FINITE GROUP ACTION AND VANISHING OF $N^G_*$\([F]\)$

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Let $G$ be a finite group (not necessarily abelian). The object of this paper is to describe a $G$-bordism theory which vanishes. We construct a family $F$ of $G$ slice types, such that the $N_\ast$-module $N^G_\ast[F]$ is zero. Kosniowski has proved a similar result earlier for a finite abelian group. The present work is a generalisation of his result by using basically the same technique. A recent result of Khare is obtained as a corollary to the vanishing of $N^G_\ast[F]$.

1. Preliminaries and statement of the main theorem. Let $G$ be a finite group with centre $C(G)$ and $G_2$ be the subgroup generated by the elements of order 2 in $C(G)$. We also assume that $G_2$ is nontrivial. By a $G$-manifold $M$ we mean throughout a closed differentiable manifold on which $G$ acts smoothly. $G_x$ denotes the isotropy subgroup at $x \in M$. For every $x \in M$, there exists a $G_x$-module $\overline{V}_x$ which is equivariantly diffeomorphic to a $G_x$-invariant neighbourhood of $x$. $\overline{V}_x$ has a submodule $V'_x$ in which $G_x$ acts trivially and a complementary submodule $V_x$ in which no nonzero element is fixed by all of $G_x$. By the $G$-slice type of $x$ we mean the pair $[G_x; V_x]$. By a $G$-slice type we mean a pair $[H; U]$ where $U$ is a $H$-module in which no nonzero element is fixed by all of $H$ (equivalently $U$ contains no trivial $H$-submodule). A family $F$ of $G$-slice types is a collection of $G$-slice types such that if $[H; U] \in F$ then for every $x \in G \times_H U$ the $G$-slice type $[G_x; V_x] \in F$. A $G$-manifold is said to be of type $F$ if for all $x \in M$, $[G_x, V_x] \in F$. Bordism relation is defined in the usual way. Two $n$-dimensional closed $G$-manifolds $M_1$, $M_2$ of type $F$ are said to be $F$-bordant if there exists an $(n + 1)$-dimensional compact differentiable $G$-manifold $W$ of type $(F, F)$ such that the disjoint union of $M_1$ and $M_2$ is $G$ equivariantly diffeomorphic to $\partial W$. It is clearly an equivalence relation on the set of $G$-manifolds of type $F$ and gives rise to a bordism theory $N^G_\ast[F]$. We note that $N^G_\ast[F]$ is a graded $N_\ast$-module, $N_\ast$ being the unoriented bordism ring.

Kosniowski has described a family $\tilde{F}(\hat{G})$ in [4] for an abelian group $G$ such that $N^G_\ast[\tilde{F}(\hat{G})] = 0$, $\hat{G}$ being a subgroup of $G$ containing $G_2$. As a consequence he proved that if $M$ is a $G$-manifold ($G$ abelian) in which $G_2$ acts without fixed points then $M$ is a $G$-boundary—a result obtained earlier by Khare using a different technique [1]. The main theorem of this
paper is a generalisation of Kosniowski’s theorem in [4] for an arbitrary finite group. Once again another result of Khare [2] is obtained as a corollary of this theorem.

The subgroup $G_2$ consisting of the identity and elements of order two in the centre of $G$ is isomorphic to $\mathbb{Z}_2^k$ for some $k > 0$. Kosniowski has studied $\mathbb{Z}_2^k$-bordism in [3] and the techniques used here are generalized from his techniques. We choose once for all a basis $g_1, g_2, \ldots, g_k$ of $G_2$ and order the elements by

$$g_1 < g_2 < \cdots < g_k < g_1 g_2 < \cdots < g_1 g_k < \cdots < g_1 g_2 \cdots g_k.$$ 

Now let $[G_x; V_x]$ be the $G$-slice type of a point $x$ in a $G$-manifold $M$ and $G(x)$ be the orbit of $x$. Then $G(x)$ is a closed and compact submanifold of $M$. Consider the normal bundle $\nu(i)$ of the canonical embedding of $G(x)$ in $M$. This is a $G$-vector bundle and its disc bundle is a closed $G$-invariant tubular neighborhood of $G(x)$. Further $G$ acts as a group of bundle maps on the normal bundle and the fibre over $x$ is $G_x$-invariant and contains no $G_x$-trivial subspace. It is precisely $V_x$ the $G_x$-module present in the $G$-slice type $[G_x; V_x]$ of $x$. Let $g_\ast$ be the map on the total space $E(\nu(i))$ induced by the action of $g$ on the base space $G(x)$. The $G$-slice type of $gx \in G(x)$ is $[gG_x g_1; g_\ast V_x]$. The underlying vector space of $V_x$ and $g_\ast V_x$ are same and the action of $ghg_1$, $h \in G_x$ on $v \in g_\ast V_x$ is same as the action of $h$ on $v \in V_x$. Again if $F$ be a family of $G$-slice types and $[H; V] \in F$ then from the definition of family the $G$-slice type $[G_x; V_x]$ of every point $x \in G \times_H V$ belongs to $F$. Now the $G$-slice type of $[e, 0] \in G \times_H V$ is $[H; V]$ and the $G$ slice type of $[g, 0] \in G \times_H V$ is $[gHg_1; g_\ast V]$. The $G$-slice type $[H; V]$ will be denoted by $\rho$ and the collection

$$\{ [gHg_1; g_\ast V] | g \in G \}$$

termed as a conjugate class of $G$-slice types will be denoted by $\overline{\rho}$ or $[H; V]^g$.

