FINITE GROUP ACTION AND VANISHING OF $N^G_*$

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Let $G$ be a finite group (not necessarily abelian). The object of this paper is to describe a $G$-bordism theory which vanishes. We construct a family $F$ of $G$ slice types, such that the $N^G_\ast$-module $N^G_\ast[F]$ is zero. Kosniowski has proved a similar result earlier for a finite abelian group. The present work is a generalisation of his result by using basically the same technique. A recent result of Khare is obtained as a corollary to the vanishing of $N^G_\ast[F]$.

1. Preliminaries and statement of the main theorem. Let $G$ be a finite group with centre $C(G)$ and $G_2$ be the subgroup generated by the elements of order 2 in $C(G)$. We also assume that $G_2$ is nontrivial. By a $G$-manifold $M$ we mean throughout a closed differentiable manifold on which $G$ acts smoothly. $G_x$ denotes the isotropy subgroup at $x \in M$. For every $x \in M$, there exists a $G_x$-module $\overline{V}_x$ which is equivariantly diffeomorphic to a $G_x$-invariant neighbourhood of $x$. $\overline{V}_x$ has a submodule $V'_x$ in which $G_x$ acts trivially and a complementary submodule $V_x$ in which no nonzero element is fixed by all of $G_x$. By the $G$-slice type of $x$ we mean the pair $[G_x; V_x]$. By a $G$-slice type we mean a pair $[H; U]$ where $U$ is a $H$-module in which no nonzero element is fixed by all of $H$ (equivalently $U$ contains no trivial $H$-submodule). A family $F$ of $G$-slice types is a collection of $G$-slice types such that if $[H; U] \in F$ then for every $x \in G \times_H U$ the $G$-slice type $[G_x; V_x] \in F$. A $G$-manifold is said to be of type $F$ if for all $x \in M$, $[G_x, V_x] \in F$. Bordism relation is defined in the usual way. Two $n$-dimensional closed $G$-manifolds $M_1$, $M_2$ of type $F$ are said to be $F$-bordant if there exists an $(n + 1)$-dimensional compact differentiable $G$-manifold $W$ of type $(F, F)$ such that the disjoint union of $M_1$ and $M_2$ is $G$ equivariantly diffeomorphic to $\partial W$. It is clearly an equivalence relation on the set of $G$-manifolds of type $F$ and gives rise to a bordism theory $N^G_\ast[F]$. We note that $N^G_\ast[F]$ is a graded $N_\ast$-module, $N_\ast$ being the unoriented bordism ring.

Kosniowski has described a family $\tilde{F}(\hat{G})$ in [4] for an abelian group $G$ such that $N^G_\ast[\tilde{F}(\hat{G})] = \hat{0}$, $\hat{G}$ being a subgroup of $G$ containing $G_2$. As a consequence he proved that if $M$ is a $G$-manifold ($G$ abelian) in which $G_2$ acts without fixed points then $M$ is a $G$-boundary—a result obtained earlier by Khare using a different technique [1]. The main theorem of this
paper is a generalisation of Kosniowski’s theorem in [4] for an arbitrary finite group. Once again another result of Khare [2] is obtained as a corollary of this theorem.

The subgroup $G_2$ consisting of the identity and elements of order two in the centre of $G$ is isomorphic to $\mathbb{Z}_2^k$ for some $k > 0$. Kosniowski has studied $\mathbb{Z}_2^k$-bordism in [3] and the techniques used here are generalized from his techniques. We choose once for all a basis $g_1, g_2, \ldots, g_k$ of $G_2$ and order the elements by

$$g_1 < g_2 < \cdots < g_k < g_1g_2 < \cdots < g_1g_k < \cdots < g_1g_2 \cdots g_k.$$ 

Now let $[G_x; V_x]$ be the $G$-slice type of a point $x$ in a $G$-manifold $M$ and $G(x)$ be the orbit of $x$. Then $G(x)$ is a closed and compact submanifold of $M$. Consider the normal bundle $\nu(i)$ of the canonical embedding of $G(x)$ in $M$. This is a $G$-vector bundle and its disc bundle is a closed $G$-invariant tubular neighborhood of $G(x)$. Further $G$ acts as a group of bundle maps on the normal bundle and the fibre over $x$ is $G_x$-invariant and contains no $G_x$-trivial subspace. It is precisely $V_x$ the $G_x$-module present in the $G$-slice type $[G_x; V_x]$ of $x$. Let $g \ast$ be the map on the total space $E(\nu(i))$ induced by the action of $g$ on the base space $G(x)$. The $G$-slice type of $gx \in G(x)$ is $[gG_xg^{-1}; g \ast V_x]$. The underlying vector space of $V_x$ and $g \ast V_x$ are same and the action of $ghg^{-1}, h \in G_x$ on $v \in g \ast V_x$ is same as the action of $h$ on $v \in V_x$. Again if $F$ be a family of $G$-slice types and $[H; V] \in F$ then from the definition of family the $G$-slice type $[G_x; V_x]$ of every point $x \in G \times_H V$ belongs to $F$. Now the $G$-slice type of $[e,0] \in G \times_H V$ is $[H; V]$ and the $G$ slice type of $[g,0] \in G \times_H V$ is $[gHg^{-1}; g \ast V]$. The $G$-slice type $[H; V]$ will be denoted by $\rho$ and the collection

$$\{ [gHg^{-1}; g \ast V] | g \in G \}$$

termed as a conjugate class of $G$-slice types will be denoted by $\bar{\rho}$ or $[H; V]^g$.

