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Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ -finite measure spaces, $i = 1, 2$, and let E be a Hilbert space. If the Bochner spaces $L^p(\Omega_1, \Sigma_1, \mu_1, E)$ and $L^p(\Omega_2, \Sigma_2, \mu_2, E)$ are nearly isometric, for either $p = 1$ or $p = \infty$, then $L^1(\Omega_1, \Sigma_1, \mu_1, E)$ is isometric to $L^1(\Omega_2, \Sigma_2, \mu_2, E)$ and hence $L^\infty(\Omega_1, \Sigma_1, \mu_1, E)$ is isometric to $L^\infty(\Omega_2, \Sigma_2, \mu_2, E)$.

Throughout this paper the letter E will denote a Banach space which will often be taken to be Hilbert space. Interaction between elements of a Banach space and those of its dual will be denoted by $\langle \cdot, \cdot \rangle$. We will write $E_1 \cong E_2$ to indicate that the Banach spaces E_1 and E_2 are isometric.

Following Banach, [2, p. 242], we will call the Banach spaces E_1 and E_2 nearly isometric if $1 = \inf\{\|T\|\|T^{-1}\|\}$, where T runs through all isomorphisms of E_1 onto E_2 . It is of course equivalent to suppose that $1 = \inf\{\|T\|\}$, where $\|T^{-1}\| = 1$, and hence T is a norm-increasing isomorphism of E_1 onto E_2 . For if T is any continuous isomorphism of one Banach space onto another, we obtain an isomorphism \hat{T} having the desired properties by defining \hat{T} to be equal to $\|T^{-1}\|T$.

If (Ω, Σ, μ) is a positive measure space and E a Banach space, the Bochner spaces $L^p(\Omega, \Sigma, \mu, E)$ will be denoted by $L^p(\mu, E)$ when there is no danger of confusing the underlying measurable spaces involved, and by $L^p(\mu)$ when E is the scalar field. For the definitions and properties of these spaces we refer to [8].

It has been noted by Benyamini [4] that, as a consequence of known properties of spaces of continuous functions, if two spaces $L^p(\mu_1)$ and $L^p(\mu_2)$ are nearly isometric, for either $p = 1$ or $p = \infty$, then they are isometric. What we wish to show is that the same conclusion can be drawn for near isometries of certain Bochner spaces. We will prove the following:

THEOREM. *Let $(\Omega_i, \Sigma_i, \mu_i)$ be σ -finite measure spaces, $i = 1, 2$, and E a Hilbert space. If there exists an isomorphism T , with $\|T^{-1}\| = 1$ and $\|T\| < 3/(2\sqrt{2})$, mapping $L^p(\Omega_1, \Sigma_1, \mu_1, E)$ onto $L^p(\Omega_2, \Sigma_2, \mu_2, E)$ for either $p = 1$ or $p = \infty$, then $L^1(\Omega_1, \Sigma_1, \mu_1, E) \cong L^1(\Omega_2, \Sigma_2, \mu_2, E)$ and $L^\infty(\Omega_1, \Sigma_1, \mu_1, E) \cong L^\infty(\Omega_2, \Sigma_2, \mu_2, E)$.*

In the scalar case, Banyamini's theorem follows from an analogous result for spaces of continuous functions obtained independently by D. Amir [1] and the author [5], [6]. And we note that if E is finite-dimensional with orthonormal basis $\{e_n: n = 1, \dots, N\}$, and X_i denotes the maximal ideal space of $L^\infty(\mu_i)$, $i = 1, 2$, then it can be shown that $L^\infty(\mu_i, E)$ is isometrically isomorphic to $C(X_i, E)$, the space of continuous functions on X_i to E , under the map $\sum_{n=1}^N f_n e_n \rightarrow \sum_{n=1}^N \hat{f}_n e_n$, where $f \rightarrow \hat{f}$ is the Gelfand representation of $L^\infty(\mu_i)$. In this case the theorem of this article can be obtained from what is known about isomorphisms of continuous vector-valued functions [7], the result for vectorial L^∞ following directly from [7] and that for L^1 then following by arguments analogous to those given here in the proof of Lemma 8. However when E is infinite dimensional, the continuity on X_i of the coordinate functions \hat{f}_n no longer implies continuity for $\sum_n \hat{f}_n e_n$, even in the presence of separability, and thus the problem requires different methods of approach.

