ONE-DIMENSIONAL ALGEBRAIC FORMAL GROUPS

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Let $K$ be an algebraically closed field of characteristic zero. We shall call an element of $K[[x_1, \ldots, x_n]]$ algebraic if it is algebraic over $K(x_1, \ldots, x_n)$. Thus a one-dimensional algebraic formal group is an element $F \in K[[x_1, x_2]]$ such that $F$ is a formal group and $F$ is algebraic.

As is well known, such formal groups arise from one-dimensional algebraic groups. Our intention is to show that this is the only way they arise. All formal groups mentioned in this note shall be one-parameter formal groups.

DEFINITION. Two algebraic formal groups $F, F' \in K[[x_1, x_2]]$ are said to be algebraically isomorphic if there exists an algebraic element $f \in xK[[x]]$ such that $f \neq 0$ and

$$f(F(x_1, x_2)) = F'(f(x_1), f(x_2)).$$

It is easy to see that there exists a unique element $f^* \in xK[[x]]$ such that $f \circ f^* = x$. It then follows that

$$f^*F'(x_1, x_2) = F(f^*(x_1), f^*(x_2))$$

and that $f^*$ is algebraic.

Now suppose $(X, e, +)$ is a one-dimensional algebraic group over $K$. Let $z \in K(X)$ be a local parameter at $e$. Let $\rho_1, \rho_2 : X \times X \to X$ be the natural projections. Then $\{z \circ \rho_1, z \circ \rho_2\}$ is a set of local parameters at $e \times e$ in $X \times X$, and so there exists a unique power series $H(x, y) \in K[[x, y]]$ such that

$$H(z \circ \rho_1, z \circ \rho_2) = z(\rho_1[+] \rho_2)$$

as elements of the complete local ring at $e \times e$ on $X \times X$. It is easy to see that $H$ is an algebraic formal group. We shall call such a formal group a formal algebraic group.

PROPOSITION A. Every algebraic formal group is algebraically isomorphic to a formal algebraic group.
We will prove a stronger statement than Proposition A. We call a differential \( \omega \in K[[x]] dx \) algebraic if \( \omega/dx \) is an algebraic element of \( K[[x]] \). If \( H(x, y) \) is a formal group and

\[
g(x) = \frac{d}{dy} H(x, y) \bigg|_{y=0},
\]

then \( g(0) = 1 \), and

\[
\omega = g dx
\]

is the invariant differential of \( H \). If \( H \) is an algebraic, then so is \( \omega \). We will prove

**Proposition B.** Let \( \omega \) be an algebraic differential. Suppose that there exist nonzero algebraic elements \( f_1, f_2 \) of \( xK[[x]] \) such that

\[
f_1^*(\omega) = af_2^*(\omega)
\]

where \( a \in \mathbb{C}^* \), \( a \) is not a root of unity. Then there exist a formal algebraic group with invariant differential \( \omega' \) and an algebraic element \( u \) of \( K[[x]] \) such that

\[
eu^*(\omega') = \omega
\]

where \( e = \text{Res}_0(\omega/x) \).

To deduce Proposition A from Proposition B, let \( F \) be an algebraic formal group, \( \omega \) its invariant differential, \( f_2(x) = x, f_1(x) = F(x, x) \). Then

\[
f_1^*(\omega) = 2\omega = 2f_2^*(\omega).
\]

It follows that there exists a formal algebraic group \( H \) with invariant differential \( \omega' \) and an algebraic element \( g \in xK[[x]] \) such that

\[
g^*(\omega') = \omega.
\]

We claim

\[
g(F(x, y)) = H(g(x), g(y)).
\]

Indeed, if \( \lambda, \lambda' \in xK[[x]] \), \( d\lambda = \omega, d\lambda' = \omega' \), then (1) implies \( \lambda' \circ g = \lambda \). On the other hand,

\[
\lambda F(x, y) = \lambda(x) + \lambda(y)
\]

\[
\lambda' H(x, y) = \lambda'(x) + \lambda'(y),
\]
so that
\[ gF(x, y) = \lambda' \circ \lambda F(x, y) = \lambda' (\lambda(x) + \lambda(y)) \]
\[ = H(\lambda' \circ \lambda(x) + \lambda'(\lambda(y))) = H(g(x), g(y)) \]
as required.