Suppose that $K$ is a subgroup of $H$. We write $K \subset_2 H$ if $H = (x) \times K$ where $x \in G_2$. Quite a number of elements of $G_2$ may yield $H$ when a direct product of above type is formed. We take the minimal element $x$ according to the total order fixed at the beginning of this article. We now have a homomorphism

$$p = p_{H,K} : H \to K.$$ 

which is the projection onto the second factor. This is termed as the distinguished projection. It enables us to obtain an $H$-module $p^*U$ from a $K$-module $U$. The modules $p^*U$ and $U$ have the same underlying vector
space and $H$ acts on $p^*U$ via the map $p$. Corresponding to a $G$ slice type $[K; U]$ such that $K \subset H$ we have an extension function $e = e_{K, H}$ given by

$$e_{K, H}[K; U] = [H; V(K) \oplus p^*U]$$

where $V(K)$ is one dimensional real representation of $H$ in which $h \in H$ acts by multiplication with 1 if $h \in K$ and multiplication with $-1$ if $h \notin K$. Since $gHg^{-1} = (x) \times gKg^{-1}$ when $H = (x) \times K$, we have

$$e[gKg^{-1}; g_*U] = [gHg^{-1}; V(gKg^{-1}) \oplus p^*(g_*U)]$$

Thus $e_{K, H}$ induces a map $e^g = e_{K, H}^g$ on the collection of conjugate classes of $G$ slice types $[K; U]^g$ and

$$e_{K, H}^g[K; U]^g = [H; V(K) \oplus p^*U]^g.$$ 

Corresponding to a subgroup $\hat{G}$ of $G$ containing $G_2$ we have three families of $G$ slice types.

$$F(\hat{G}) = \{[gHg^{-1}; g_*V]|[H, V] \text{ is a } G \text{ slice type with } H \text{ contained in } \hat{G}, g \in G\}$$

$$F'(\hat{G}) = \{[K; U] \in F(\hat{G})|K \cap G_2 \neq G_2\}$$

and

$$\tilde{F}(\hat{G}) = F'(\hat{G}) \cup \{e_{K, H}[K; U]|[K; U] \in F'(\hat{G})\}$$

and $K \subset H$ with $H \cap G_2 = G_2$.

That each collection is a family is clear. Now we are in a position to state the main theorem of this paper.

**Theorem 1.** If $G$ be a finite group and $\hat{G}$ be a subgroup of $G$ which contains $G_2$ then $N_*^G[\tilde{F}(\hat{G})] = 0$.

**Corollary (Khare [2]).** Suppose that $G$ is a finite group. If $M$ is a $G$-manifold on which $G_2$ acts without fixed points then $M$ is a $G$-boundary.

The corollary follows because if $G_2$ acts without fixed points then an isotropy subgroup $H$ of a point in $M$ satisfies the condition $H \cap G_2 \neq G_2$ so that $M$ is of the type $F'(G)$ and consequently of the type $\tilde{F}(G)$.

The proof of the theorem will be given in §7. In §2, §3, §4 and §5, we develop the necessary tools and results.
2. Vector bundles of type \( \bar{\rho} \). Let \( F' \subseteq F \) be two families of \( G \) slice types with \( F = F' \cup \bar{\rho} \) where \( \bar{\rho} \) is a class of conjugate \( G \) slice types. By a \( G \)-vector bundle of type \( \bar{\rho} \) we mean a \( G \)-vector bundle \( \xi: E(\xi) \rightarrow B(\xi) \) where the set of points of \( E(\xi) \) having \( G \) slice type in \( \bar{\rho} \) is precisely the zero section. We have the bundle bordism groups \( N^G_n[\bar{\rho}] \) obtained by defining a bordism relation on the set of all \( G \) vector bundles of type \( \bar{\rho} \) having total dimension \( n \).

Let \( M^n \) be a \( G \)-manifold of type \( F \) and \( F_\bar{\rho} \) be the set of all points in \( M^n \) with slice type in \( \bar{\rho} \). Then the normal bundle over \( F_\bar{\rho} \) is a \( G \) vector bundle of type \( \bar{\rho} \). This assignment of the normal bundle over \( F_\bar{\rho} \) in \( M^n \) leads to a \( N_* \)-homomorphism

\[
\nu_\bar{\rho}: N^G_n[F] \rightarrow N^G_n[\bar{\rho}].
\]

We have the following proposition and lemmas involving the bundle bordism groups.

**Proposition 2.** There exists a long exact sequence

\[
\cdots \rightarrow N^G_n[F'] \rightarrow N^G_n[F] \rightarrow N^G_n[\bar{\rho}] \rightarrow N^G_{n-1}[F'] \rightarrow \cdots
\]

where \( F' \subseteq F \) are families of \( G \) slice types such that \( F - F' = \bar{\rho} \).

For proof we refer to 1.4.2 of [3].

**Lemma 3.** Suppose that \( K \subset H \) and \( \bar{\rho} = [H; V]^g, \bar{\rho}' = [K; U]^g \) be two classes of conjugate \( G \) slice types such that \( e^g(\bar{\rho}') = \bar{\rho} \). Then there exists an \( N_* \)-isomorphism

\[
N^G_n[\bar{\rho}] \rightarrow N^G_{n-1}[\bar{\rho}']
\]

given by \([\xi] \rightarrow [\nu_\bar{\rho} S(\xi)]\), where \( S(\xi) \) is the sphere bundle of \( \xi \).

