q.w*.n.s</td>
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<td>( \Gamma ) is countable</td>
<td>w.n.s.</td>
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<td>( C(H) )</td>
<td>( T(H) )</td>
<td>( B(H) )</td>
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<td>No n.s.</td>
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<td>q.w.n.s.</td>
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<tr>
<td>( H ) is separable</td>
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3. **On Lim's condition.** Let \( E \) be a Banach space. Then \( E^* \) is said to satisfy *Lim's condition* if whether \( \{ \phi_\alpha \} \) is a bounded net in \( E^* \), \( \phi_\alpha \) converges to 0 in the weak* topology and \( \lim_\alpha \| \phi_\alpha \| = s \), then \( \lim_\alpha \| \phi_\alpha + \psi \| = s + \| \psi \| \) for any \( \psi \in E^* \).

In [6], Lim showed that \( l_1 \) satisfies this condition for sequences.

**Lemma 4.** Let \( E \) be a Banach space. If \( E^* \) satisfies Lim's condition, then \( E^* \) has the following properties:

(a) Norm and weak* topology agree on \( S = \{ \phi \in E^*; \| \phi \| = 1 \} \)

(b) For any \( 0 < \epsilon < 2 \), if \( \{ \phi_\alpha \} \) is a net in \( E^* \), \( \| \phi_\alpha \| \leq 1 \), \( \phi_\alpha \to \phi \) in the weak*-topology and \( \| x_\alpha - x_\beta \| \geq \epsilon \) for all \( \alpha \neq \beta \), then \( \| \phi \| \leq 1 - \epsilon/2 \).

In particular, \( E^* \) has the Radon Nikodym Property and weak* normal structure.

**Proof.** (a) Let \( \{ \phi_\alpha \} \) be a net in \( S \), \( \phi \in S \) such that \( \phi_\alpha \to \phi \) in the weak*-topology. Suppose \( \| \phi_\alpha - \phi \| \to 0 \). Then we may assume, by passing to a subnet if necessary, that \( \| \phi_\alpha - \phi \| \geq \epsilon \) for each \( \alpha \). Since \( \{ \| \phi_\alpha - \phi \| \} \) is bounded by 2, we may further assume that \( \lim_\alpha \| \phi_\alpha - \phi \| = s \geq \epsilon > 0 \).
Let $\psi_\alpha = \phi_\alpha - \phi$. Then $\psi_\alpha \to 0$ in the weak*-topology but
\[
1 = \lim_{\alpha} \|\phi + (\phi_\alpha - \phi)\| = \|\phi\| + s > 1
\]
which is impossible.

(b) We may assume that $\|\phi_\alpha - \phi\| \geq \varepsilon/2$ for each $\alpha$, and
\[
\lim_{\alpha} \|\phi_\alpha - \phi\| = s.
\]
Then by Lim's condition,
\[
\lim_{\alpha} \|\phi_\alpha\| = \lim_{\alpha} \| (\phi_\alpha - \phi) + \phi\| = s + \|\phi\|
\]
i.e. $s + \|\phi\| \leq 1$ or $\|\phi\| \leq 1 - s \leq 1 - \varepsilon/2$.

The last statement follows from Corollary 8 and Proposition 9 in [8], and the proof of Theorem 3 [4] (That $E^*$ has weak* normal structure also follows simple modification of Lim's proof of Theorem 3 in [6]).

Given a locally compact Hausdorff space $X$, let $C_0(X)$ denote the $C^*$-algebra of complex-valued continuous functions $f$ on $X$ such that for any $\varepsilon > 0$ there exists a compact subset $\sigma$ of $X$ such that $|f(x)| \leq \varepsilon$ for $x \in X \sim \sigma$ with the supremum norm.

**Theorem 4.** Let $X$ be a locally compact Hausdorff space. The dual Banach space $C_0(X)^*$ satisfies Lim's condition if and only if $C_0(X)^*$ is isometric isomorphic to $l_1(\Gamma)$ for some non-empty set $\Gamma$.

