ABELIAN GROUPS AND PACKING BY SEMICROSSES

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Motivated by a question about geometric packings in \( n \)-dimensional Euclidean space, \( \mathbb{R}^n \), we consider the following problem about finite abelian groups. Let \( n \) be an integer, \( n \geq 3 \), and let \( k \) be a positive integer. Let \( g(k, n) \) be the order of the smallest abelian group in which there exist \( n \) elements, \( a_1, a_2, \ldots, a_n \), such that the \( kn \) elements \( ia_i \), \( 1 \leq i \leq k \), are distinct and not 0. We will show that for \( n \) fixed, \( g(k, n) \sim 2\cos(\pi/n)k^{3/2} \).

The geometric question concerns certain star bodies, called “semicrosses”, which are defined as follows:

If \( k \) and \( n \) are positive integers, a \((k, n)\)-semicross consists of \( kn + 1 \) unit cubes in \( \mathbb{R}^n \), a “corner” cube parallel to the coordinate axes together with \( n \) arms of length \( k \) attached to faces of the cube, one such arm pointing in the direction of each positive coordinate axis. Let \( K \), the “semicross at the origin”, be the semicross whose corner cube is \([0,1]^n\). Then every semicross is a translate of \( K \); i.e. has the form \( v + K \) for some vector \( v \).

A family of translates \( \{v + K: v \in H\} \) is called an integer lattice packing if \( H \) is an \( n \)-dimensional subgroup of \( \mathbb{Z}^n \) and, for any two vectors \( v \) and \( w \) in \( H \), the interiors of \( v + K \) and \( w + K \) are disjoint. We shall examine how densely such packings pack \( \mathbb{R}^n \) for large \( k \), and show that, for \( n \geq 3 \), this density is asymptotic to

\[
\frac{n \sec \pi/n}{2\sqrt{k}}.
\]

(For \( n = 1 \) or 2 the density is 1 for every \( k \).)

This result contrasts with the already known result for crosses. (A \((k, n)\)-cross consists of \( 2kn + 1 \) unit cubes, a center cube with an arm of length \( k \) attached to each face.) As shown in [St1], for \( n \geq 2 \) the integer lattice packing density of the \((k, n)\)-cross is asymptotic to \( 2n/k \).

**0. Preliminary matters.** Suppose \( M \) is a set of nonzero integers, \( G \) is an abelian group, and \( n \) is a positive integer. We say that \( M \) \( n \)-packs \( G \) if there is a set \( S \subseteq G \) such that \( |S| = n \) and the elements \( ms \) with \( m \in M \) and \( s \in S \) are distinct and nonzero. Such a set \( S \) is called a packing set.
Let $S(k) = \{1, \ldots, k\}$ and $F(k) = \{-1, \ldots, +k\}$. Then, as shown in [St1], there is a relation between integer lattice packings by the $(k, n)$-semicross (resp. cross) and $n$-packings of finite abelian groups by $S(k)$ (resp. $F(k)$). We now develop this connection.

We will designate each unit cube in $\mathbb{R}^n$ with edges parallel to the coordinate axes by its vertex with minimal coordinates. Thus $K$, the $(k, n)$-semicross at the origin, is the union of the $kn + 1$ cubes designated by $(0,0,\ldots,0), (i,0,\ldots,0),\ldots,$ and $(0,\ldots,0,i)$ with $1 \leq i \leq k$.

Let $H$ be an integer packing lattice for $K$, i.e. an $n$-dimensional subgroup of $\mathbb{Z}^n$ such that the interiors of $v + K$ for $v \in H$ are pairwise disjoint. Let $G = \mathbb{Z}^n/H$, $f : \mathbb{Z}^n \to G$ be the natural homomorphism, $e_i \in \mathbb{Z}^n$ be the unit vector in the $i$th coordinate direction, and $a_i = f(e_i)$. Then it is easy to show that the $kn$ elements $ia_j$ with $1 \leq i \leq k$ and $1 \leq j \leq n$ are distinct and nonzero; that is, $S(k)$ $n$-packs $G$ with packing set $\{a_1, \ldots, a_n\}$.

Conversely, suppose $S(k)$ $n$-packs a finite abelian group $G$ with packing set $\{a_1, \ldots, a_n\}$. Let $H = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1a_1 + \cdots + x_na_n = 0\}$. Then $H$ is an integer packing lattice for the $(k, n)$-semicross. Moreover, the density of this packing is $(kn + 1)/|G^*|$, where $G^*$ is the subgroup generated by $a_1, \ldots, a_n$.

Thus, finding the densest integer lattice packing by the $(k, n)$-semicross is equivalent to finding the smallest abelian group $G$ such that $S(k)$ $n$-packs $G$. Let $g(k, n)$ be the order of the smallest such group. Clearly $g(k, n) \geq kn + 1$, with equality if $n = 1$ or $n = 2$. Our main result is given in the following theorem.