Suppose that $K$ is a subgroup of $H$. We write $K \subset H$ if $H = (x) \times K$ where $x \in G_2$. Quite a number of elements of $G_2$ may yield $H$ when a direct product of above type is formed. We take the minimal element $x$ according to the total order fixed at the beginning of this article. We now have a homomorphism

$$p = p_{H,K}: H \to K.$$ 

which is the projection onto the second factor. This is termed as the distinguished projection. It enables us to obtain an $H$-module $p^*U$ from a $K$-module $U$. The modules $p^*U$ and $U$ have the same underlying vector
space and $H$ acts on $p^*(U)$ via the map $p$. Corresponding to a $G$ slice type $[K; U]$ such that $K \subset H$ we have an extension function $e = e_{K,H}$ given by

$$e_{K,H}[K; U] = [H; V(K) \oplus p^*(U)]$$

where $V(K)$ is one dimensional real representation of $H$ in which $h \in H$ acts by multiplication with 1 if $h \in K$ and multiplication with $-1$ if $h \notin K$. Since $gHg^{-1} = (x) \times gKg^{-1}$ when $H = (x) \times K$, we have

$$e_{K,H}[gKg^{-1}; g\ast U] = [gHg^{-1}; V(gKg^{-1}) \oplus p^*(g\ast U)]$$

Thus $e_{K,H}$ induces a map $e^g = e_{K,H}^g$ on the collection of conjugate classes of $G$ slice types $[K; U]^g$ and

$$e_{K,H}^g[K; U]^g = [H; V(K) \oplus p^*(U)]^g.$$  

Corresponding to a subgroup $\hat{G}$ of $G$ containing $G_2$ we have three families of $G$ slice types $F(\hat{G})$,

$$F(\hat{G}) = \{ [gHg^{-1}; g\ast V] | [H, V] \text{ is a } G \text{ slice type}$$

$$\text{with } H \text{ contained in } \hat{G}, \ g \in G \}$$

and

$$F'(\hat{G}) = \{ [K; U] \in F(\hat{G}) | K \cap G_2 \neq G_2 \}$$

and

$$\hat{F}(\hat{G}) = F'(\hat{G}) \cup \{ e_{K,H}[K; U][[K; U]]' \in F'(\hat{G})$$

$$\text{and } K \subset H \text{ with } H \cap G_2 = G_2 \}.$$  

That each collection is a family is clear. Now we are in a position to state the main theorem of this paper.

**THEOREM 1.** If $G$ be a finite group and $\hat{G}$ be a subgroup of $G$ which contains $G_2$ then $N^G_F(\hat{F}(\hat{G})) = 0$.

**COROLLARY (Khare [2]).** Suppose that $G$ is a finite group. If $M$ is a $G$-manifold on which $G_2$ acts without fixed points then $M$ is a $G$-boundary.

The corollary follows because if $G_2$ acts without fixed points then an isotropy subgroup $H$ of a point in $M$ satisfies the condition $H \cap G_2 \neq G_2$ so that $M$ is of the type $F'(G)$ and consequently of the type $\hat{F}(G)$.

The proof of the theorem will be given in §7. In §2, §3, §4 and §5, we develop the necessary tools and results.
2. Vector bundles of type $\bar{\rho}$. Let $F' \subseteq F$ be two families of $G$ slice types with $F = F' \cup \bar{\rho}$ where $\bar{\rho}$ is a class of conjugate $G$ slice types. By a $G$-vector bundle of type $\bar{\rho}$ we mean a $G$-vector bundle $\xi$: $E(\xi) \rightarrow B(\xi)$ where the set of points of $E(\xi)$ having $G$ slice type in $\bar{\rho}$ is precisely the zero section. We have the bundle bordism groups $N_n^G[\bar{\rho}]$ obtained by defining a bordism relation on the set of all $G$ vector bundles of type $\bar{\rho}$ having total dimension $n$.

Let $M^n$ be a $G$-manifold of type $F$ and $F_\bar{\rho}$ be the set of all points in $M^n$ with slice type in $\bar{\rho}$. Then the normal bundle over $F_\bar{\rho}$ is a $G$ vector bundle of type $\bar{\rho}$. This assignment of the normal bundle over $F_\bar{\rho}$ in $M^n$ leads to a $N_\bullet$-homomorphism

$$v_\bar{\rho}: N_n^G[F] \rightarrow N_n^G[\bar{\rho}].$$

We have the following proposition and lemmas involving the bundle bordism groups.

**Proposition 2.** There exists a long exact sequence

$$\cdots \rightarrow N_n^G[F'] \rightarrow N_n^G[F] \rightarrow N_n^G[\bar{\rho}] \rightarrow N_{n-1}^G[F'] \rightarrow \cdots$$

where $F' \subseteq F$ are families of $G$ slice types such that $F - F' = \bar{\rho}$.

For proof we refer to 1.4.2 of [3].