**Proof of Proposition B.** Let \( P^1 \) denote the projective line over \( K \) and regard \( x \) as the standard parameter on \( P^1 \). In doing this we will identify \( K[[x]] \) with the formal completion of the ring of functions on \( P^1 \) regular at 0, \( \mathcal{O}_{P^1,0} \).

Let \( f_0 = \omega / dx \). Then for \( i = 0, 1, 2 \) there exist complete pointed curves \( (X_i, e_i) \) over \( K \) together with morphisms
\[ x_i, \tilde{f}_i: Y_i \to P^1 \]
such that \( x_i \) is a local uniformizing parameter at \( e_i \) and \( x_i \circ f_i \) is the formal expansion of \( \tilde{f}_i \) in \( x_i \) at \( e_i \). In other words, \( x_i \circ f_i \) is the image of \( f_i \) in \( \mathcal{O}_{Y_i, e_i} \).

Now set \( \tilde{\omega} = f_0 \, dx_0 \in \Omega^1_{Y_0/k} \). Also note that \( f_i(e_i) = 0 \) as \( f_i(0) = 0 \), \( i = 1, 2 \). Let \( (Z_i, e_i') \) denote the fiber product of \( (Y_0, e_0) \) and \( (Y_i, e_i) \) over \( (P^1, 0) \) with respect to the morphisms \( x_0 \) and \( \tilde{f}_i, i = 1, 2 \). Thus \( (Z_i, e_i') \) fits into a commutative diagram
\[
\begin{array}{ccc}
(Z_i, e_i') & \xrightarrow{y_i} & (Y_i, e) \\
\downarrow \tilde{f}_i & & \downarrow f_i \\
(Y_0, e_0) & \xrightarrow{x_0} & (P^1, 0).
\end{array}
\]
Moreover, \( (x_i \circ y_i) \circ f_i \circ \omega \) is the formal expansion of \( \tilde{f}_i \circ \tilde{\omega} \) at \( e_i \) in \( x_i \circ y_i \).

Now let \( (W, e) \) denote the fiber product of \( (Z_1, e_1') \) and \( (Z_2, e_2') \) with respect to the morphisms \( x_1 \circ y_1 \) and \( x_2 \circ y_2 \). Thus we have a commutative diagram
\[
\begin{array}{ccc}
(W, e) & \xrightarrow{z_2} & (Z_2, e_2') \\
\downarrow z_1 & & \downarrow x_2 \circ y_2 \\
(Z_1, e_1) & \xrightarrow{x_1 \circ y_1} & (P^1, 0).
\end{array}
\]
Let \( (W^c, e) \) denote the connected component of \( (W, e) \) passing through \( e \). Let
\[ \tilde{f}_i: (W^c, e) \to (Y_0, e_0) \]
denote the restriction of \( \tilde{f}_i \circ z_i \) to \( W^c \). Then
\[ (x_i \circ y_i \circ z_i) \circ f_i \circ \omega \]
is the formal expansion of $\hat{f}_1^*\omega$ at $e$ in $x_i \circ y_i \circ z_i$. Since $x_1 \circ y_1 \circ z_1 = x_2 \circ y_2 \circ z_2$, it follows from the hypothesis that

$$\hat{f}_1^*\omega = af^*_2\omega.$$  

Taking $X_1 = X_0, X_2 = W^c$ and $\omega_1 = \omega$ we see that Proposition B follows from:

**Proposition C.** Let $X_1, X_2$ be two curves. Let $\omega_1$ be a nonzero differential on $X_1$ and $f_1, f_2$ two nonconstant morphisms from $X_2$ to $X_1$ such that

$$(2) \quad f_1^*(\omega_1) = af_2^*(\omega_1)$$

for some $a \in K^*,$ a not a root of unity. Then there exists a one-dimensional algebraic group $G$ with invariant differential $\omega$, and a morphism $f: X_1 \to G$ such that

$$f^*(\omega) = \omega_1.$$  

**Proof.** For a curve $C$ let $\bar{C}$ denote its complete nonsingular model. Let $\omega_2 = f_2^*(\omega_1)$. Let $S_i$ denote the set of poles of $\omega_i$ on $\bar{X}_i$. Clearly, $|S_1| \leq |S_2|$, $|S_i|$ denotes the order of $S_i$. We also claim:

$$g(X_1) < g(X_2) \quad \text{or} \quad g(X_2) \leq 1$$

where $g(X_i)$ denotes the genus of $X_i$. Indeed, if this is not the case, then by the Hurwitz genus formula we see that $g(X_1) = g(X_2) > 1$ and $1 = \deg(f_1) = \deg(f_2)$, but then $f_i: \bar{X}_2 \to \bar{X}_1$ is birational ($f_i$ is the "lifting" of $f_i$), so that $\alpha = f_2^{-1} \circ f_1$ is an automorphism of $X_2$. But $\alpha$ is of finite order since $g(X_2) > 1$. On the other hand, the hypotheses of the lemma imply

$$\alpha^*(\omega_2) = a\omega_2.$$  

Since $a$ is not a root of unity, we obtain a contradiction, so we have our claim.

We also claim that there exists a curve $X_0$ with a differential $\omega_0$ and two morphisms $g_1, g_2: X_1 \to X_0$ such that $g_2^*(\omega_0) = \omega_1$ and $g_1^*(\omega_0) = a\omega_2^*(\omega_0)$. Thus $(X_0, \omega_0)$ satisfies the same hypotheses as $(X_1, \omega_1)$, so once we establish this claim, we will be able to use induction to suppose that $|S_1| = |S_2|$ and $g(X_2) \leq 1$.

For the results on generalized Jacobians used below, see [S].

**Proof of Claim.** Without loss of generality $X_i$ is nonsingular, $\omega_i$ has no poles on $X_i$, and $f_i X_2 = X_1$, for $i = 1, 2$.  

Let $i = 1$ or $2$ in the following: Let $M_i$ denote the polar divisor of $\omega_i$. Let $J_i$ denote the generalized Jacobian of $X_i$ corresponding to $M_i$. There exists a unique invariant differential $v_i$ on $J_i$ and an embedding of $X_i$ in $J_i$ (as $\omega_i \neq 0$) well defined up to translation such that $\omega_i$ is the pullback of $v_i$ to $X_i$. Henceforth we will view $X_i$ as a subvariety of $J_i$. From the functoriality of generalized Jacobians there exists a canonical affine transformation

$$f_i': J_2 \to J_1$$

whose restriction to $X_2$ is $f_i$. Let $T_i$ denote translation on $J_2$ by $[-] f_i'(0)$ where $[-]$ denotes inversion on $J_1$. Set $f_i'' = T_i \circ f_i'$. Then $f_i''$ is a homomorphism from $J_2$ to $J_1$. It follows that

$$(f_i'')^* v_1 = a (f_2'')^* v_1 = av_2.$$

There also exists a homomorphism $h: J_1 \to J_2$ such that

$$f_2'' \circ h = [d]$$

where $d$ denotes the degree of $f_2$ and $[d]$ denotes multiplication by $d$ on $J_1$. Let

$$e = (f_1'' \circ h \circ f_2'' - [d] \circ f_1'') : J_2 \to J_1.$$ 

Then $e$ is a homomorphism and

$$e^* v_1 = (f_2'')^* h^* (f_i'')^* v_1 - g_1^* [d] v_1 = a (f_2'')^* h^* v_2 - d g_1^* v_1 = a (f_2'')^* h^* v_2 - d a v_2 = a (f_2'')^* [d] v_1 - d a v_2 = 0.$$ 

Let $A$ denote the quotient of $J_1$ by $e(J_2)$ and $\rho: J_1 \to A$ the quotient morphism. Since $e^* v_1 = 0$, it follows that there exists an invariant differential $v_0$ on $A$ such that $\rho^* v_0 = v_1$. Let

$$X_0 = (\rho \circ [d] \circ T_1)(X_1) \subseteq A.$$ 

As $\rho \circ e = 0$ we have $\rho \circ [d] \circ f_i'' = \rho \circ f_i'' \circ h \circ f_2''$. Hence as $f_i'(X_2) = f_2'(X_2) = X_v$,

$$X_0 = (\rho \circ [d] \circ T_1 \circ f_i')(X_2) = (\rho \circ f_i'' \circ h \circ f_2')(X_2) = (\rho \circ f_i'' \circ h \circ T_2)(X_1).$$ 