Consequently, in what follows, E will represent an infinite-dimensional Hilbert space. Although the proofs presented here require only that the dimension of E be greater than two, for all finite-dimensional Hilbert spaces E not only does our theorem follow from [7], but it follows with the bound $3/(2\sqrt{2})$ replaced by the better bound $\sqrt{2}$.

Our approach here will be to replace the measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ by measure spaces in which we have a topology, and on which measurable vector-valued functions are very close to being continuous. For this we will require the notion of a perfect measure. Thus, following [3], if X is an extremally disconnected compact Hausdorff space we will call a nonnegative, extended real-valued measure μ defined on the Borel sets $\mathcal{B}(X)$ of X *perfect* if

- (i) every nonempty clopen set has positive measure,
- (ii) every nowhere dense Borel set has measure zero, and
- (iii) every nonempty clopen set contains another nonempty clopen set with finite measure.

The proof of our theorem is now completed by means of a sequence of lemmas.

LEMMA 1. *Let (Ω, Σ, μ) be a σ -finite measure space, and let X be the Stonean space of the measure algebra Σ/μ . (Equivalently, X is the maximal ideal space of $L^\infty(\mu)$.) For $A \in \Sigma$ let \hat{A} denote the clopen subset of X which represents the equivalence class of A . Then the measure $\hat{\mu}$ defined on the algebra \mathcal{A} of clopen subsets of X by $\hat{\mu}(\hat{A}) = \mu(A)$, $A \in \Sigma$, can be extended to a perfect measure, also denoted by $\hat{\mu}$, on $\mathcal{B}(X)$ such that $L^1(\Omega, \Sigma, \mu, E) \cong L^1(X, \mathcal{B}(X), \hat{\mu}, E)$, and hence $L^\infty(\Omega, \Sigma, \mu, E) \cong L^\infty(X, \mathcal{B}(X), \hat{\mu}, E)$.*

Proof. The set function $\hat{\mu}$ defined above is, indeed, countably additive on \mathcal{A} , [8, p. 11]. Thus, by the Carathéodory extension theorem, $\hat{\mu}$ has a unique extension to the σ -algebra generated by \mathcal{A} . This σ -algebra clearly contains the Baire sets of X .

First suppose that μ is finite. Then, [9, p. 351], $\hat{\mu}$ can be further extended to a regular measure on $\mathcal{B}(X)$, which is clearly perfect. (The proof that every nowhere dense Borel set has measure zero is contained in [10, p. 18, Lemma 9.4].)

If μ is σ -finite but not finite, let Ω be the disjoint union $\Omega = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \Sigma$ and $0 < \mu(A_n) < \infty$ for all n . Then define the finite measure μ_0 on Σ by $\mu_0(A) = \sum_{n=1}^{\infty} \mu(A \cap A_n) / (2^n \cdot \mu(A_n))$. Since the μ_0 -null and μ -null sets of Σ coincide, the measure algebras Σ/μ and Σ/μ_0 have the same Stonean space X . The measure $\hat{\mu}_0$ defined as above on \mathcal{A} extends to a perfect regular Borel measure on X . And since for sets $A \in \Sigma$ we have

$$\mu(A) = \sum_n \mu(A \cap A_n) = \sum_n 2^n \cdot \mu(A_n) \mu_0(A \cap A_n),$$

it follows that for $\hat{A} \in \mathcal{A}$,

$$\hat{\mu}(\hat{A}) = \sum_n \hat{\mu}(\hat{A} \cap \hat{A}_n) = \sum_n 2^n \cdot \hat{\mu}(\hat{A}_n) \hat{\mu}_0(\hat{A} \cap \hat{A}_n).$$

Thus if we define, for $B \in \mathcal{B}(X)$, $\hat{\mu}(B) = \sum_n 2^n \cdot \hat{\mu}(\hat{A}_n) \hat{\mu}_0(B \cap \hat{A}_n)$, the set function so defined is an extension of $\hat{\mu}$ to a perfect measure on $\mathcal{B}(X)$.