The proof of this lemma is similar to that given for Lemma 4.5.8 of [3].

**Lemma 4.** Let \( F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \) be a sequence of families of \( G \)-slice types with

(i) \( F_0 = \bar{\rho}_0 = \{[e; R^0]\} \)
(ii) \( F_i = F_{i-1} \cup \bar{\rho}_i \) for all \( i \geq 1 \)
(iii) \( \bigcup_{i \geq 0} F_i = F \)

and

(iv) \( e^g(\bar{\rho}_{2i}) = \bar{\rho}_{2i+1} \) for all \( i \geq 0 \). Then \( N_*^G[F] = 0 \).

**Proof.** Using Proposition 2 and Lemma 3 we get

\[
N_*^G[F_{2i}] = N_*^G[\bar{\rho}_{2i}]
\]

and \( N_*^G[F_{2i+1}] = 0 \).
Taking direct limit

\[ N_*^G[F] = \lim_{\rightarrow} N_*^G[F_i] = 0. \]

The rest of the paper is aimed to show that the family \( \tilde{F}(\hat{G}) \) satisfies the conditions laid down in Lemma 4. The \( G \) slice types of \( \tilde{F}(\hat{G}) \) are to be ordered suitably now in order to get the families \( F_0 \subset F_1 \subset \cdots \).

3. Ordering the conjugate classes of \( G \) slice types. We define three distinct relations \(<\) on the collection \( \tilde{A} \) of all subgroups conjugates to subgroups of \( \hat{G} \), on the collection of all \( H \)-modules, \( H \in \tilde{A} \) and finally on the collection of all conjugate classes of \( G \) slice types of the family \( \tilde{F}(\hat{G}) \) and extend each of these relations into a total order on the respective collection. We note that the elements of \( G_2 \) are totally ordered by

\[ g_1 < g_2 < \cdots < g_k < g_1g_2 < \cdots < g_1g_k < \cdots < g_1g_2 \cdots g_k \]

and a subgroup \( H_2 \) of \( G_2 \) has a distinguished base \( h_1 < h_2 < \cdots < h_m \) such that \( h_1 \) (≠ identity) is the least element in \( H \) and for \( i > 1 \), \( h_i \) is the least element in \( H \) which is not present in \( (h_1, h_2, \ldots, h_{i-1}) \), the subgroup generated by \( h_1, h_2, \ldots, h_{i-1} \). The subgroups of \( G_2 \) are now totally ordered first by the order of the subgroup and then lexicographically on the distinguished base:

\[ (e) < (g_1) < (g_2) < \cdots < (g_1g_2 \cdots g_k) < (g_1, g_2) < \cdots. \]

Rule A. Let \( H \) and \( K \) belong to \( \tilde{A} \). We define \( \leq \) by:

(i) if \( |H| \leq |K| \) Then \( H \leq K \),
(ii) if \( |H| = |K| \) and \( |K_2| \leq |H_2| \). Then \( H \leq K \) where \( K_2 = K \cap G_2 \) and \( H_2 = H \cap G_2 \),
(iii) if \( |H| = |K|, |K_2| = |H_2| \) but \( H_2 \leq K_2 \) then \( H \leq K \) and
(iv) if \( |H| = |K|, H_2 = K_2 \) then we order them arbitrarily so as to make the relation \( \leq \) a total ordering on \( \tilde{A} \).

Next a relation \( \preceq \) is introduced on the collection of all nontrivial irreducible \( H \)-modules \( H \in \tilde{A} \). We write \( U \preceq V \) if \( U = V \) or else there exists \( K \subset H \) such that \( U = p^*i^*V \) where \( i: K \to H \) is the natural inclusion and \( p: H \to K \) is the distinguished projection. We now have the following lemma whose proof is similar to Lemma 8 of [4].

**Lemma 5.** The relation \( \preceq \) is a partial order on the collection of all nontrivial irreducible \( H \)-modules.

We now choose a total ordering on the set of all nontrivial irreducible \( H \)-modules having the same dimension compatible with the partial ordering introduced. The total ordering is now extended to all irreducible
H-modules by writing $U \leq V$ if and only if $\dim U \leq \dim V$. Since any
$H$-module can be expressed uniquely as the sum of irreducible $H$-mod-
ules, we can extend this total ordering on all $H$-modules by lexicography.
The following rule expresses the whole rule coincisely.

Rule B. Let $U$ and $V$ be two $H$-modules.
(i) If $\dim U \leq \dim V$ then $U \leq V$
(ii) If $\dim U = \dim V$ and $V$ follows $U$ lexicographically then $U \leq V$.

Finally Rule C given as below defines the order $\leq$ on the collection
of all classes of conjugate $G$ slice types of the family $\tilde{F}(\tilde{G})$.

Rule C. Let $\tilde{\rho} = [H; U]^g$, $\tilde{\rho}' = [K; V]^g$ be two classes of conjugate $G$
slice types of $\tilde{F}(\tilde{G})$
(i) If $\dim U \leq \dim V$ then $\tilde{\rho} \leq \tilde{\rho}'$.
(ii) If $\dim U = \dim V$ and $H \leq K$ then $\tilde{\rho} \leq \tilde{\rho}'$.
(iii) If $\dim U = \dim V$, $H = K$ and $U \leq V$ then $\tilde{\rho} \leq \tilde{\rho}'$.

We now proceed to prove some algebraic results relating to the
extension map $e$.

