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ESTIMATES FOR PARTIAL SUMS OF CONTINUED FRACTION PARTIAL QUOTIENTS

HAROLD GEORGE DIAMOND AND JEFFREY D. VAALER

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Metric-type estimates are given for a class of partial sums involving continued fraction partial quotients. These results extend a well known theorem of Khinchin and yield an almost-everywhere estimate for the quantity in the title.

1. Introduction. For α an irrational number in $(0, 1)$ let

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} = \langle 0, a_1, a_2, \dots \rangle$$

be the representation of α as a regular continued fraction ([4, Ch. X], [5]). The numbers $a_n = a_n(\alpha)$ are called the *partial quotients* of α .

A well-known theorem of Khinchin [5], [6] in the metric theory of continued fractions asserts that if F is an arithmetic function satisfying $F(r) \ll r^{1/2-\delta}$ for some $\delta > 0$ and if $S_N(F, \alpha) := F(a_1(\alpha)) + \dots + F(a_N(\alpha))$ for each positive integer N , then

$$(1) \quad \lim_{N \rightarrow \infty} \frac{1}{N} S_N(F, \alpha) = \frac{1}{\log 2} \sum_{r=1}^{\infty} F(r) \log \left\{ 1 + \frac{1}{r(r+2)} \right\}$$

holds for almost all α in $(0, 1)$. This result has been extended by others ([2, §4], [7, Theorem 4]). In particular, we note that the Birkhoff Ergodic Theorem implies that (1) holds if its right-hand side is absolutely convergent.

Here we shall establish analogues of (1) for arithmetic functions F which grow more rapidly than is allowed by Khinchin's theorem. In particular we shall consider the case $F(r) = I(r) = r$ and estimate

$$S_N(I, \alpha) = a_1(\alpha) + \dots + a_N(\alpha).$$

Khinchin noted at the end of his book *Continued Fractions* [5] that $S_N(I, \alpha)/N$ could not have a finite limit for most values of α . Indeed, for almost all α the inequality $a_n(\alpha) > n \log n$ holds for an infinite sequence of integers n in consequence of the following result of Borel and Bernstein

[4, Theorem 197], [5, Theorem 30], [1, Theorem 4.1]:

LEMMA 1. *Let $\varphi(1), \varphi(2), \dots$ be a sequence of positive numbers. For almost all $\alpha \in (0, 1)$ the inequality $a_n(\alpha) > \varphi(n)$ has a finite number of solutions n if and only if $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$.*

Khinchin showed in [6] that

$$(2) \quad (b_1 + \dots + b_N)/(N \log N) \rightarrow 1/\log 2$$

in measure as $N \rightarrow \infty$, where

$$b_n = b_n(\alpha) = \begin{cases} a_n, & \text{if } a_n < n(\log n)^{4/3} \\ 0, & \text{otherwise.} \end{cases}$$

The limit (2) cannot hold a.e., since for almost all $\alpha \in (0, 1)$ the inequality $b_n > n \log n \log \log n$ holds for an infinite sequence of integers n by Lemma 1.

The obstacle to a.e. convergence, as we shall see, is the occurrence of a single large value of a_n . Here we shall establish an analogue of (1) by excluding at most one summand.

THEOREM 1. *Suppose that F is a positive valued arithmetic function satisfying the bound*

$$(3) \quad \left\{ \sum_{j \leq N} F(j)^2/j^2 \right\} / \left\{ \sum_{j \leq N} F(j)/j^2 \right\}^2 \leq N(\log N)^{-3/2-\varepsilon}$$

for some $\varepsilon > 0$. Then for almost all $\alpha \in (0, 1)$ and for all N exceeding a number $N_0(\alpha)$, we have

$$S_N(f, \alpha) = (1 + o(1)) \frac{N}{\log 2} \sum_{r \leq N} F(r) \log \left\{ 1 + \frac{1}{r(r+2)} \right\} \\ + \vartheta_+ \max_{1 \leq n \leq N} F(a_n(\alpha)).$$

Here $0 \leq \vartheta_+ = \vartheta_+(N, \alpha, F) \leq 1$.