Finally, the map $\sum_{j=1}^n e_j \chi_{A_j} \rightarrow \sum_{j=1}^n e_j \chi_{\hat{A}_j}$ carries the dense subspace of $L^1(\Omega, \Sigma, \mu, E)$ consisting of simple functions isometrically into the corresponding subspace of $L^1(X, \mathcal{B}(X), \hat{\mu}, E)$. Since every $B \in \mathcal{B}(X)$ differs from a clopen set by a set of $\hat{\mu}$ -measure zero [3, p. 1], the map is actually onto the subspace of simple functions in $L^1(X, \mathcal{B}(X), \hat{\mu}, E)$ and thus extends to an isometry of $L^1(\Omega, \Sigma, \mu, E)$ onto $L^1(X, \mathcal{B}(X), \hat{\mu}, E)$.

LEMMA 2. *Let X and $\hat{\mu}$ be as in Lemma 1. Then given a measurable E -valued function F on X there exists an open dense subset U_F of X such that $F|_{U_F}$ is continuous, and $\hat{\mu}(X - U_F) = 0$.*

Proof. First assume that $\hat{\mu}$ is finite. Here we follow the argument given by Peter Greim in [11, p. 124]. Take a sequence $\{F_n\}$ of simple functions converging a.e. to F . Again using the fact that each set in $\mathcal{B}(X)$ differs from a clopen set by a set of measure zero, we may suppose that each F_n is continuous. Then Egoroff's theorem shows that F is the almost uniform limit of continuous functions. Hence for each $\varepsilon > 0$ there is a

measurable set U_ε such that the restriction of F to U_ε is continuous and $\hat{\mu}(X - U_\varepsilon) < \varepsilon$. Using the facts that $\hat{\mu}$ is regular and that an open set and its closure have the same measure, we may assume that U_ε is clopen. If then U_F is the union of all the U_ε 's it has the required properties, for its complement is closed and has measure zero, and thus can contain no non-void open set.

If $\hat{\mu}$ is σ -finite but not finite let $\hat{\mu}_0$ be the finite measure that appears in the proof of Lemma 1. The argument of the preceding paragraph with $\hat{\mu}$ replaced by $\hat{\mu}_0$ then shows that F is continuous on a dense open set U_F with $\hat{\mu}_0(X - U_F) = 0$. Since $\hat{\mu}$ and $\hat{\mu}_0$ have the same null sets, the proof is complete.

As a consequence of Lemma 1 it suffices to prove our theorem for two σ -finite perfect Borel measures defined on extremally disconnected compact Hausdorff spaces. Accordingly, we shall henceforth assume that X and Y are extremally disconnected compact Hausdorff spaces and that μ (resp. ν) is a σ -finite perfect measure on $\mathcal{B}(X)$ (resp. $\mathcal{B}(Y)$). Until further notice, T will denote a norm-increasing isomorphism of $L^\infty(X, \mathcal{B}(X), \mu, E)$ onto $L^\infty(Y, \mathcal{B}(Y), \nu, E)$ with $\|T\| < 3/(2\sqrt{2})$ and $\|T^{-1}\| = 1$.

LEMMA 3. *If $F \in L^\infty(\mu, E)$ and $\|F(x)\| = 1$ for almost all $x \in X$, then, for almost all $y \in Y$, $(63/64)^{1/2} \leq \|T(F)(y)\|$.*

Proof. Suppose, to the contrary, that there exists a set $A \in \mathcal{B}(Y)$ with $\nu(A) > 0$ such that $\|T(F)(y)\| < (63/64)^{1/2}$ for $y \in A$. Again using [3, p. 1], $A = B \Delta C$ with B clopen and C of first category. We may assume that $T(F) = 0$ on the ν -null set $B \cap C$ and hence that $\|T(F)(y)\| < (63/64)^{1/2}$ on the clopen set B with $\nu(B) = \nu(A) > 0$. Let $U_{T(F)}$ be an open dense subset of Y on which $T(F)$ is continuous, and whose complement has ν -measure zero. Then $\nu(B \cap U_{T(F)}) = \nu(B) > 0$, $B \cap U_{T(F)}$ is open and $T(F)$ is continuous on this set.

Let $k = \|T(F)\|_\infty$. Choose $y_0 \in B \cap U_{T(F)}$ and take $e \in E$ with $\|e\| = 1$ perpendicular to $T(F)(y_0)$. Then for all scalars α with $|\alpha| = 1$,

$$\begin{aligned} & \left\| T(F)(y) + \alpha(k^2 - 63/64)^{1/2} \cdot e \right\|^2 \\ & \leq \|T(F)(y)\|^2 + 2(k^2 - 63/64)^{1/2} |\langle e, T(F)(y) \rangle| \\ & \quad + k^2 - (63/64). \end{aligned}$$

For $y = y_0$ the expression on the right is less than k^2 , and since it is continuous on $B \cap U_{T(F)}$, there exists a clopen set D containing y_0 such that for all $y \in D$ we have $\|T(F)(y) + \alpha(k^2 - 63/64)^{1/2} \cdot e\|^2 < k^2$.