**Lemma 3.** Suppose that $K \subset H$ and $\bar{\rho} = [H; V]^g$, $\bar{\rho}' = [K; U]^g$ be two classes of conjugate $G$ slice types such that $e^g(\bar{\rho}') = \bar{\rho}$. Then there exists an $N_\bullet$-isomorphism

$$N_n^G[\bar{\rho}] \rightarrow N_{n-1}^G[\bar{\rho}']$$

given by $[\xi] \rightarrow [v_\bar{\rho}S(\xi)]$, where $S(\xi)$ is the sphere bundle of $\xi$.

The proof of this lemma is similar to that given for Lemma 4.5.8 of [3].

**Lemma 4.** Let $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ be a sequence of families of $G$-slice types with

(i) $F_0 = \bar{\rho}_0 = \{[e; R^0]\}$
(ii) $F_i = F_{i-1} \cup \bar{\rho}_i$ for all $i \geq 1$
(iii) $\bigcup_{i \geq 0} F_i = F$

and

(iv) $e^g(\bar{\rho}_{2i}) = \bar{\rho}_{2i+1}$ for all $i \geq 0$. Then $N_*^G[F] = 0$.

**Proof.** Using Proposition 2 and Lemma 3 we get

$$N_*^G[F_{2i}] = N_*^G[\bar{\rho}_{2i}]$$

and $N_*^G[F_{2i+1}] = 0$. 

Taking direct limit

\[ N^G_*[F] = \lim_{\to} N^G_*[F_i] = 0. \]

The rest of the paper is aimed to show that the family \( \tilde{F}(\hat{G}) \) satisfies the conditions laid down in Lemma 4. The \( G \) slice types of \( \tilde{F}(\hat{G}) \) are to be ordered suitably now in order to get the families \( F_0 \subset F_1 \subset \cdots \).

### 3. Ordering the conjugate classes of \( G \) slice types.

We define three distinct relations \(<\) on the collection \( \hat{A} \) of all subgroups conjugates to subgroups of \( \hat{G} \), on the collection of all \( H \)-modules, \( H \in \hat{A} \) and finally on the collection of all conjugate classes of \( G \) slice types of the family \( \tilde{F}(\hat{G}) \) and extend each of these relations into a total order on the respective collection. We note that the elements of \( G_2 \) are totally ordered by

\[ g_1 < g_2 < \cdots < g_k < g_1g_2 < \cdots < g_1g_k < \cdots < g_1g_2 \cdots g_k \]

and a subgroup \( H_2 \) of \( G_2 \) has a distinguished base \( h_1 < h_2 < \cdots < h_m \)
such that \( h_1 \) (\( \neq \) identity) is the least element in \( H \) and for \( i > 1 \), \( h_i \) is the least element in \( H \) which is not present in \( (h_1, h_2, \ldots, h_{i-1}) \), the subgroup generated by \( h_1, h_2, \ldots, h_{i-1} \). The subgroups of \( G_2 \) are now totally ordered first by the order of the subgroup and then lexicographically on the distinguished base:

\[ (e) < (g_1) < (g_2) < \cdots < (g_1g_2 \cdots g_k) < (g_1, g_2) < \cdots. \]

**Rule A.** Let \( H \) and \( K \) belong to \( \hat{A} \). We define \( \leq \) by:

(i) if \( |H| \leq |K| \) Then \( H \leq K \),

(ii) if \( |H| = |K| \) and \( |K_2| \leq |H_2| \). Then \( H \leq K \) where \( K_2 = K \cap G_2 \)
and \( H_2 = H \cap G_2 \),

(iii) if \( |H| = |K|, |K_2| = |H_2| \) but \( H_2 \leq K_2 \) then \( H \leq K \) and

(iv) if \( |H| = |K|, H_2 = K_2 \) then we order them arbitrarily so as to make the relation \( \leq \) a total ordering on \( \hat{A} \).

Next a relation \( \leq \) is introduced on the collection of all nontrivial irreducible \( H \)-modules \( H \in \hat{A} \). We write \( U \leq V \) if \( U = V \) or else there exists \( K \subset H \) such that \( U = p^*i^*V \) where \( i: K \to H \) is the natural inclusion and \( p: H \to K \) is the distinguished projection. We now have the following lemma whose proof is similar to Lemma 8 of \([4]\).

**Lemma 5.** The relation \( \leq \) is a partial order on the collection of all nontrivial irreducible \( H \)-modules.

We now choose a total ordering on the set of all nontrivial irreducible \( H \)-modules having the same dimension compatible with the partial ordering introduced. The total ordering is now extended to all irreducible
$H$-modules by writing $U \leq V$ if and only if $\dim U \leq \dim V$. Since any $H$-module can be expressed uniquely as the sum of irreducible $H$-modules, we can extend this total ordering on all $H$-modules by lexicography. The following rule expresses the whole rule coincisely.

**Rule B.** Let $U$ and $V$ be two $H$-modules.

(i) If $\dim U \leq \dim V$ then $U \leq V$

(ii) If $\dim U = \dim V$ and $V$ follows $U$ lexicographically then $U \leq V$.

Finally Rule C given as below defines the order $\leq$ on the collection of all classes of conjugate $G$ slice types of the family $\tilde{F}(\hat{G})$.

**Rule C.** Let $\tilde{\rho} = [H; U]^g$, $\tilde{\rho}' = [K; V]^g$ be two classes of conjugate $G$ slice types of $\tilde{F}(\hat{G})$

(i) If $\dim U \leq \dim V$ then $\tilde{\rho} \leq \tilde{\rho}'$.