4. Algebraic lemmas and extension map. The following lemmas are
generalisations of propositions of $Z_2^k$ bordism given in 4.5 of [3]

Lemma 6. Let $(e) \subset K \subset_2 H \subset G$ and

$g_1 < g_2 < \cdots < g_k,$
$h_1 < h_2 < \cdots < h_m,$

and

$k_1 < k_2 < \cdots < k_{m-1}$

be the distinguished bases of $G_2$, $H_2$ and $K_2$ respectively and $r$ be the
largest integer for which $k_i = h_i$ for all $i < r$. Then $K$ is not contained in a
predecessor of $H$ if and only if $h_i = g_i$ for all $i < r$. (By a predecessor of $H$
we mean a subgroup $H' = H$ such that $H'_2 < H_2$.)

Proof. We have $(e) \subset K_2 \subset_2 H_2 \subset G_2$. If $K \subset_2 H'$, a predecessor of
$H$ then by definition $K_2 \subset_2 H'_2$, a predecessor of $H_2$. Further if $K_2 \subset_2 H'_2$, a
predecessor of $H_2$, then $H'_2 = (x) \times K_2$ ($x$ being chosen minimally)
and $K \subset_2 (x) \times K$, a predecessor of $H$.

Thus $K$ is not contained in a predecessor of $H$ if and only if $K_2$ is
not contained in a predecessor of $H_2$. The latter statement implies and is
implied by $h_i = g_i$ for all $i < r$ and this follows from 4.5.12 of [3].
**Lemma 7.** Let $K \subset_2 H,$ $K' \subset_2 H$ with $K$ and $K'$ not contained in a predecessor of $H.$ If

$$H = (x) \times K = (x') \times K'$$

where $x$ and $x'$ are chosen minimally, $x \in K',$ $x' \in K$ and $K$ precedes $K'$ then $K \cap K'$ is not contained in a predecessor of $K.$

**Proof.** We have

$$H_2 = (x) \times K_2 = (x') \times K'_2,$$

and $K_2$ precedes $K'_2.$ By the Proposition 4.5.13 of [3], $K_2 \cap K'_2$ is not contained in a predecessor of $K_2$ and this in turn implies that $K \cap K'$ is not contained in a predecessor of $K.$

In order to proceed further we need the following constructions and lemmas.

$S(H) =$ collection of all conjugate classes of $G$ slice types with isotropy subgroup $H.$ For any $K \subset_2 H$ we have the extension function

$$e^g = e_{K,H}^g: S(K) \to S(H)$$

and consequently a function

$$E^g: \bigcup_{K \subset_2 H, K \subset_2 P(H)} S(K) \to S(H).$$

where by $P(H)$ one means a predecessor of $H.$ Let

$$\tilde{S}(K) = S(K) - \text{image} \left\{ E^g: \bigcup_{L \subset_2 K, L \subset_2 P(K)} S(L) \to S(K) \right\}.$$  

The function

$$\tilde{E}^g: \bigcup_{K \subset_2 H, K \subset_2 P(H)} \tilde{S}(K) \to S(H)$$

is the restriction of $E^g.$

**Lemma 8.** Image $\tilde{E}^g = \text{image } E^g.$

**Proof.** Clearly image $\tilde{E}^g \subseteq \text{image } E^g.$

Let $\tilde{p} \in \text{im } E^g$ i.e. $\tilde{p} = e^g(\tilde{p}')$ for some $\tilde{p}' \in S(K)$ where $K \subset_2 H$ and $K \not\subset_2 P(H).$
If \( \bar{\rho}' \notin \bar{S}(K) \) then \( \bar{\rho}' = e^g(\bar{\rho}'') \) for some \( \bar{\rho}'' \in S(L) \) where \( L \subset 2 K \) and \( L \not\subset 2 P(K) \). By Lemma 6 we have the following distinguished bases of \( H_2, K_2 \) and \( L_2 \):

- \( L_2: g_1 < g_2 < \cdots < g_{s-1} < l_s < \cdots \)
- \( K_2: g_1 < g_2 < \cdots < g_{s-1} < g_s < \cdots < g_{r-1} < k_r < \cdots \)
- \( H_2: g_1 < g_2 < \cdots < g_{r-1} < g_r < h_{r+1} < \cdots \).

We note that \( l_s \neq g_s \) and \( k_r \neq g_r \). So

\[
H = (g_r) \times K \quad \text{and} \quad K = (g_s) \times L.
\]

Writing \( \bar{\rho}'' = [L; U]^g \) we get

\[
\bar{\rho}' = e^g(\bar{\rho}'') = [K; V(L) \oplus q^*U]^g,
\]

and

\[
\bar{\rho} = e^g(\bar{\rho}') = [H; V(K) \oplus p^*(V(L) \oplus q^*U)]^g
\]

\[
= [H; V(K) \oplus V((g_s) \times L) \oplus p^*q^*U]^g
\]

\( q: K \to L \) and \( p: H \to K \) are the distinguished projections.