Thus if we define $G \in L^\infty(\nu, E)$ by $G = (k^2 - 63/64)^{1/2} \cdot e \cdot \chi_D$, then G is a nonzero element of $L^\infty(\nu, E)$ such that for all scalars α with $|\alpha| = 1$ $\|T(F) + \alpha G\|_\infty = k$.

We can suppose that $\|F(x)\| = 1$ for all $x \in X$. We must have

$$\|T^{-1}(G)\|_\infty \geq (1/\|T\|)(k^2 - 63/64)^{1/2} > ((2\sqrt{2})/3)(k^2 - 63/64)^{1/2}.$$

And since the complement of $U_F \cap U_{T^{-1}(G)}$ has μ -measure zero, we can choose $x_0 \in U_F \cap U_{T^{-1}(G)}$ with

$$\|T^{-1}(G)(x_0)\| > ((2\sqrt{2})/3)(k^2 - 63/64)^{1/2}.$$

Next note that if α is a scalar with $|\alpha| = 1$ such that $\operatorname{Re} \alpha \langle T^{-1}(G)(x_0), F(x_0) \rangle \geq 0$, then

$$\|F(x_0) + \alpha T^{-1}(G)(x_0)\|^2 > 1 + (8/9)(k^2 - 63/64).$$

Since $\|F(x) + \alpha T^{-1}(G)(x)\|$ is continuous on $U_F \cap U_{T^{-1}(G)}$, there is a clopen set W containing x_0 such that

$$\|F(x) + \alpha T^{-1}(G)(x)\|^2 > 1 + (8/9)(k^2 - 63/64) \quad \text{on } W.$$

Thus

$$\|F + \alpha T^{-1}(G)\|_\infty^2 > 1 + (8/9)(k^2 - 63/64),$$

and we will have obtained a contradiction to the fact that T^{-1} is norm-decreasing if the quantity on the right is greater than k^2 -equivalently if $63/64 < (9 - k^2)/8$. But since $k^2 \leq \|T\|^2 < 9/8$, we indeed have $63/64 < (9 - k^2)/8$ and this contradiction completes the proof of the lemma.

LEMMA 4. *Let $F \in L^\infty(\mu, E)$ with $(63/64)^{1/2} \leq \|F(x)\| \leq \|T\|$ a.e. For $A \in \mathcal{B}(X)$ define $\phi(A) \in \mathcal{B}(Y)$ by $\phi(A) = \{y \in Y: \|T(\chi_A F)(y)\| \geq 31/32\}$.*

(i) *If A and B are disjoint measurable subsets of X then $\phi(A) \cap \phi(B)$ is a ν -null set and, modulo a ν -null set, $\phi(A') = [\phi(A)]'$ (where for any set A , A' denotes its complement).*

(ii) *If we furthermore assume that $\|F\|_\infty \leq 1$ then $\|T(\chi_A F)(y)\| < .44$ a.e. on $\phi(A')$.*

Proof. (i). If $\phi(A) \cap \phi(B)$ had positive measure then, proceeding as in the proof of the previous lemma we could find a nonempty clopen set $C \subseteq Y$ on which $\|T(\chi_A F)(y)\| > 15/16$ and $\|T(\chi_B F)(y)\| > 15/16$, and

on which both $T(\chi_A F)$ and $T(\chi_B F)$ are continuous. By choosing first a point $y_0 \in C$ and then a scalar α such that

$$\operatorname{Re} \langle T(\chi_B F)(y_0), T(\chi_A F)(y_0) \rangle \geq 0,$$

it would then follow that $\|T(\chi_A F) + \alpha T(\chi_B F)\|_\infty > (15\sqrt{2})/16 > 1.3$. But since for all scalars α with $|\alpha| = 1$ we have $\|\chi_A F + \alpha \chi_B F\|_\infty \leq \|T\|$, $\|T(\chi_A F) + \alpha T(\chi_B F)\|_\infty$ must be less than $\|T\|^2 < 1.2$, and thus $\phi(A)$ and $\phi(B)$ must be a.e. disjoint.