(ii) If $\dim U = \dim V$ and $H \leq K$ then $\tilde{\rho} \leq \tilde{\rho}'$.

(iii) If $\dim U = \dim V$, $H = K$ and $U \leq V$ then $\tilde{\rho} \leq \tilde{\rho}'$.

We now proceed to prove some algebraic results relating to the extension map $e$.

**4. Algebraic lemmas and extension map.** The following lemmas are generalisations of propositions of $\mathbb{Z}_2^k$ bordism given in 4.5 of [3]

**Lemma 6.** Let $(e) \subset K \subset H \subset G$ and

$$g_1 < g_2 < \cdots < g_k,$$

$$h_1 < h_2 < \cdots < h_m,$$

and

$$k_1 < k_2 < \cdots < k_{m-1}$$

be the distinguished bases of $G_2$, $H_2$ and $K_2$ respectively and $r$ be the greatest integer for which $k_i = h_i$ for all $i < r$. Then $K$ is not contained in a predecessor of $H$ if and only if $h_i = g_i$ for all $i < r$. (By a predecessor of $H$ we mean a subgroup $H' \prec H$ such that $H'_2 < H_2$.)

**Proof.** We have $(e) \subset K \subset H \subset G$. If $K \subset H'$, a predecessor of $H$ then by definition $K_2 \subset H'_2$, a predecessor of $H_2$. Further if $K_2 \subset H'_2$, a predecessor of $H_2$, then $H' = (x) \times K_2$ ($x$ being chosen minimally) and $K \subset (x) \times K$, a predecessor of $H$.

Thus $K$ is not contained in a predecessor of $H$ if and only if $K_2$ is not contained in a predecessor of $H_2$. The latter statement implies and is implied by $h_i = g_i$ for all $i < r$ and this follows from 4.5.12 of [3].
Lemma 7. Let $K \subset H$, $K' \subset H$ with $K$ and $K'$ not contained in a predecessor of $H$. If
\[ H = (x) \times K = (x') \times K' \]
where $x$ and $x'$ are chosen minimally, $x \in K'$, $x' \in K$ and $K$ precedes $K'$ then $K \cap K'$ is not contained in a predecessor of $K$.

Proof. We have
\[ H_2 = (x) \times K_2 = (x') \times K'_2. \]
and $K_2$ precedes $K'_2$. By the Proposition 4.5.13 of [3], $K_2 \cap K'_2$ is not contained in a predecessor of $K_2$ and this in turn implies that $K \cap K'$ is not contained in a predecessor of $K$.

In order to proceed further we need the following constructions and lemmas.

$S(H)$ = collection of all conjugate classes of $G$ slice types with isotropy subgroup $H$. For any $K \subset H$ we have the extension function
\[ e^g = e^g_{K,H} : S(K) \to S(H) \]
and consequently a function
\[ E^g : \bigcup_{K \subset H} S(K) \to S(H). \]
where by $P(H)$ one means a predecessor of $H$. Let
\[ \bar{S}(K) = S(K) - \text{image } \left( \bigcup_{L \subset P(H)} E^g : S(L) \to S(K) \right). \]
The function
\[ \bar{E}^g : \bigcup_{K \subset H} \bar{S}(K) \to S(H) \]
is the restriction of $E^g$.

Lemma 8. Image $\bar{E}^g = \text{image } E^g$.

Proof. Clearly $\text{image } \bar{E}^g \subseteq \text{image } E^g$.

Let $\bar{\rho} \in \text{im } E^g$ i.e. $\bar{\rho} = e^g(\bar{\rho}')$ for some $\bar{\rho}' \in S(K)$ where $K \subset H$ and $K \not\subset P(H)$. 
If $p' \not\in \tilde{S}(K)$ then $p' = e^s(\tilde{p}')$ for some $\tilde{p}' \in S(L)$ where $L \subset H$ and $L \not\subset P\langle H\rangle$. By Lemma 6 we have the following distinguished bases of $H_2$, $K_2$ and $L_2$

$L_2: g_1 < g_2 < \cdots < g_{s-1} < l_s < \cdots$

$K_2: g_1 < g_2 < \cdots < g_{s-1} < g_s < \cdots < g_{r-1} < k_r < \cdots$

$H_2: g_1 < g_2 < \cdots < g_{r-1} < g_r < h_{r+1} < \cdots$.

We note that $l_s \neq g_s$ and $k_r \neq g_r$. So

$H = (g_r) \times K$ and $K = (g_s) \times L$.

Writing $\tilde{p}'' = [L; U]^g$ we get

$$\tilde{p}' = e^s(\tilde{p}'') = [K; V(L) \oplus q^*U]^g,$$

and

$$\tilde{p} = e^s(\tilde{p}') = [H; V(K) \oplus p^*(V(L) \oplus q^*U)]^g$$

$$= [H; V(K) \oplus V((g_r) \times L) \oplus p^*q^*U]^g$$

$q: K \to L$ and $p: H \to K$ are the distinguished projections.

Taking $K' = (g_r) \times L$ we note that $K' \subset H$ and $K$ precedes $K'$. Moreover $K' \not\subset P\langle H\rangle$. Extending $\tilde{p}''$ through $K'$ we get

$$\tilde{p}'' = e^s_{K',H}(\tilde{p}'') = [K'; V(L) \oplus q'^*U]^g \in S(K')$$

and

$$e^s_{K',H}(\tilde{p}'') = [H; V(K') \oplus V((g_s) \times L) \oplus p'^*q'^*U]^g$$

where $p': H \to K'$ and $q': K' \to L$ are the distinguished projections.