Taking \( K' = (g_r) \times L \) we note that \( K' \subset 2 H \) and \( K \) precedes \( K' \). Moreover \( K' \not\subset 2 P(H) \). Extending \( \bar{\rho}'' \) through \( K' \) we get

\[
\bar{\rho}''' = e_{K',H}^g(\bar{\rho}'') = [K'; V(L) \oplus q^*U]^g \in S(K')
\]

and

\[
e_{K',H}^g(\bar{\rho}'''') = [H; V(K') \oplus V((g_s) \times L) \oplus p^*q^*U]^g
\]

where \( p': H \to K' \) and \( q': K' \to L \) are the distinguished projections. Since \( qp = q'p' \), we have

\[
e_{K',H}^g(\bar{\rho}''') = [H; V(K') \oplus V(K) \oplus p^*q^*U]^g = \bar{\rho}.
\]

If \( \bar{\rho}''' \in \bar{S}(K') \) then \( \bar{\rho} \in \text{image} \bar{E}^g \). If not then by arguing as before we get a conjugate class of \( G \) slice type \( \bar{\rho}^{(v)} \in S(K'') \) such that \( \bar{\rho} = e^g(\bar{\rho}^{(v)}) \) where \( K'' \subset 2 H \) and \( K < K' < K'' \not\subset 2 P(H) \).

Continuing this way we exhaust all the finite number of possibilities and find some \( \bar{\rho}^{(2n+1)} \in \bar{S}(K^{(n)}) \) such that \( K^{(n)} \subset 2 H, K^{(n)} \not\subset 2 P(H) \) and \( \bar{\rho} = e^g(\bar{\rho}^{(2n+1)}) \) i.e. \( \bar{\rho} \in \text{image} \bar{E}^g \).

**Lemma 9.** The function

\[
\bar{E}^g: \bigcup_{K \subset 2 H} S(K) \to S(H)
\]

is injective.
Proof. Suppose that $$\bar{p} = [K; U]^g$$, $$\bar{p}' = [K'; U']^g$$
where $K$ and $K' \subset H$, $K$ and $K' \not\subset \mathcal{P}(H)$, $K$ precedes $K'$ and
$$e^g(\bar{p}) = e^g(\bar{p}') = [H; V]^g.$$ 
From Lemma 6 we get
$$H = (g_r) \times K = (g_s) \times K'$$
where $g_r$ and $g_s$ are the minimal possible choices and $s < r$. We have
$$[H; V(K) \oplus p^*U]^g = [H; V]^g = [H; V(K') \oplus p'^*U']^g$$
where $p: H \to K$, $p': H \to K'$ are the distinguished projections. Writing $U = \sum n_i U_i$ and $U' = \sum n'_j U'_j$ where $U_i$ and $U'_j$ are nontrivial irreducible $K$ and $K'$ modules respectively we get
$$V(K) \oplus \sum n_i p^*U_i = V(K') \oplus \sum n'_j p'^*U'_j.$$ 
Since $K \neq K'$, $V(K) = p'^*U'$ for some $t$ and $n'_t = 1$. The underlying vector space of these modules is $R$.
We write $g_s = g^a_sk$, $a_i \in \{0, 1\}$ and $k \in K$ and consider its action on $x \in V(K) = p'^*U'_t$. We get $g_s x = x$ i.e. $(-1)^{a_i} x = x$ i.e. $a_1 = 0$. So $g_s \in K$. Similarly $g_r \in K'$. By Lemma 7, $L = K \cap K' \not\subset \mathcal{P}(K)$ and $K = (g_s) \times L$ ($L$ is the intersection of two normal subgroups of $H$). We have also the restriction function
$$r^g = r^g_{\mathcal{H}, K}: S(H) \to S(K)$$
such that $r^g[H; V]^g = [K; I^*V]^g$ where $I^*V$ is the nontrivial part of $i^*V$, $i: K \to H$ being the natural inclusion. Note that
$$r^g_{\mathcal{H}, K} e^g_{K,H} [K; U]^g = r^g_{\mathcal{H}, K}[H; V(K) \oplus p^*U]^g$$
$$= [K; I^*(V(K) \oplus p^*U)]^g$$
$$= [K; I^*p^*U]^g = [K; i^*p^*U]^g = [K; U]^g$$
i.e. $r^g_{\mathcal{H}, K} e^g_{K,H} = \text{identity}$.
Therefore
$$\bar{p} = [K, U]^g = r^g_{\mathcal{H}, K} e^g_{K,H} [K; U]^g = r^g_{\mathcal{H}, K} e^g_{K',H} [K'; U']^g$$
$$= r^g_{\mathcal{H}, K}[H; V(K') \oplus p'^*U']^g$$
$$= [K; V(K' \cap K) \oplus I^*p'^*U']^g$$
$$= [K; V(L) \oplus NTq^*j^*U']^g$$
where \( i: K \rightarrow H, \quad i': K' \rightarrow H, \quad j: L \rightarrow K, \quad j': L \rightarrow K' \) are the natural inclusions and \( p: H \rightarrow K, \quad p': H \rightarrow K', \quad q: K \rightarrow L, \quad q': K' \rightarrow L \) are the distinguished projections. We have \( p'i = j'q \) and \( NT \) stands for the nontrivial part. Also

\[
\rho^g_{K,L}(\bar{\rho}) = [L; NTj^*q^*U']^g = [L; NTj'^*U']^g
\]

(since \( qj = \text{id} \)). So

\[
e_{L,K}^g \rho^g_{K,L}(\bar{\rho}) = [K; V(L) \oplus NTq^*j'^*U']^g = \bar{\rho}.
\]

Thus \( \bar{\rho} = e(\rho'') \) for \( \rho'' = r_{K,L}(\bar{\rho}) \in S(L) \) and \( L \subset_2 K, \quad L \not\subset_2 P(K) \) i.e.

\[
\bar{\rho} \in \text{im} \left( E^g: \bigcup_{L \subset_2 K \atop L \not\subset_2 P(K)} S(L) \rightarrow S(K) \right)
\]

i.e. \( \bar{\rho} \not\in \bar{S}(K) \)—a contradiction.