We wish next to show that the union of $\phi(A)$ and $\phi(A')$ is almost all of Y . Suppose, to the contrary, that on some Borel set $D \subseteq Y$ with $\nu(D) > 0$ we had $\|T(\chi_A F)(y)\| < 31/32$ and $\|T(\chi_{A'} F)(y)\| < 31/32$. We may suppose that D is clopen and that both $T(\chi_A F)$ and $T(\chi_{A'} F)$ are continuous on D . Let $k_1 = \|T(\chi_A F)\|_\infty$, $k_2 = \|T(\chi_{A'} F)\|_\infty$ and $k = \max\{k_1, k_2\}$. Then arguing as in the second paragraph of the proof of Lemma 3, we could find a $G \in L^\infty(\nu, E)$ with $\|G\|_\infty = (k^2 - (31/32)^2)^{1/2}$ and such that $\|T(\chi_A F) + \alpha G\|_\infty \leq k$ and $\|T(\chi_{A'} F) + \alpha G\|_\infty \leq k$ for all scalars α with $|\alpha| = 1$.

Then $\|T^{-1}(G)\|_\infty > ((2\sqrt{2})/3)(k^2 - (31/32)^2)^{1/2}$ so that by an argument analogous to that given in the third paragraph of the proof of Lemma 3, we can find a scalar α with $|\alpha| = 1$ such that

$$\|F + \alpha T^{-1}(G)\|_\infty^2 > 63/64 + (8/9)(k^2 - (31/32)^2).$$

This latter quantity will be greater than k^2 iff $(9 \cdot 63 - 64 \cdot k^2)/8 \cdot 64 > (31/32)^2$ an inequality which in fact holds since here $\|F\|_\infty \leq \|T\|$ gives $k \leq \|T\|^2$ and hence $k^2 \leq \|T\|^4 < 81/64$. Thus $\|F + \alpha T^{-1}(G)\|_\infty > k$.

But since $\|T(\chi_A F) + \alpha G\|_\infty \leq k$ and $\|T(\chi_{A'} F) + \alpha G\|_\infty \leq k$ and T^{-1} is norm-decreasing, we must have $\|\chi_A F + \alpha T^{-1}(G)\|_\infty \leq k$ and $\|\chi_{A'} F + \alpha T^{-1}(G)\|_\infty \leq k$. Since, for any $x \in X$, $F(x) + \alpha T^{-1}(G)(x)$ is equal either to $\chi_A(x)F(x) + \alpha T^{-1}(G)(x)$ or to $\chi_{A'}(x)F(x) + \alpha T^{-1}(G)(x)$ we have a contradiction and thus, modulo a null set, $\phi(A') = [\phi(A)]'$.

(ii): We know that $\|T(\chi_{A'} F)(x)\| \geq 31/32$ on $\phi(A')$ and thus on this set we must have $\|T(\chi_A F)(x)\|^2 < 9/8 - (31/32)^2 < .19$ so that $\|T(\chi_A F)(x)\| < .44$ a.e. on $\phi(A')$. Otherwise an argument analogous to that of the first paragraph of this proof would provide a contradiction. This concludes the proof of the lemma.

Now fix an $F \in L^\infty(\mu, E)$ with $\|F(x)\| = 1$ a.e. $[\mu]$. Then by Lemma 4(i) we obtain a map ϕ , defined modulo null sets, from $\mathcal{B}(X)$ to $\mathcal{B}(Y)$ determined, for $A \in \mathcal{B}(X)$, by $\phi(A) = \{y \in Y: \|T(\chi_A F)(y)\| \geq 31/32\}$

and satisfying $\phi(A') = [\phi(A)]'$. Next note that $R = \|T\|T^{-1}$ is a norm-increasing isomorphism of $L^\infty(\nu, E)$ onto $L^\infty(\mu, E)$ satisfying $\|R\| < 3/(2\sqrt{2})$ and $\|R^{-1}\| = 1$, and that by Lemma 3,

$$(63/64)^{1/2} \leq \|T(F)(y)\| \leq \|T\| = \|R\| \quad \text{a.e. } [\nu].$$

Thus, interchanging the roles of T and R , of F and $T(F)$, and those of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, by Lemma 4(i) we obtain a map ψ from $\mathcal{B}(Y)$ to $\mathcal{B}(X)$ satisfying $\psi(B') = [\psi(B)]'$, modulo null sets, for $B \in \mathcal{B}(Y)$ and determined by $\psi(B) = \{x \in X: \|R(\chi_B \cdot T(F))(x)\| \geq 31/32\}$.