Since $qp = q'p'$, we have

$$e^s_{K',H}(\tilde{p}'') = [H; V(K') \oplus V(K) \oplus p^*q^*U]^g = \tilde{p}.$$

If $\tilde{p}'' \in \tilde{S}(K')$ then $\tilde{p} \in \text{image } \tilde{E}^g$. If not then by arguing as before we get a conjugate class of $G$ slice type $\tilde{p}^{(n)} \in S(K''')$ such that $\tilde{p} = e^s(\tilde{p}^{(n)})$ where $K''' \subset H$ and $K < K' < K'' \not\subset P\langle H\rangle$.

Continuing this way we exhaust all the finite number of possibilities and find some $\tilde{p}^{(2n+1)} \in \tilde{S}(K^{(n)})$ such that $K^{(n)} \subset H$, $K^{(n)} \not\subset P\langle H\rangle$ and $\tilde{p} = e^s(\tilde{p}^{(2n+1)})$ i.e. $\tilde{p} \in \text{image } \tilde{E}^g$.

**Lemma 9.** The function

$$\tilde{E}^g: \bigcup_{K \subset H} S(K) \to S(H)$$

$$K \notin \tilde{p}(H)$$

is injective.
Proof. Suppose that
\[ \bar{p} = [K; U]^g, \bar{p}' = [K'; U']^g \]
where \( K \) and \( K' \subset H \), \( K \) and \( K' \not\subset P(H) \), \( K \) precedes \( K' \) and
\[ e^g(\bar{p}) = e^g(\bar{p}') = [H; V]^g. \]
From Lemma 6 we get
\[ H = (g_r) \times K = (g_s) \times K' \]
where \( g_r \) and \( g_s \) are the minimal possible choices and \( s < r \). We have
\[ [H; V(K) \oplus p^*U]^g = [H; V]^g = [H; V(K') \oplus p'^*U']^g \]
where \( p: H \to K, p': H \to K' \) are the distinguished projections. Writing
\[ U = \sum_i n_i U_i \] and \( U' = \sum_j n'_j U'_j \) where \( U_i \) and \( U'_j \) are nontrivial irreducible \( K \) and \( K' \) modules respectively we get
\[ V(K) \oplus \sum_i n_i p^*U_i = V(K') \oplus \sum_j n'_j p'^*U'_j. \]
Since \( K \neq K' \), \( V(K) = p'^*U'_t \) for some \( t \) and \( n'_t = 1 \). The underlying vector space of these modules is \( \mathbb{R} \).

We write \( g_s = g^s \circ k, \alpha_1 \in \{0, 1\} \) and \( k \in K \) and consider its action on \( x \in V(K) = p'^*U'_t \). We get \( g_s x = x \) i.e. \((-1)^{\alpha_1}x = x \) i.e. \( \alpha_1 = 0 \). So \( g_s \in K \). Similarly \( g_r \in K' \). By Lemma 7, \( L = K \cap K' \not\subset P(K) \) and \( K = (g_s) \times L \) (\( L \) is the intersection of two normal subgroups of \( H \)). We have also the restriction function
\[ r^g = r_{H,K}^g: S(H) \to S(K) \]
such that \( r^g[H; V]^g = [K; I^*V]^g \) where \( I^*V \) is the nontrivial part of \( i^*V \), \( i: K \to H \) being the natural inclusion. Note that
\[ r_{H,K}^g e_{k,H}^g[K; U]^g = r_{H,K}^g[H; V(K) \oplus p^*U]^g \]
\[ = [K; I^*(V(K) \oplus p^*U)]^g \]
\[ = [K; I^*p^*U]^g = [K; i^*p^*U]^g = [K; U]^g \]
i.e. \( r_{H,K}^g e_{k,H}^g = \text{identity} \).

Therefore
\[ \bar{p} = [K, U]^g = r_{H,K}^g e_{k,H}^g[K; U]^g = r_{H,K}^g e_{k',H}^g[K'; U']^g \]
\[ = r_{H,K}^g[H; V(K') \oplus p'^*U']^g \]
\[ = [K; V(K' \cap K) \oplus I^*p'^*U']^g \]
\[ = [K; V(L) \oplus NTq^*j'^*U']^g \]
where \( i : K \hookrightarrow H, i' : K' \hookrightarrow H, j : L \hookrightarrow K, j' : L \hookrightarrow K' \) are the natural inclusions and \( p : H \rightarrow K, p' : H \rightarrow K', q : K \rightarrow L, q' : K' \rightarrow L \) are the distinguished projections. We have \( p'i = j'q \) and \( NT \) stands for the nontrivial part. Also

\[
r_{k,l}^*((\bar{\rho}) = [L; NTj^*q^*j'^*U^*]^8 = [L; NTj'^*U^*]^8
\]

(since \( qj = \text{id} \)). So

\[
e_{L,K}^*r_{k,l}^*((\bar{\rho}) = [K; V(L) \oplus NTq^*j'^*U^*]^8 = \bar{\rho}.
\]

Thus \( \bar{\rho} = e(\bar{\rho}'' \) for \( \bar{\rho}'' = r_{k,l}^*((\bar{\rho}) \in S(L) \) and \( L \subset K, L \not\subset P(K) \) i.e.

\[
\bar{\rho} \in \text{im} \left\{ E^8 : \bigcup_{L \subset K \atop L \not\subset P(K)} S(L) \rightarrow S(K) \right\}
\]

i.e. \( \bar{\rho} \not\in \bar{S}(K) \) — a contradiction.