Now let $g_1, g_2: X_1 \to X_0$ denote the restrictions of

$$\rho \circ f_i'' \circ h \circ T_2 \quad \text{and} \quad \rho \circ [d] \circ T_1.$$
respectively to $X_1$. Also let $\omega_0$ denote the restriction of $v_0/d$ to $X_0$. Since $(\rho \circ f'' \circ h)^* v_0 = (f'' \circ h)^* v_1 = \text{ad} v_1 = a(\rho \circ [d])^* v_0$ it follows that
\[ g_1^* \omega_0 = a g_2^* \omega_0 = a \omega_1 \]
and so we have our claim. Thus by induction we may suppose
\[ g(X_1) = g(X_2) \leq 1 \quad \text{and} \quad |S_2| = |S_1|. \]
We also have $f_i^{-1}(S_1) = S_2$, so that $f_i$ induces a bijection from $S_2$ onto $S_1$.

**Case 1.** $g(X_i) = 1$. Then $\tilde{X}_i$ has a unique group structure with origin at some point $P_i$. It follows that $f_2$ and $T_R \circ f_1$ are affine transformations from $X_2$ to $X_1$. Now since $f_i|_{S_2}$ $S_2 \to S_1$ is a bijection and $f_i^{-1}(S_1) = S_2$, it follows that either
\[ S_2 = S_1 = \emptyset \]
or degree $f_i = 1$, $i = 1, 2$, because $f_i$ is étale. In the second case, $f_2^{-1}$ exists and $\alpha = f_2^{-1} \circ f_1$ is an automorphism of $X_1$ such that $\alpha S_2 = S_2$. But if $S_2 \neq \emptyset$, $\alpha$ is of finite order. This contradicts
\[ \alpha^* \omega_2 = a \omega_2. \]
Thus $S_1 = S_2 = \emptyset$, and $\omega_1$ is an invariant differential on $X_1$ as required.

**Case 2.** $g(X_i) = 0$. Then $|S_i| \geq 1$. Let
\[ A = \begin{cases} \{ \infty \} & \text{if } |S_i| = 1, \\ \{ \infty, 0 \} & \text{if } |S_i| = 2, \\ \{ \infty, 0, 1 \} & \text{if } |S_i| \geq 3. \end{cases} \]
After composing with linear fractional transformations, we may suppose $A \subseteq S_2$ and $A \subseteq S_1$.

If $|S| = 1$, then $\omega_1 = b \, dx$ for some $b \in K^*$, and so is an invariant differential on $G_a$. Now suppose $|S_2| \geq 1$. Let $h_i$ be a linear fractional transformation such that
\[ h_i \circ f_i(p) = p, \quad p \in A. \]
Because $(h_i \circ f_i)^{-1}(p) = \{ p \}$, $p \in A$, it follows that $h_i \circ f_i$ takes the value $p$ with multiplicity $n_i$, where $n_i$ is the degree of $f_i$. As $\{ 0, \infty \} \subseteq A$ we must have
\[ h_i \circ f_i = c_i x^{n_i}, \]
where $c_i \in K^*$. If $|S_2| > 2$, then $1 \in A$. It follows that $c_i = 1$, and since $(h_i \circ f_i)^{-1}(1) = 1$, that $n_i = 1$. That is, $f_i = h_i^{-1}$. But then $\alpha = h_2^{-1} \circ h$, takes
$S_2$ onto itself, and $a^*\omega_2 = a\omega_2$. As the group of linear fractional transformations preserving $S_2$ is finite this contradicts the hypothesis that $a$ is not a root of unity. Thus $S_2 = S_1 = \{0, \infty\}$,

$$f_i = r_i x^{m_i} \quad \text{and} \quad \omega_1 = s \, dx + t \frac{dx}{x}$$

for some $r_i, t \in K^*, m_i \in \mathbb{Z}, m_i \neq 0$ and $s \in K$. So,

$$f_i^*(\omega_1) = sr_i m_i x^{m_i - 1} \, dx + tm_i \frac{dx}{x}.$$

Since $a \neq 1$, the hypothesis $f_1^*(\omega) = af_2^*(\omega_1)$ implies $s = 0$. Thus $\omega_1$ is an invariant differential on $G_m$ as required.

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