LEMMA 5. $\|T^{-1}(\chi_{B'} \cdot T(F))(x)\| < .44$ a.e. on $\psi(B)$.

Proof. For $B \in \mathcal{B}(Y)$ we have $\|R(\chi_B \cdot T(F))(x)\| \geq 31/32$ on $\psi(B)$ and thus

$$\|T^{-1}(\chi_B \cdot T(F))(x)\| = \|R(\chi_B \cdot T(F))(x)\|/\|T\| \geq .9 \quad \text{on } \psi(B).$$

If we let

$$P = \text{ess sup}_{x \in \psi(B)} \|T^{-1}(\chi_{B'} \cdot T(F))(x)\|$$

then since $F = T^{-1}(\chi_B \cdot T(F)) + T^{-1}(\chi_{B'} \cdot T(F))$ we must have $(.9)^2 + P^2 \leq 1 = \|F\|_\infty^2$ and hence $P < .44$ as claimed.

LEMMA 6. If $B \in \mathcal{B}(Y)$ then, modulo a ν -null set, $\phi(\psi(B)) = B$. Hence ϕ is a mapping, defined modulo null sets, of $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$.

Proof. Recall that $\phi(\psi(B))$ is the set on which $\|T(\chi_{\psi(B)} \cdot F)(y)\| \geq 31/32$. We have

$$\chi_{\psi(B)} \cdot F = \chi_{\psi(B)} \cdot T^{-1}(\chi_B \cdot T(F)) + \chi_{\psi(B)} \cdot T^{-1}(\chi_{B'} \cdot T(F)).$$

Thus for $x \in \psi(B)$, $\chi_{\psi(B)}(x) \cdot F(x)$ differs from $T^{-1}(\chi_B \cdot T(F))(x)$ by $\chi_{\psi(B)}(x) \cdot T^{-1}(\chi_{B'} \cdot T(F))(x)$ which, by Lemma 5, has norm $< .44$ for almost all x . And for $x \in \psi(B')$, $\chi_{\psi(B)}(x) \cdot F(x) = 0$ and so can differ from $T^{-1}(\chi_B \cdot T(F))(x)$ by this latter function itself which, again by Lemma 5, has norm a.e. $< .44$ on $\psi(B')$. Hence

$$\|\chi_{\psi(B)} \cdot F - T^{-1}(\chi_B \cdot T(F))\|_\infty \leq .44$$

and thus

$$(*) \quad \|T(\chi_{\psi(B)} \cdot F) - \chi_B \cdot T(F)\|_\infty \leq .44\|T\| < .47.$$

If we suppose that $\phi(\psi(B)) - B$ has positive ν -measure, we have, for $x \in \phi(\psi(B)) - B$, $\|T(\chi_{\psi(B)} \cdot F)(x)\| \geq 31/32$ and $\chi_B(x)T(F)(x) = 0$, which contradicts (*) above. And if we suppose that $B - \phi(\psi(B))$ has

positive ν -measure then, by Lemma 3, $\chi_B(x) \cdot T(F)(x)$ has norm $\geq (63/64)^{1/2} > .99$ a.e. on this set, while by Lemma 4(ii) $T(\chi_{\psi(B)} \cdot F)(x)$ has norm $< .44$ a.e. on $B - \phi(\psi(B)) \subseteq \phi(\psi(B'))$. This again contradicts (*) and so completes the proof of the lemma.

Recall that a mapping ϕ , defined modulo null sets, of $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ is called a *regular set isomorphism* if it satisfies the properties

$$\phi(A') = [\phi(A)]'$$

$$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \phi(A_n)$$

and

$$\nu[\phi(A)] = 0 \quad \text{if, and only if, } \mu(A) = 0,$$

for all sets A, A_n in $\mathcal{B}(X)$, [12].

LEMMA 7. ϕ is a regular set isomorphism of $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$.