With this we come to an end of this section.

5. **Decomposition of the collection of conjugate classes of \( G \) slice types of a family.** If we now define the dimension of a conjugate class of \( G \)-slice types as dimension of the module present therein then it is clear that there are only a finite number of conjugate classes of \( G \) slice types of a given dimension. The classes of the family \( \bar{F}(\hat{G}) \) are totally ordered by the Rule \( C \) and we index them by nonnegative integers as

\[
\bar{\rho}_0 < \bar{\rho}_1 < \bar{\rho}_2 < ...
\]

where \( \bar{\rho}_0 = \{([e], R^0)\} \) and \( \bar{\rho}_0 \) is clearly a family of \( G \) slice types. Corresponding to the family \( F_j \) we form the collection \( \bar{F}_j = \{\bar{\rho}_0, \bar{\rho}_1, \ldots, \bar{\rho}_j\} \) and define inductively three subcollections \( A_j, B_j \) and \( C_j \) of \( \bar{F}_j \) such that \( \bar{F}_j = A_j \cup B_j \cup C_j \). For \( j = 0 \), \( \bar{F}_j = \{\bar{\rho}_0\} \) and we set

\[
A_j = \{\bar{\rho}_0\}, \quad B_j = \emptyset, \quad C_j = \emptyset
\]

Let \( A_{j-1}, B_{j-1}, C_{j-1} \) be defined for some \( j \geq 1 \). We have

\[
\bar{F}_{j-1} = A_{j-1} \cup B_{j-1} \cup C_{j-1}
\]

and

\[
\bar{F}_j = \bar{F}_{j-1} \cup \{\bar{\rho}_j\}.
\]

There are two possibilities:

(i) either \( \bar{\rho}_j = e^8(\bar{\rho}) \) for some \( \bar{\rho} \in A_{j-1} \) or

(ii) \( \bar{\rho}_j \neq e^8(\bar{\rho}) \) for any \( \bar{\rho} \in A_{j-1} \).

In case of (i) We define

\[
A_j = A_{j-1} - \{\bar{\rho}\}, \quad B_j = B_{j-1} \cup \{\bar{\rho}_j\}, \quad C_j = C_{j-1} \cup \{\bar{\rho}\}
\]
and in case of (ii) 
\[ A_j = A_{j-1} \cup \{ \tilde{\rho}_j \}, \quad B_j = B_{j-1}, \quad C_j = C_{j-1}. \]

We now establish an analogue of Lemma 9 of [4].

**Lemma 10.** There is at most one conjugate class of G slice types \( \tilde{\rho} \in A_{j-1} \) such that \( e^g(\tilde{\rho}) = \tilde{\rho}_j \).

**Proof.** The proof of this lemma is given by induction. Clearly the lemma holds for \( j = 1 \). Let it be true for all \( i < j \).

Let \( \tilde{\rho}_j = e^g(\tilde{\rho}_m) \) and take \( \tilde{\rho}_j \in S(H) \) and \( \tilde{\rho}_m \in S(K) \) with \( K \subset H \).

We claim that \( K \not\subset P(H) \). If \( K \subset P(H) \) then we choose \( J \) to be the least of all predecessors of \( H \). We get \( K \subset J \) and

\[ \tilde{\rho}_i = e^g_{K,J}(\tilde{\rho}_m) < \tilde{\rho}_j = e^g_{K,H}(\tilde{\rho}_m). \]

By the induction hypothesis there exists at most one such \( \tilde{\rho}_m \) such that \( \tilde{\rho}_i = e^g_{K,J}(\tilde{\rho}_m) \). Consequently neither \( \tilde{\rho}_m \) nor \( \tilde{\rho}_i \) belongs to \( A_{j-1} \). So

\[ K \not\subset P(H) \quad \text{and} \quad \rho_m \in \bigcup_{K \subset P(H)} S(K). \]

By Lemma 8, this implies

\[ \tilde{\rho}_j \in \text{image } E^g = \text{image } \bar{E}^g. \]

If now

\[ \rho_m \in \text{image } \left( E^g : \bigcup_{L \subset K} S(L) \to S(K) \right) \]

then \( \tilde{\rho}_m = e^g(\tilde{\rho}') \) for \( \tilde{\rho}' \in S(L), \ L \subset K \) and \( L \not\subset P(K) \). From the construction of the families \( A_j \) it follows that \( \tilde{\rho}_m \notin A_{j-1} \). So

\[ \tilde{\rho}_m \in S(K) = S(K) - \text{image } \left( E^g : \bigcup_{L \subset K} S(L) \to S(K) \right). \]

By Lemma 9, \( \bar{E}^g \) is injective and this establishes our lemma.

The next theorem further characterises the families \( A_j \).

**Lemma 11.** If \( N \) is sufficiently large compared to \( n \) then \( A_N \) consists of conjugate classes of G slice types of dimension greater than \( n \).