If we take $F(r) = I(r) = r$ we obtain

COROLLARY 1. *For almost all $\alpha \in (0, 1)$ there exists a number $N_0 = N_0(\alpha)$ such that*

$$S_N(I, \alpha) = \frac{1 + o(1)}{\log 2} N \log N + \vartheta_+ \max_{1 \leq n \leq N} a_n(\alpha)$$

holds for all $N \geq N_0$.

An immediate consequence of Lemma 1 and Corollary 1 is

COROLLARY 2. *Let $0 < \varphi(1) < \varphi(2) < \dots$ satisfy $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$.*

Then

$$S_N(I, \alpha) = \frac{1 + o(1)}{\log 2} N \log N + \vartheta_+ \varphi(N)$$

holds for almost all $\alpha \in (0, 1)$ and all $N \geq N_0(\alpha)$.

There are two main steps in the proof of the theorem. First, we show that for most α there can be at most one “large” $a_n(\alpha)$. Next we estimate the variance of a truncated form of S_N . The theorem follows easily from these estimates.

2. Auxiliary results. Let $[x]$ denote the integer part of a real number x and let $\{x\} = x - [x]$ denote the fractional part. Define $T: (0, 1) \rightarrow [0, 1)$ by $Tx = \{1/x\}$. Then the partial quotients of (an irrational number) α are given by the formulas

$$a_1(\alpha) = [1/\alpha], \quad a_{n+1}(\alpha) = a_1(T^n \alpha), \quad n \geq 1.$$

(Rational numbers have terminating continued fraction expansions and require slight alteration of the formulas. This is not needed here, since the rationals form a set of measure zero.)

The so called *Gauss measure* μ is defined on Borel subsets of $(0, 1)$ by

$$\mu(E) = \frac{1}{\log 2} \int_{t \in E} \frac{dt}{1+t}.$$

This measure satisfies the *invariance* relation

$$\mu(T^{-1}E) = \mu(E), \quad E \text{ a Borel set.}$$

Note that μ and Lebesgue measure have the same null sets.

For r and k_1, k_2, \dots, k_r positive integers, set

$$E_r = E\left(\begin{matrix} 1 & 2 \cdots r \\ k_1 & k_2 \cdots k_r \end{matrix}\right) = \{\alpha: a_1(\alpha) = k_1, \dots, a_r(\alpha) = k_r\}.$$

E_r is called a *fundamental interval of rank r* . If r and s are positive integers, B is any Borel set, and E_r any fundamental interval of rank r , then μ satisfies the *mixing relation* [1, Chapter 1, §4]

$$\mu(E_r \cap T^{-r-s}B) = \mu(E_r)\mu(B)\{1 + O(q^s)\}$$

uniformly in r, s, B , and E_r . Here q is some number in $(0, 1)$. Together

the preceding relations imply that

$$\begin{aligned}
 & \mu \{ \alpha : a_r(\alpha) = m \text{ and } a_{r+s}(\alpha) = n \} \\
 &= \mu \{ \alpha : a_1(\alpha) = m \text{ and } a_{1+s}(\alpha) = n \} \\
 &= \mu \left\{ \left(\frac{1}{m+1}, \frac{1}{m} \right] \cap T^{-s-1} \left(\frac{1}{n+1}, \frac{1}{n} \right] \right\} \\
 &= (\log 2)^{-2} \log \frac{(m+1)(m+1)}{m(m+2)} \log \frac{(n+1)(n+1)}{n(n+2)} \{1 + O(q^s)\}.
 \end{aligned}$$

LEMMA 2. *Let $c > 1/2$, and for given $N \in \mathbf{Z}^+$ set $N' = N(\log N)^c$. For almost all $\alpha \in (0, 1)$ there exist at most finitely many positive integers N for which the inequalities*

$$(4) \quad a_m(\alpha) > N', \quad a_n(\alpha) > N'$$

hold for two distinct indices $m, n \leq N$.