**Proof.** If $C_0(X)^*$ satisfies Lim's condition, then, by Lemma 4, $C_0(X)^*$ has the Radon Nikodym Property. Since $C_0(X)^{**} = M$ is the enveloping von Neumann algebra of the $C^*$-algebra $C_0(X)$, it follows from Theorem 4 in [3] that $M$ is the direct sum of Type I factors i.e. $M$ is isomorphic to $\sum_{\alpha \in \Gamma} \mathcal{B}(H_\alpha)$. Since $M$ is commutative, $H_\alpha = C$ for each $\alpha \in \Gamma$. In particular, $C_0(X)^* \cong l_1(\Gamma)$.

Suppose $C_0(X)^*$ is isometric isomorphic to $l_1(\Gamma)$ for some non empty set $\Gamma$. We may assume that $\Gamma$ is infinite. Let $\{f_\alpha\}$ be a bounded net in $l_1(\Gamma)$ such that $f_\alpha \to 0$ in the weak*-topology and $\lim_{\alpha} \|f_\alpha\| = s$. Let $g \in l_1(\Gamma)$. Since $\|f_\alpha - g\| \leq \|f_\alpha\| + \|g\|$ for each $\alpha$, we may assume, by passing to a subnet if necessary, that $\lim_{\alpha} \|f_\alpha - g\| = t$ exists. Clearly we have $t \leq s + \|g\|$. To see that we actually have equality, let $\varepsilon > 0$. Observe that in $l_1(\Gamma)$,
\[
\|f_\alpha - g\| \geq \|f_\alpha\| - \|g\| + 2 \sum_{s \in \sigma} (|g(s)| - |f_\alpha(s)|)
\]
for any subset \( \sigma \) of \( \Sigma \). Now let \( \sigma \) be a finite subset such that \( \sum_{s \in \sigma} |g(s)| > \|g\| - \epsilon \). For this \( \sigma \), we can choose \( \alpha_0 \), using the weak* convergence of \( f_\alpha \) and the convergence of \( \|f_\alpha\| \), so that for all \( \alpha \geq \alpha_0 \) we have \( \sum_{s \in \sigma} |f_\alpha(s)| < \epsilon \) and \( \|f_\alpha\| > s - \epsilon \). Then for all \( \alpha \geq \alpha_0 \) we have from (1)
\[
\|f_\alpha - g\| \geq s - \epsilon - \|g\| + 2\|g\| - 2\epsilon - 2\epsilon = s + \|g\| - 5\epsilon.
\]
Thus \( t \geq s + \|g\| \).

**Problem 2.** Let \( X \) be a locally compact Hausdorff space. When does \( C_0(X)^* \) have the weak* normal structure?

**Theorem 5.** Let \( H \) be a Hilbert space. Then \( \mathcal{T}(H) \) satisfies Lim's condition if and only if \( H \) is finite dimensional.

**Proof.** If \( H \) is finite dimensional, then \( \mathcal{T}(H) \) is finite dimensional. Hence \( \mathcal{T}(H) \) satisfies Lim's condition.

If \( H \) is infinite dimensional, let \( \{\xi_n, n = 1, 2, \ldots\} \) be an orthonormal sequence in \( H \). For each \( n = 1, 2, \ldots \), define \( \phi_n(T) = \langle T\xi_1, \xi_n \rangle \). Then \( \phi_n \in \mathcal{T}(H), \|\phi_n\| = 1 \) and \( \phi_n \to 0 \) weakly. Indeed, if \( T \in \mathcal{B}(H) \), then
\[
\infty > \|T\xi_1\|^2 = \sum_{\alpha \in I} |\langle T\xi_1, \xi_\alpha \rangle|^2 \geq \sum_{n=1}^{\infty} |\langle T\xi_1, \xi_n \rangle|^2 = \sum_{n=1}^{\infty} |\phi_n(T)|^2
\]
where \( \{\xi_\alpha\}_{\alpha \in I} \) is a complete orthonormal set of \( H \) containing \( \{\xi_n\} \). So \( \phi_n(T) \to 0 \). Also \( \|\phi_n - \phi_1\| \leq \sqrt{2} \) for each \( n \). Hence \( \lim_n \|\phi_n - \phi_1\| \leq \sqrt{2} \) i.e. \( \lim_n \|\phi_n - \phi_1\| \neq \lim \|\phi_n\| + \|\phi_1\| \). In particular, \( \mathcal{T}(H) \) does not satisfy Lim's condition.

**References**


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