**Theorem 1.** For $n \geq 3$,

$$\lim_{k \to \infty} \frac{g(k, n)}{k^{3/2}} = 2 \cos \frac{\pi}{n}.$$ 

Since the integer lattice packing density of the $(k, n)$-semicross is $(kn + 1)/g(k, n)$, this density is asymptotic to $n \sec(\pi/n)/2\sqrt{k}$ as $k \to \infty$.

This result should be compared with the corresponding result for crosses. Let $h(k, n)$ be the order of the smallest abelian group $G$ such that $F(k)$ $n$-packs $G$. Clearly $h(k, n) \geq 2kn + 1$, with equality if $n = 1$. As shown in [St1] for $n \geq 2$,

$$\lim_{k \to \infty} \frac{h(k, n)}{k^2} = 1.$$
Since the integer lattice packing density of the \((k,n)\)-cross is \((2kn + 1)/h(k, n)\), this density is asymptotic to \(2n/k\) as \(k \to \infty\).

Throughout the remaining sections, \(C(m)\) denotes the cyclic group of order \(m\), \(Z/mZ\).

1. Motivation. In [St1] it was shown that for any integer \(b > 1\), \(S(b^2 - b)\) 3-packs \(C(b^3 + 1)\) with packing set \(\{1, -b, (-b)^2\}\). Since \((-b)^3 = 1\) in \(C(b^3 + 1)\), the packing set is a subgroup of the multiplicative structure of the ring \(Z/[(b^3 + 1)Z]\). In these 3-packings, \(k = b^2 - b\) and the order of the group is \(b^3 + 1\), which is asymptotic to \(k^{3/1}\) for large \(k\).

This method also gives some information in the case of 4-packings and 6-packings. It can be shown that for an odd integer \(b\) greater than 1, \(S((b^2 - 1)/2)\) 4-packs \(C((b + 1)(b^2 + 1)/2)\). The packing set is the (multiplicative) subgroup \(\{1, -b, (-b)^2, (-b)^3\}\), with \((-b)^4 = 1\) since \((b + 1)(b^2 + 1)/2\) divides \(b^4 - 1\). Observe that, since \(k = (b^2 - 1)/2\) and the order of the group is \((b + 1)(b^2 + 1)/2\), the order of the group is asymptotic to \(\sqrt{2} k^{3/2}\).

Similarly, for \(b \equiv 1 \pmod{6}\) and greater than 1, \(S((b^2 + b - 2)/3)\) 6-packs \(C((b^2 + b + 1)(b + 1)/3)\) with packing set \(\{1, -b, (-b)^2, (-b)^3, (-b)^4, (-b)^5\}\), again a group since \((-b)^6 = 1\). In this case, the order of the group is asymptotic to \(\sqrt{3} k^{3/2}\).

In these cases the order \(m\) of the group is a polynomial of degree 3 in \(b\) and the number \(k\) is a polynomial of degree 2 in \(b\). Since these polynomials have rational coefficients, \(\lim_{b \to \infty} m^2/k^3\) is necessarily rational. However, according to Theorem 1, only in the cases \(n = 3, 4,\) and 6 is

\[
\lim_{k \to \infty} \frac{g(k, n)^2}{k^3}
\]

rational, since only for these \(n \geq 3\) is \(\cos^2 \pi/n\) rational.

To obtain Theorem 1, we will modify this approach. While we will still consider packing sets in cyclic groups of the form \(\{1, -b, (-b)^2, \ldots, (-b)^{n-1}\}\), we do not demand that they form a subgroup, that is, that \((-b)^n = 1\). Our argument is motivated by a relation between pairs of elements in these packings. To express their relation we introduce the diagram in Fig. 1.1:

\[
\text{FIGURE 1.1}
\]
In this diagram $g$ and $h$ are elements in some abelian group and $x$ and $y$ are positive integers such that $xg + yh = 0$.

In the 3-, 4-, 6-packings mentioned earlier, the relations expressed by the three diagrams in Fig. 1.2 are valid:

Along each edge $x = (1 - \alpha)b + \alpha$ and $y = \alpha b + (1 - \alpha)$ for some rational $\alpha \in [0, 1]$. (For $r = 3$, $\alpha = 0$ or 1; for $r = 4$, $\alpha = 0, 1/2, \text{ or } 1$; for $r = 6$, $\alpha = 0, 1/3, 1/2, 2/3, \text{ or } 1$.) Furthermore, in any triangle in Fig. 1.2 labelled as in Fig. 1.3, we have $xx'x'' + yy'y'' = m$, the order of the group.