Proof. Fix $m < n$. By a weak form of the mixing condition we have

$$\begin{aligned}
 & \mu \{ \alpha \in (0, 1) : a_m(\alpha) > N', a_n(\alpha) > N' \} \\
 & \ll \mu \{ \alpha : a_m(\alpha) > N' \} \cdot \mu \{ \alpha : a_n(\alpha) > N' \} \\
 & = \mu \{ \alpha : a_1(\alpha) > N' \}^2 \ll (N')^{-2} = N^{-2} (\log N)^{-2c}.
 \end{aligned}$$

It follows that the measure of the set on which (4) holds for *some* distinct indices $m, n \leq 2N$ is of order at most $(\log N)^{-2c}$. For $K = 1, 2, \dots$ let

$$\begin{aligned}
 U_k &= \bigcup_{k \geq K} \{ \alpha \in (0, 1) : a_m(\alpha) > (2^k)', a_n(\alpha) > (2^k)' \} \\
 & \quad \text{for some distinct } m, n \leq 2^{k+1} \}.
 \end{aligned}$$

Then

$$\mu(U_k) \ll \sum_{k \geq K} k^{-2c} \rightarrow 0 \quad \text{as } K \rightarrow \infty.$$

For $\alpha \notin U_k$ and $N \geq 2^K$ there exists at most one index $n \leq N$ for which

$$a_n(\alpha) > N(\log N)^c. \quad \square$$

3. Proof of the theorem. Given $\varepsilon > 0$ and $N \in \mathbf{N}$, set

$$a_n^* = a_{n,N}^*(\alpha) = \begin{cases} a_n & \text{if } a_n \leq N(\log N)^{1/2+\varepsilon/4} =: N' \\ 0 & \text{otherwise.} \end{cases}$$

Define $F(0) = 0$,

$$S_N^*(\alpha) = S_N^*(F, \alpha) = \sum_{n \leq N} F(a_n^*(\alpha)),$$

$$J_N = \int_0^1 S_N^*(\alpha) d\mu(\alpha),$$

and

$$V_N = \int_0^1 (S_N^*(\alpha) - J_N)^2 d\mu(\alpha).$$

We have

$$\begin{aligned} J_N &= \sum_{n=1}^N \int_0^1 F(a_n^*(\alpha)) d\mu(\alpha) = N \int_0^1 F(a_1^*(\alpha)) d\mu(\alpha) \\ &= N \sum_{j=1}^{N'} F(j) \mu\{\alpha: a_1(\alpha) = j\} \\ &= \frac{N}{\log 2} \sum_{j=1}^{N'} F(j) \log\left(1 + \frac{1}{j(j+2)}\right) \asymp N \sum_{j=1}^{N'} F(j)/j^2. \end{aligned}$$

(We say that $f \asymp g$ if $f \leq K_1 g$ and $g \leq K_2 f$ for suitable K_1 and K_2 .)

Next we show that $V_N \ll N \sum_{j \leq N'} F(j)^2/j^2$. We begin by writing

$$\begin{aligned} V_N + J_N^2 &= \int_0^1 |S_N^*(\alpha)|^2 d\mu(\alpha) \\ &= \sum_{m,n=1}^N \int_0^1 F(a_m^*(\alpha)) F(a_n^*(\alpha)) d\mu(\alpha) = \sum_{m,n=1}^N b_{mn}, \end{aligned}$$

say. For $1 \leq m < n \leq N$ we have

$$\begin{aligned} b_{mn} &= \sum_{j,k \leq N'} F(j) F(k) \mu\{\alpha: a_m(\alpha) = j, a_n(\alpha) = k\} \\ &= \sum_{j,k \leq N'} F(j) F(k) \mu\left(\frac{1}{j+1}, \frac{1}{j}\right] \mu\left(\frac{1}{k+1}, \frac{1}{k}\right] (1 + O(q^{n-m})) \\ &= J_N^2 N^{-2} (1 + O(q^{n-m})). \end{aligned}$$