**Proof.** Let \( F_i \) be the family which contains all conjugate G slice types of dimension \( \leq n \) and

\[ A_i = \{ \tilde{\rho}_{i_1}, \tilde{\rho}_{i_2}, \ldots, \tilde{\rho}_{i_k} \} \]
with \( \bar{\rho}_t = \{K_t; U_t\}^g, 1 \leq t \leq k \). Then \( K_t \cap G_2 \neq G_2 \) because \( K_t \cap G_2 = G_2 \Rightarrow \bar{\rho}_t = e^g(\bar{\rho}') \) for some \( \bar{\rho}' \). We take
\[
\rho_j = e^g(\rho_i)
\]

If \( N \geq \max\{j_1, \ldots, j_k\} \) then clearly \( A_N \) does not contain any conjugate class of \( G \) slice types of dimension \( \leq n \).

The next theorem reveals the necessity of ordering the conjugate classes of \( G \) slice types.

**Theorem 12.** If \([H; U]\) is a \( G \) slice type and \( \bar{\rho} \in A_j \) is a conjugate class of \( G \) slice types of an orbit of a point of \( G \times_H U \), then either \( \bar{\rho} = [H; U]^g \) or \([H; U]^g \notin \bar{F}_j \).

**Proof.** Let \( \bar{\rho} \neq [H; U]^g \). Then \( \bar{\rho} \) is not the conjugate class of \( G \) slice types of the orbit of \([e,0] \in G \times_H U \). So \( \bar{\rho} \) is a conjugate class of \( G \) slice types of the orbit of a point \([e,u] \in G \times_H U, 0 \neq u \in U \). The isotropy subgroup of \([e,u] \) is a proper subgroup \( K \) of \( H \). We can write \( \bar{\rho} = [K; I^*U]^g \) where \( I^*U \) is the nontrivial part of \( i^*U, i: K \hookrightarrow H \) being the natural inclusion. Clearly \( \dim I^*U \leq \dim i^*U = \dim U \). We now discuss the two possible cases separately.

**Case I.** \( K \subset_2 H \) i.e. \( H = (x) \times K \).

We have
\[
e_{K,H}(\bar{\rho}) = [H; V(K) \oplus p^*I^*U]^g
\]
where \( p: H \to K \) is the distinguished projection.

Since \( K \) fixes \( u \in U \), \( K \) has trivial action on the one dimensional subspace \( L(u) \) spanned by \( u \). Also \( H \) has nontrivial action on \( L(u) \). So \( (x) \) acts on \( L(u) \) nontrivially and we get \( V(K) = L(u) \subset U \). If
\[
dim(V(K) \oplus p^*I^*U) < \dim U
\]
then
\[\bar{\rho} < \bar{\rho}_k = e_{K,H}^g(\bar{\rho}) \leq [H; U]^g = \bar{\rho}_i.\]

If
\[
dim(V(K) \oplus p^*I^*U) = \dim U
\]
then \( \dim I^*U \) is just one less than \( \dim U \) and by writing \( U = V(K) \oplus U' \) we get \( I^*U = i^*U' \). So \( p^*I^*U = p^*i^*U' \leq U' \) by the ordering of irreducible \( H \)-modules and its extension by lexicography i.e. \( V(K) \oplus p^*I^*U \leq V(K) \oplus U' = U \). Again we have
\[\bar{\rho} < \bar{\rho}_k = e_{K,H}^g(\bar{\rho}) \leq [H; U]^g = \rho_i.\]
Case II. Let $K \not\subset 2H$ i.e. $K < H$ but $H \neq (x) \times K$ for any $x \in G_2$.
If $K_2 = G_2$ then the class $\bar{\rho}$ is the $e^g$-image of some conjugate class of $G$ slice types occurring earlier according to the order so constructed. But this means $\bar{\rho} \not\in A_j$—a contradiction. So $K_2 \subseteq G_2$ and there exists an element $x \in G_2$ such that $(x) \times K$ can be formed. Since $K$ is a proper subgroup of $H$, $|(x) \times K| \leq |H|$. If $|(x) \times K| < |H|$ then by (i) of Rule A

$$\bar{\rho} < \bar{\rho}_k < \bar{\rho}_l.$$ 

If $|(x) \times K| = |H|$ then $|H: K| = \text{index of } K \text{ in } H = 2$. Since $K \not\subset 2H$, $x \not\in H$. Also there does not exist $y \in G_2$ such that $y \in H$ but $y \notin K$.

Hence $K_2 = H_2$ and $|(x) \times K_2| > |H_2|$. By (ii) of Rule A, $(x) \times K < H$ and

$$\bar{\rho} < \bar{\rho}_k < \bar{\rho}_l.$$ 

Now

$$\bar{\rho}_i = [H; U]^g \in \bar{F}_j \Rightarrow \bar{\rho}_i < \bar{\rho}_j$$

$$\Rightarrow \bar{\rho} < e^g(\bar{\rho}) = \bar{\rho}_k < \bar{\rho}_l < \bar{\rho}_j$$

$$\Rightarrow \bar{\rho} \in A_{k-1} \text{ and } \bar{\rho} \notin A_k \quad \text{(Lemma 10)}$$

$$\Rightarrow \bar{\rho} \notin A_j$$—a contradiction.