With this we come to an end of this section.

5. Decomposition of the collection of conjugate classes of \( G \) slice types of a family. If we now define the dimension of a conjugate class of \( G \)-slice types as dimension of the module present therein then it is clear that there are only a finite number of conjugate classes of \( G \) slice types of a given dimension. The classes of the family \( \hat{F}(\hat{G}) \) are totally ordered by the Rule C and we index them by nonnegative integers as

\[
\bar{\rho}_0 < \bar{\rho}_1 < \bar{\rho}_2 <
\]

where \( \bar{\rho}_0 = \{[(e), R^0]\} \). We define \( F_j = \bigcup_{i \leq j} \bar{\rho}_i \). \( F_j \) is clearly a family of \( G \) slice types. Corresponding to the family \( F_j \) we form the collection \( \bar{F}_j = \{ \bar{\rho}_0, \bar{\rho}_1, \ldots, \bar{\rho}_j \} \) and define inductively three subcollections \( A_j, B_j \) and \( C_j \) of \( \bar{F}_j \) such that \( \bar{F}_j = A_j \cup B_j \cup C_j \). For \( j = 0 \), \( \bar{F}_j = \{ \bar{\rho}_0 \} \) and we set

\[
A_j = \{ \bar{\rho}_0 \}, \quad B_j = \emptyset, \quad C_j = \emptyset
\]

Let \( A_{j-1}, B_{j-1}, C_{j-1} \) be defined for some \( j \geq 1 \). We have

\[
\bar{F}_{j-1} = A_{j-1} \cup B_{j-1} \cup C_{j-1}
\]

and

\[
\bar{F}_j = \bar{F}_{j-1} \cup \{ \bar{\rho}_j \}.
\]

There are two possibilities:

(i) either \( \bar{\rho}_j = e^g(\bar{\rho}) \) for some \( \bar{\rho} \in A_{j-1} \) or

(ii) \( \bar{\rho}_j \neq e^g(\bar{\rho}) \) for any \( \bar{\rho} \in A_{j-1} \).

In case of (i) We define

\[
A_j = A_{j-1} - \{ \bar{\rho} \}, \quad B_j = B_{j-1} \cup \{ \bar{\rho}_j \}, \quad C_j = C_{j-1} \cup \{ \bar{\rho} \}
\]
and in case of (ii)

\[ A_j = A_{j-1} \cup \{ \bar{\rho}_j \}, \quad B_j = B_{j-1}, \quad C_j = C_{j-1}. \]

We now establish an analogue of Lemma 9 of [4].

**Lemma 10.** There is at most one conjugate class of \( G \) slice types \( \bar{\rho} \in A_{j-1} \) such that \( e^g(\bar{\rho}) = \bar{\rho}_j \).

**Proof.** The proof of this lemma is given by induction. Clearly the lemma holds for \( j = 1 \). Let it be true for all \( i < j \).

Let \( \bar{\rho}_j = e^g(\bar{\rho}_m) \) and take \( \bar{\rho}_j \in S(H) \) and \( \bar{\rho}_m \in S(K) \) with \( K \subseteq H \). We claim that \( K \not\subseteq P(H) \). If \( K \subseteq P(H) \) then we choose \( J \) to be the least of all predecessors of \( H \). We get \( K \subseteq J \) and

\[ \bar{\rho}_t = e_{k,j}^g(\bar{\rho}_m) < \bar{\rho}_j = e_{k,H}^g(\bar{\rho}_m). \]

By the induction hypothesis there exists at most one such \( \bar{\rho}_m \) such that \( \bar{\rho}_t = e_{k,j}^g(\bar{\rho}_m) \). Consequently neither \( \bar{\rho}_m \) nor \( \bar{\rho}_t \) belongs to \( A_{j-1} \). So

\[ K \not\subseteq P(H) \quad \text{and} \quad \rho_m \in \bigcup_{K \subset P(H)} S(K). \]

By Lemma 8, this implies

\[ \bar{\rho}_j \in \text{image } E^g = \text{image } \overline{E}^g. \]

If now

\[ \rho_m \in \text{image } \left( E^g : \bigcup_{L \subseteq K} S(L) \to S(K) \right) \]

then \( \bar{\rho}_m = e^g(\bar{\rho}') \) for \( \bar{\rho}' \in S(L), \ L \subseteq K \) and \( L \not\subseteq P(K) \). From the construction of the families \( A_j \) it follows that \( \bar{\rho}_m \not\in A_{j-1} \). So

\[ \bar{\rho}_m \in \overline{S}(K) = S(K) - \text{image } \left( E^g : \bigcup_{L \subseteq K} S(L) \to S(K) \right). \]

By Lemma 9, \( \overline{E}^g \) is injective and this establishes our lemma.

The next theorem further characterises the families \( A_j \).

**Lemma 11.** If \( N \) is sufficiently large compared to \( n \) then \( A_N \) consists of conjugate classes of \( G \) slice types of dimension greater than \( n \).