The diagonal terms satisfy

$$\begin{aligned} b_{nn} &= \int_0^1 F(a_n^*(\alpha))^2 d\mu(\alpha) = \int_0^1 F(a_1^*(\alpha))^2 d\mu(\alpha) \\ &= \sum_{j \leq N'} F(j)^2 \mu\{\alpha: a_1(\alpha) = j\} \ll \sum_{j \leq N'} F(j)^2/j^2. \end{aligned}$$

Thus we have

$$\begin{aligned}
 V_N &= \sum_{m,n=1}^N b_{mn} - J_N^2 \ll J_N^2 N^{-2} \sum_{m \leq n \leq N} q^{n-m} + N \sum_{j \leq N'} F(j)^2 / j^2 \\
 &\ll J_N^2 / N + N \sum_{j \leq N'} F(j)^2 / j^2 \\
 &\ll N \left(\sum_{j \leq N'} F(j) / j^2 \right)^2 + N \sum_{n \leq N'} F(j)^2 / j^2 \\
 &\ll N \sum_{j \leq N'} F(j)^2 / j^2.
 \end{aligned}$$

The last relation follows from the Cauchy-Schwarz inequality.

Now we apply the estimate of V_N to show that

$$S_N^*(\alpha) = (1 + o(1))J_N$$

for most values of α . Let

$$c(k) = \lceil \exp k^{1-\varepsilon/4} \rceil, \quad k = 1, 2, \dots$$

We have

$$\begin{aligned}
 \int_{\alpha=0}^1 \sum_{k=1}^{\infty} (S_{c(k)}^*(\alpha) - J_{c(k)})^2 \left(c(k) \sum_{j \leq c(k)'} \frac{F(j)^2}{j^2} k^{1+\varepsilon/4} \right)^{-1} d\mu(\alpha) \\
 \ll \sum_{k=1}^{\infty} k^{-1-\varepsilon/4} < \infty.
 \end{aligned}$$

It follows that the integrand in the last integral is finite a.e. and hence

$$S_{c(k)}^*(\alpha) - J_{c(k)} = o \left\{ c(k) \sum_{j \leq c(k)'} \frac{F(j)^2}{j^2} k^{1+\varepsilon/4} \right\}^{1/2}$$

for almost all α . The hypothesis of Theorem 1 and a small calculation show that

$$c(k) \sum_{j \leq c(k)'} \frac{F(j)^2}{j^2} k^{1+\varepsilon/4} \ll J_{c(k)}^2 / (\log c(k))^{\varepsilon/12} = o(J_{c(k)}^2),$$

provided that $\varepsilon < 1$. Thus

$$S_{c(k)}^*(\alpha) = (1 + o(1))J_{c(k)} \quad \text{a.e.}$$

Suppose that $c(k-1) < N \leq c(k)$. Then

$$S_{c(k-1)}^*(\alpha) \leq S_N^*(\alpha) \leq S_{c(k)}^*(\alpha),$$

and so, off a set of measure 0,

$$(1 + o(1))J_{c(k-1)} \leq S_N^*(\alpha) \leq (1 + o(1))J_{c(k)}.$$

Now we show that $J_{c(k)} \sim J_{c(k-1)}$. Recall that

$$J_N = \frac{N}{\log 2} \sum_{j \leq N'} F(j) \log \left(1 + \frac{1}{j(j+2)} \right).$$

Another small calculation shows that

$$c(k-1) = (1 + O(k^{-\varepsilon/4}))c(k),$$

so $c(k-1) \sim c(k)$ as $k \rightarrow \infty$. It remains to show that

$$(5) \quad \sum_{j \leq c(k)'} F(j) \log \left(1 + \frac{1}{j(j+2)} \right) \sim \sum_{j \leq c(k-1)'} F(j) \log \left(1 + \frac{1}{j(j+2)} \right).$$