A consequence of this theorem is:

**Corollary 13.** The union of all conjugate classes of $G$ slice types of $B_j$ and $C_j$ is a family.

**Proof.** Let $[H; U]^g \in B_j \cup C_j \subseteq F_j$ and $\bar{\rho}$ is a conjugate class of $G$-slice types of an orbit of a point of $G \times_H U$. Clearly $\bar{\rho} \subset F_j$. If $\bar{\rho} \notin B_j \cup C_j$ then $\bar{\rho} \notin A_j$ and this contradicts Theorem 12.

6. **Proof of the main theorem.** We denote the elements of $C_j$ by $\vec{\sigma}_0$, $\vec{\sigma}_2, \ldots, \vec{\sigma}_{2k}$ where $k = |C_j|$ and $\vec{\sigma}_{2t} \leq \vec{\sigma}_{2m}$ if and only if $t \leq m$. We have $B_j = \{ e^g(\vec{\sigma}_i) \mid 0 \leq i \leq k \}$ and write $e^g(\vec{\sigma}_i) = \vec{\sigma}_{2i+1}$.

By Corollary 13, $\bar{F}_k = \bigcup_{i=0}^k \bar{\sigma}_i$ is a family when $k$ is odd. When $k$ is even $\bar{F}_k$ is again a family because the $G$ slice types of $\vec{\sigma}_k$ are 'maximal' in $\bar{F}_k$. By Lemma 11 we see that $\bar{F}(\hat{G})$ satisfies all the conditions of Lemma 4 and so

$$N^G_\bullet [\bar{F}(\hat{G})] = 0.$$ 

An alternative proof of Theorem 1 can be given by generalising Theorem 4.5.11 of [3].
THEOREM 14. There is an isomorphism
\[ \bigoplus v_i : N_n^G[F_j] \to \bigoplus \left\{ N_n^G[\bar{\rho}_i] \right\}, \]

Proof. We prove the result by induction. Clearly the result is true for \( j = 0 \). Now suppose it is true for \( j - 1 \) i.e.
\[ \bigoplus v_i : N_n^G[F_{j-1}] \to \bigoplus \left\{ N_n^G[\bar{\rho}_i] \right\}. \]

From the long exact sequence of Proposition 2 we have the composite
\[ v_i \partial_j : N_n^G[\bar{\rho}_j] \to N_n^G[\bar{\rho}_{j-1}]. \]

If \( v_i \partial_j \neq 0 \) then \( \bar{\rho}_j \) is a conjugate class of \( G \) slice types of \( G \times_H V \) where \([H, V] = \{0\}\) and by Theorem 12 \( \rho_j \notin A_j \).

Now for the class \( \bar{\rho}_j \) there exists almost one conjugate class of \( G \) slice types \( \bar{\rho}_i \) such that \( e^\mathcal{S}(\bar{\rho}_i) = \rho_j \). If there does not exist any such \( \bar{\rho}_i \in A_{j-1} \) then for any \( \bar{\rho}_i \in A_{j-1} \) both \( \rho_j \) and \( \bar{\rho}_i \) belong to \( A_j \) and \( v_i \partial_j = 0 \) for every \( \bar{\rho}_i \in A_{j-1} \). Thus \( \left( \bigoplus_{\bar{\rho}_i \in A_{j-1}} v_i \right) \partial_j = 0 \) and consequently \( \partial_j = 0 \). We have a short exact sequence
\[ 0 \to N_n^G[F_{j-1}] \to N_n^G[F_j] \xrightarrow{\rho_j} N_n^G[\bar{\rho}_j] \to 0. \]

If again for \( \bar{\rho}_j \) we have \( \bar{\rho}_i \in A_{j-1} \) s.t. \( \rho_j = e^\mathcal{S}(\bar{\rho}_i) \) then neither \( \rho_j \) nor \( \bar{\rho}_i \) belong to \( A_j \) and by Lemma 3
\[ v_i \partial_j : N_n^G[\bar{\rho}_j] \to N_n^G[\bar{\rho}_{j-1}] \]
is an isomorphism and we have again a short exact sequence
\[ 0 \to N_n^G[\bar{\rho}_j] \to N_n^G[F_{j-1}] \to N_n^G[F_j] \to 0. \]

Both the short exact sequences split as the modules involved are vector spaces over \( \mathbb{Z}_2 \). So
\[ N_n^G[F_j] \cong \bigoplus_{\bar{\rho}_i \in A_j} N_n^G[\bar{\rho}_i]. \]

COROLLARY 15. \( N_n^G(\hat{F}(\hat{G})) = 0. \)

Proof. Corresponding to the positive integer \( n \) we take all conjugate classes of \( G \) slice types of dimension \( \leq n + 1 \). If \( F_N \) be the union of all these classes then
\[ N_n^G[\hat{F}(\hat{G})] = N_n^G[F_N] = \bigoplus_{\rho \in A_n} N_n^G[\bar{\rho}_i]. \]

If now \( N \) is made sufficiently large compared to \( n \) then by Lemma 11 \( A_N \) consists of all conjugate classes of \( G \) slice types of dimension \( > n \) and hence the isomorphism \( \bigoplus v_i \) is zero.
COROLLARY 16.

\[ N^G_\bullet[F'(\hat{G})] = N^G_{\bullet+1}[\tilde{F}'(\hat{G}), F'(\hat{G})]. \]

This follows from the main theorem and the long exact sequence for the pair \( F'(\hat{G}) \subset \tilde{F}(\hat{G}) \) of families of \( G \)-slice types.

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