**Proof.** Let \( F_i \) be the family which contains all conjugate \( G \) slice types of dimension \( \leq n \) and

\[ A_i = \{ \bar{\rho}_{i_1}, \bar{\rho}_{i_2}, \ldots, \bar{\rho}_{i_k} \}. \]
with \( \overline{\rho}_i = \{ K_t; U_t \}^g, 1 \leq t \leq k \). Then \( K_t \cap G_2 \neq G_2 \) because \( K_t \cap G_2 = G_2 \Rightarrow \overline{\rho}_i = e^g(\overline{\rho}') \) for some \( \overline{\rho}' \). We take

\[
\rho_i = e^g(\rho_i)
\]

If \( N \geq \max\{ j_1, \ldots, j_k \} \) then clearly \( A_N \) does not contain any conjugate class of \( G \) slice types of dimension \( \leq n \).

The next theorem reveals the necessity of ordering the conjugate classes of \( G \) slice types.

**Theorem 12.** If \([H; U]\) is a \( G \) slice type and \( \overline{\rho} \in A_j \) is a conjugate class of \( G \) slice types of an orbit of a point of \( G \times_H U \), then either \( \overline{\rho} = [H; U]^g \) or \([H; U]^g \not\in \overline{F}_j \).

**Proof.** Let \( \overline{\rho} \neq [H; U]^g \). Then \( \overline{\rho} \) is not the conjugate class of \( G \) slice types of the orbit of \([e, 0] \in G \times_H U \). So \( \overline{\rho} \) is a conjugate class of \( G \) slice types of the orbit of a point \([e, u] \in G \times_H U, 0 \neq u \in U \). The isotropy subgroup of \([e, u]\) is a proper subgroup \( K \) of \( H \). We can write \( \overline{\rho} = [K; I^*U]^g \) where \( I^*U \) is the nontrivial part of \( i^*U \), \( i: K \rightarrow H \) being the natural inclusion. Clearly \( \dim I^*U \leq \dim i^*U = \dim U \). We now discuss the two possible cases separately.

**Case I.** \( K \subset H \) i.e. \( H = \langle x \rangle \times K \).

We have

\[
e_{K,H}(\overline{\rho}) = [H; V(K) \oplus p^*I^*U]^g
\]

where \( p: H \rightarrow K \) is the distinguished projection.

Since \( K \) fixes \( u \in U \), \( K \) has trivial action on the one dimensional subspace \( L(u) \) spanned by \( u \). Also \( H \) has nontrivial action on \( L(u) \). So \( \langle x \rangle \) acts on \( L(u) \) nontrivially and we get \( V(K) = L(u) \subset U \). If

\[
\dim(V(K) \oplus p^*I^*U) < \dim U
\]

then

\[
\overline{\rho} < \overline{\rho}_k = e_{K,H}^g(\overline{\rho}) \leq [H; U]^g = \overline{\rho}_i.
\]

If

\[
\dim(V(K) \oplus p^*I^*U) = \dim U
\]

then \( \dim I^*U \) is just one less than \( \dim U \) and by writing \( U = V(K) \oplus U' \) we get \( I^*U = i^*U' \). So \( p^*I^*U = p^*i^*U' \leq U' \) by the ordering of irreducible \( H \)-modules and its extension by lexicography i.e. \( V(K) \oplus p^*I^*U \leq V(K) \oplus U' = U \). Again we have

\[
\overline{\rho} < \overline{\rho}_k = e_{K,H}^g(\overline{\rho}) \leq [H; U]^g = \rho_i.
\]
Case II. Let \( K \triangleleft_2 H \) i.e. \( K < H \) but \( H \neq (x) \times K \) for any \( x \in G_2 \).

If \( K_2 = G_2 \) then the class \( \bar{\rho} \) is the \( e^g \)-image of some conjugate class of \( G \) slice types occurring earlier according to the order so constructed. But this means \( \bar{\rho} \not\in A_j \)—a contradiction. So \( K_2 \nsubseteq G_2 \) and there exists an element \( x \in G_2 \) such that \( (x) \times K \) can be formed. Since \( K \) is a proper subgroup of \( H \), \( |(x) \times K| \leq |H| \). If \( |(x) \times K| < |H| \) then by (i) of Rule A

\[ \bar{\rho} < \bar{\rho}_k < \bar{\rho}_t. \]

If \( |(x) \times K| = |H| \) then \( |H: K| = \text{index of } K \text{ in } H = 2 \). Since \( K \triangleleft_2 H \), \( x \not\in H \). Also there does not exist \( y \in G_2 \) such that \( y \in H \) but \( y \not\in K \).

Hence \( K_2 = H_2 \) and \( |(x) \times K_2| > |H_2| \). By (ii) of Rule A, \( (x) \times K < H \) and

\[ \bar{\rho} < \bar{\rho}_k < \bar{\rho}_t. \]

Now

\[ \bar{\rho}_i = [H; U]^g \in \bar{F}_j \Rightarrow \bar{\rho}_t < \bar{\rho}_j \]

\[ \Rightarrow \bar{\rho} < e^g(\bar{\rho}) = \bar{\rho}_k < \bar{\rho}_t < \bar{\rho}_j \]

\[ \Rightarrow \bar{\rho} \in A_{k-1} \text{ and } \bar{\rho} \not\in A_k \text{ (Lemma 10)} \]

\[ \Rightarrow \bar{\rho} \not\in A_j \text{—a contradiction.} \]

A consequence of this theorem is:

**Corollary 13.** The union of all conjugate classes of \( G \) slice types of \( B_j \) and \( C_j \) is a family.

**Proof.** Let \( [H; U]^g \in B_j \cup C_j \subseteq F_j \) and \( \bar{\rho} \) is a conjugate class of \( G \)-slice types of an orbit of a point of \( G \times_H U \). Clearly \( \bar{\rho} \in F_j \). If \( \bar{\rho} \not\in B_j \cup C_j \) then \( \bar{\rho} \in A_j \) and this contradicts Theorem 12.