We shall show (5) and the final approximation of S_N^* by using

LEMMA 3. *Let F satisfy the hypotheses of Theorem 1. Then, as $N \rightarrow \infty$,*

$$\sum_{N < r \leq N \log N} \frac{F(r)}{r^2} = o \left(\sum_{r \leq N} \frac{F(r)}{r^2} \right).$$

Proof. The Cauchy Schwarz inequality and condition (3) yield

$$\begin{aligned} \sum_{N < r \leq N \log N} \frac{F(r)}{r^2} &\leq \left\{ \sum_{r \leq N \log N} \frac{F(r)^2}{r^2} \right\}^{1/2} \left\{ \sum_{r > N} \frac{1}{r^2} \right\}^{1/2} \\ &\leq \sum_{r \leq N \log N} \frac{F(r)}{r^2} (\log N)^{-1/4 - \varepsilon/2}. \end{aligned}$$

Thus, as $N \rightarrow \infty$,

$$(1 - o(1)) \sum_{N < r \leq N \log N} \frac{F(r)}{r^2} \leq o(1) \sum_{r \leq N} \frac{F(r)}{r^2},$$

and the lemma follows. \square

Returning to the proof of Theorem 1, we see first that (5) holds, and hence

$$S_N^*(\alpha) = (1 + o(1))J_N \quad \text{a.e.}$$

The lemma also implies that

$$S_N^*(\alpha) = (1 + o(1)) \frac{N}{\log 2} \sum_{j \leq N} F(j) \log \left(1 + \frac{1}{j(j+2)} \right) \quad \text{a.e.}$$

since

$$\sum_{N < j \leq N'} F(j) \log \left(1 + \frac{1}{j(j+2)} \right)$$

is negligible.

Finally, we have by Lemma 2, for almost all α and all sufficiently large N ,

$$0 \leq S_N(\alpha) - S_N^*(\alpha) \leq F \left(\max_{1 \leq n \leq N} a_n(\alpha) \right) \leq \max_{1 \leq n \leq N} F(a_n(\alpha)).$$

This inequality and the last estimate of S_N^* establish the theorem. \square

It would be interesting to learn whether Theorem 1 could be established by ergodic methods.

4. Further results. In this section we consider cases in which $S_N(I, \alpha)$ can be estimated by $N(\log N)/(\log 2)$ alone and when by $\varphi(N)$ alone, where $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$.

First we note that for any $\varepsilon > 0$ and fixed $N \in \mathbf{Z}^+$ we have

$$(6) \quad \mu \left\{ \alpha \in (0, 1) : \left| \frac{S_N(I, \alpha)}{N \log N} - \frac{1}{\log 2} \right| > \varepsilon \right\} \ll \frac{1}{\varepsilon \log N}.$$

(The implied constant here is absolute.)

This bound is achieved by setting

$$a_n^{**} = a_{n, N, \varepsilon}^{**}(\alpha) = \begin{cases} a_n, & \text{if } a_n < \varepsilon N \log N =: N'' \\ 0, & \text{otherwise.} \end{cases}$$

We compute the variance of the sum function S_N^{**} as before and apply Chebyshev's estimate to obtain

$$\mu \left\{ \alpha : \left| S_N^{**}(\alpha) - \frac{N \log N}{\log 2} \right| > \varepsilon N \log N \right\} \ll \frac{1}{\varepsilon \log N}.$$

Also, for each $n \leq N$,

$$\mu \{ \alpha : a_n(\alpha) > \varepsilon N \log N \} \ll \frac{1}{\varepsilon N \log N},$$

and estimate (6) follows.

Next, we show directly a sharp one sided estimate of Pruitt [9, Theorem 5.2].