Proof. We have seen that ϕ is a mapping, defined modulo null sets, of $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ satisfying

$$\phi(A') = [\phi(A)]', \quad A \in \mathcal{B}(X).$$

Note that for $A \in \mathcal{B}(X)$, $\mu(A) \neq 0$ iff $\chi_A \cdot F \neq 0$ in $L^\infty(\mu, E)$ which is true iff $T(\chi_A \cdot F) \neq 0$ in $L^\infty(\nu, E)$ which holds (since T is norm-increasing) iff $\nu[\phi(A)] = \nu(\{y \in Y: \|T(\chi_A \cdot F)(y)\| \geq 31/32\}) > 0$. Thus

$$\nu[\psi(A)] = 0 \quad \text{if } \mu(A) = 0.$$

Now suppose that A and B are disjoint set in $\mathcal{B}(X)$. Then by Lemma 4(i) $\phi(A)$ and $\phi(B)$ are a.e. disjoint. Thus if B is a measurable subset of the measurable set A , then B and A' are disjoint so that $\phi(B)$ and $\phi(A')$ are disjoint. Hence $B \subseteq A$ implies that $\phi(B) \subseteq \phi(A)$. The sentence before last also implies that A and B are disjoint iff $\phi(A)$ and $\phi(B)$ are disjoint.

Next assume that $\{A_1, A_2, \dots\}$ is a sequence of measurable subsets of X and let $A = \bigcup_{n=1}^{\infty} A_n$. Then since $A_n \subseteq A$ for all n we have $\phi(A_n) \subseteq \phi(A)$ for all n so that $\bigcup_{n=1}^{\infty} \phi(A_n) \subseteq \phi(A)$. Set $B = \phi(A) - \bigcup_{n=1}^{\infty} \phi(A_n)$. We would like to show that $\nu(B) = 0$.

By Lemma 6 there exists $C \in \mathcal{B}(X)$ with $\phi(C) = B$. By what we established in the paragraph before last, we must have $C \subseteq A$ in this instance. Thus if we suppose that B , hence C , has positive measure then,

for some n , C meets A_n in a set of positive measure. But $\phi(A_n)$ and $\phi(C)$ are disjoint, and this contradiction shows that we must have $\nu(B) = 0$. Thus

$$\phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \phi(A_n),$$

completing the proof of the lemma.

The proof of our Theorem is now completed by the following:

LEMMA 8. *If there exists an isomorphism T of $L^p(\mu, E)$ onto $L^p(\nu, E)$ with $\|T^{-1}\| = 1$ and $\|T\| < 3/(2\sqrt{2})$ for $p = 1$ or $p = \infty$ then $L^1(\mu, E) \cong L^1(\nu, E)$ and $L^\infty(\mu, E) \cong L^\infty(\nu, E)$.*

Proof. First suppose that T is such a mapping of $L^\infty(\mu, E)$ onto $L^\infty(\nu, E)$. We have seen that there then exists a regular set isomorphism ϕ of $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$. Then for $B \in \mathcal{B}(Y)$ define $\lambda(B) = \mu[\phi^{-1}(B)]$. If $A \in \mathcal{B}(X)$ we have $\mu(A) = \lambda[\phi(A)] = \int_{\phi(A)} d\lambda$ so that the map $\sum_{j=1}^n e_j \chi_{A_j} \rightarrow \sum_{j=1}^n e_j \chi_{\phi(A_j)}$ carries the dense subspace of simple functions in $L^1(X, \mathcal{B}(X), \mu, E)$ isometrically onto the corresponding subspace of $L^1(Y, \mathcal{B}(Y), \lambda, E)$ and can thus be extended to an isometry of $L^1(X, \mathcal{B}(X), \mu, E)$ onto $L^1(Y, \mathcal{B}(Y), \lambda, E)$. Then multiplication by the scalar function $d\lambda/d\nu$ carries this latter space isometrically onto $L^1(Y, \mathcal{B}(Y), \nu, E)$. Hence $L^1(\mu, E) \cong L^1(\nu, E)$ and consequently $L^\infty(\mu, E) \cong L^\infty(\nu, E)$.

If we start with a map T of $L^1(\mu, E)$ onto $L^1(\nu, E)$ satisfying the conditions of the lemma, then T^* is an isomorphism of $L^\infty(\nu, E)$ onto $L^\infty(\mu, E)$ with $\|T^{*-1}\| = 1$ and $\|T^*\| < 3/(2\sqrt{2})$, and the proof then follows as above.

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