**6. Proof of the main theorem.** We denote the elements of \( C_j \) by \( \bar{\sigma}_0, \bar{\sigma}_2, \ldots, \bar{\sigma}_{2k} \) where \( k = |C_j| \) and \( \bar{\sigma}_{2t} \leq \bar{\sigma}_{2m} \) if and only if \( t \leq m \). We have \( B_j = \{ e^g(\bar{\sigma}_{2i}) | 0 \leq i \leq k \} \) and write \( e^g(\bar{\sigma}_{2i}) = \bar{\sigma}_{2i+1} \).

By Corollary 13, \( \hat{F}_k = \bigcup_{i=0}^k \bar{\sigma}_i \) is a family when \( k \) is odd. When \( k \) is even \( \hat{F}_k \) is again a family because the \( G \) slice types of \( \bar{\sigma}_k \) are ‘maximal’ in \( \hat{F}_k \). By Lemma 11 we see that \( \hat{F}(\hat{G}) \) satisfies all the conditions of Lemma 4 and so

\[ N^{\mathcal{L}} \left[ \hat{F}(\hat{G}) \right] = 0. \]

An alternative proof of Theorem 1 can be given by generalising Theorem 4.5.11 of [3].
Theorem 14. There is an isomorphism

\[ \bigoplus v_i : N^G_\bullet [F_j] \to \bigoplus_{\rho_i \in A_j} N^G_\bullet [\rho_i]. \]

Proof. We prove the result by induction. Clearly the result is true for \( j = 0 \). Now suppose it is true for \( j - 1 \) i.e.

\[ \bigoplus v_i : N^G_\bullet [F_{j-1}] \to \bigoplus_{\rho_i \in A_{j-1}} N^G_\bullet [\rho_i]. \]

From the long exact sequence of Proposition 2 we have the composite

\[ v_i \partial_j : N^G_n [\bar{\rho}_j] \to N^G_{n-1} [\bar{\rho}_i]. \]

If \( v_i \partial_j \neq 0 \) then \( \bar{\rho}_i \) is a conjugate class of \( G \) slice types of \( G \times_H V \) where \([H, V]^g = \bar{\rho}_j \) and by Theorem 12 \( \rho_i \notin A_j \).

Now for the class \( \bar{\rho}_j \) there exists almost one conjugate class of \( G \) slice types \( \bar{\rho}_i \) such that \( e^g(\bar{\rho}_i) = \bar{\rho}_j \). If there does not exist any such \( \bar{\rho}_i \in A_{j-1} \) then for any \( \bar{\rho}_i \in A_{j-1} \) both \( \bar{\rho}_i \) and \( \bar{\rho}_j \) belong to \( A_j \) and \( v_i \partial_j = 0 \) for every \( \bar{\rho}_i \in A_{j-1} \). Thus \( \bigoplus_{\bar{\rho}_i \in A_{j-1}} v_i \partial_j = 0 \) and consequently \( \partial_j = 0 \). We have a short exact sequence

\[ 0 \to N^G_n [F_{j-1}] \to N^G_n [F_j] \to N^G_{n-1} [\bar{\rho}_j] \to 0. \]

If again for \( \bar{\rho}_j \) we have \( \bar{\rho}_i \in A_{j-1} \) s.t. \( \bar{\rho}_j = e^g(\bar{\rho}_i) \) then neither \( \bar{\rho}_j \) nor \( \bar{\rho}_i \) belong to \( A_j \) and by Lemma 3

\[ v_i \partial_j : N^G_n [\bar{\rho}_j] \to N^G_{n-1} [\bar{\rho}_i] \]

is an isomorphism and we have again a short exact sequence

\[ 0 \to N^G_n [\bar{\rho}_j] \to N^G_n [F_{j-1}] \to N^G_n [F_j] \to 0. \]

Both the short exact sequences split as the modules involved are vector spaces over \( \mathbb{Z}_2 \). So

\[ N^G_n [F_j] = \bigoplus_{\rho_i \in A_j} N^G_n [\rho_i]. \]

Corollary 15. \( N^G_\bullet [\tilde{F}(\hat{G})] = 0. \)

Proof. Corresponding to the positive integer \( n \) we take all conjugate classes of \( G \) slice types of dimension \( \leq n + 1 \). If \( F_N \) be the union of all these classes then

\[ N^G_n [\tilde{F}(\hat{G})] = N^G_n [F_N] = \bigoplus_{\rho_i \in A_N} N^G_n [\rho_i]. \]

If now \( N \) is made sufficiently large compared to \( n \) then by Lemma 11 \( A_N \) consists of all conjugate classes of \( G \) slice types of dimension \( > n \) and hence the isomorphism \( \bigoplus v_i \) is zero.
**Corollary 16.**

\[ N^G_*[F'(\hat{G})] \cong N^G_{*-1}[\tilde{F}(\tilde{G}), F'(\hat{G})]. \]

This follows from the main theorem and the long exact sequence for the pair \( F'(\hat{G}) \subset \tilde{F}(\tilde{G}) \) of families of \( G \)-slice types.

**References**


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**Lady Keane's College**

**Shillong, India**

AND

**North Eastern Hill University**

**Biju Campus, Bhagyakul Road**

**Laitumkhrah, Shillong, 793003**

**Meghalaya, India**
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