COROLLARY 3. For $N \geq 3$ set $\beta(N) = \exp(k \log^2 k) k \log^2 k$ for

$$(7) \quad \exp((k-1) \log^2(k-1)) < N \leq \exp(k \log^2 k).$$

Then, for almost all $\alpha \in (0, 1)$

$$\limsup_{N \rightarrow \infty} \frac{S_N(I, \alpha)}{\beta(N)} = \frac{1}{\log 2}.$$

Proof. In Corollary 2 set

$$\varphi(N) = \beta(N)/(\log \log 10k).$$

An easy calculation shows that $\sum 1/\varphi(N) < \infty$. If N satisfies (7), then by Corollary 2

$$S_N(I, \alpha) \leq \frac{1 + o(1)}{\log 2} \beta(N) + \beta(N)/\log \log 10k \quad \text{a.e.,}$$

so $\limsup S_N(I, \alpha)/\beta(N) \leq 1/\log 2$ a.e. On the sequence $N_k = \exp(k \log^2 k)$, the ratio $S_N(I, \alpha)/\beta(N)$ converges to $1/\log 2$ a.e. \square

In another direction, we show that in Corollary 2, $\varphi(N)$ dominates $N \log N$ for “most” values of N .

LEMMA 4. *Suppose that $0 < \varphi(1) \leq \varphi(2) \leq \dots$ and $\sum_{n=1}^{\infty} 1/\varphi(n) < \infty$. Let*

$$S = \{n \in \mathbf{Z}^+ : \varphi(n) < n \log n\}.$$

Then S has logarithmic density zero.

Proof. Let $T = \{\nu \in \mathbf{Z}^+ : (2^{\nu-1}, 2^\nu] \cap S \neq \emptyset\}$. Suppose $\nu \in T$. Then there exists an integer n such that $2^{\nu-1} < n \leq 2^\nu$ and $\varphi(n) < n \log n$, so

$$\sum_{n/2 < k \leq n} \frac{1}{\varphi(k)} \geq \frac{n - [n/2]}{n \log n} \geq \frac{1}{2 \log n} \geq \frac{1}{2^\nu \log 2}.$$

Thus

$$(8) \quad \sum_{\nu \in T} \frac{1}{\nu} \leq 4(\log 2) \sum_{k=1}^{\infty} 1/\varphi(k) < \infty.$$

Also, we have

$$\sum_{\substack{k \in S \\ 2^{\nu-1} < k \leq 2^\nu}} \frac{1}{k} \leq \begin{cases} \log 2, & \nu \in T \\ 0, & \nu \notin T. \end{cases}$$

With $y = (\log x)/\log 2$ we have

$$\begin{aligned} \delta_x &:= \frac{1}{\log x} \sum_{\substack{k \leq x \\ k \in S}} \frac{1}{k} \leq \frac{1}{y \log 2} \sum_{\nu \leq y+1} \sum_{\substack{k \in S \\ 2^{\nu-1} < k \leq 2^\nu}} \frac{1}{k} \\ &\leq \frac{1}{y} \sum_{\substack{\nu \leq y+1 \\ \nu \in T}} 1 = \frac{T(y+1)}{y}, \quad \text{say.} \end{aligned}$$

We have from (8) that

$$\frac{1}{N}(T(N) - T(N/2)) \leq \sum_{\substack{p \in T \\ n/2 < p \leq N}} \frac{1}{p} \rightarrow 0$$

as $N \rightarrow \infty$. Thus $T(y) = o(y)$ as $y \rightarrow \infty$. Finally,

$$\limsup_{x \rightarrow \infty} \delta_x \leq \limsup_{y \rightarrow \infty} \frac{T(y+1)}{y} = 0. \quad \square$$

COROLLARY 4. *Suppose that φ satisfies the hypotheses of Lemma 4. Then for almost all $\alpha \in (0, 1)$*

$$S_N(I, \alpha) \ll \varphi(N)$$

holds for all integers N outside a set of logarithmic density zero.

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