These observations suggest that we look for packings in cyclic groups of the form $\{-b^i | 0 \leq i \leq n - 1\}$ with the relations shown in Fig. 1.4, where $x_r = (1 - \alpha_r)b + \alpha_r$ and $y_r = \alpha_r b + (1 - \alpha_r)$. Moreover we demand the equality $xx'x'' + yy'y'' = m$ in each triangle.
Note that \( a_1 = 0 \). Denote \( a_2 \) by \( a \). Then the triangle displayed in Fig. 1.5 gives
\[
m = b^2(ab + (1 - a)) + ((1 - a)b + a),
\]
\[
m = (b + 1)(a(b - 1)^2 + b).
\]

More generally, the triangle shown in Fig. 1.6 shows that
\[
m = (b + 1)((1 - a_r)\alpha_{r+1}(b - 1)^2 + b).
\]
Thus \((1 - a_r)\alpha_{r+1} = a\), giving the recursion
\[
\alpha_{r+1} = \frac{a}{1 - a_r},
\]
which will play a central role in the argument.

With these observations in mind, the construction is straightforward:
Solve the recursion, making sure that \( 0 \leq \alpha_r \leq 1 \) for \( 1 \leq r \leq n - 1 \),
restrict \( b \) so that all \( x_r \) and \( y_r \) are integers, and then see how large \( k \) can
be for that choice of \( b \). The size of \( k \) is the substance of Lemma 2.1; note
that since in the construction \( x_r + y_r = b + 1 \), \( k \) may be as large as
\[
m/(b + 1) - 1 = a(b - 1)^2 + b - 1
\]
so, for large \( b \), \( m/k^{3/2} \approx 1/\sqrt{\alpha} \).
The proof of Theorem 1 consists of two parts. First we construct for large $k$ an $n$-packing for $S(k)$ in a cyclic group of order approximately $2 \cos(\pi/n)k^{3/2}$. This will show that
\[
\lim_{k \to \infty} \frac{g(k, n)}{k^{3/2}} \leq 2 \cos \frac{\pi}{n},
\]
which is Theorem 2. We then establish in Theorem 3 a lower bound for $g(k, n)$ which will imply that
\[
\lim_{k \to \infty} \frac{g(k, n)}{k^{3/2}} \geq 2 \cos \frac{\pi}{n}.
\]
Taken together, Theorems 2 and 3 yield Theorem 1.

2. A construction for group packings. We begin with the proofs of several lemmas. The first one gives a criterion for a 2-packing of $S(k)$ in $C(m)$. Its importance lies in the fact that a set $\{a_1, \ldots, a_n\}$ provides an $n$-packing for $S(k)$ if and only if every subset of two elements provides a 2-packing.

**Lemma 2.1.** Let $m$, $x$, and $y$ be positive integers and let $a$ and $b$ be integers such that $\gcd(a, b, m) = 1$ and $xa \equiv -yb \pmod{m}$. Let $0 < k < m/(x + y)$. Then $S(k)$ 2-packs $C(m)$, with packing set $\{a, b\}$.

**Proof.** Assume the contrary. Then we have $Xa \equiv Yb \pmod{m}$ for some integers $X$ and $Y$, with $0 \leq X, Y \leq k$ and not both 0. The congruences $xa \equiv -yb$ and $Xa \equiv Yb \pmod{m}$ imply the congruences $(Xy + Yx)a \equiv 0$ and $(Xy + Yx)b \equiv 0 \pmod{m}$. Since $\gcd(a, b, m) = 1$, it follows that $Xy + Yx \equiv 0 \pmod{m}$. However,
\[
0 < Xy + Yx \leq ky + kx = k(x + y) < m,
\]
a contradiction.

**Lemma 2.2.** Let $n \geq 3$ be an integer and let $p$ and $q$ be positive integers such that $p < q$ and $\gcd(p, q) = 1$. Let $\alpha = p/q$. Define $\alpha_1 = 0$ and $\alpha_{r+1} = \alpha/(1 - \alpha_r)$ for $r \geq 1$. Suppose $0 \leq \alpha_r \leq 1$ for $1 \leq r \leq n - 1$. Write $\alpha_r = p_r/q_r$, where $p_r$ and $q_r$ are nonnegative integers with $\gcd(p_r, q_r) = 1$. Suppose $b > 1$ is an integer such that $b \equiv 1 \pmod{L}$ and $\gcd(b, p) = 1$ where $L = \text{lcm}(q_1, q_2, \ldots, q_{n-1})$. Let $m = (b + 1)(\alpha(b - 1)^2 + b)$ and $k = \alpha(b - 1)^2 + b - 1$. Then $m$ and $k$ are integers and $S(k)$ $n$-packs $C(m)$ with packing set $\{1, -b, (-b)^2, \ldots, (-b)^{n-1}\}$. Also
\[
\lim_{b \to \infty} \frac{m^2}{k^3} = \frac{1}{\alpha}.
\]
(Some examples of this construction are given after the proof of Theorem 2.)
**Proof.** Note that $\alpha_2 = \alpha$, $p_2 = p$, and $q_2 = q$. By the definition of $L$, $b \equiv 1 \pmod{q}$. Thus

$$k = \frac{p}{q}(b-1)^2 + b - 1$$

is an integer. Since $m = (b+1)(b+1)$, $m$ is also an integer.

We next show that $\gcd(b, m) = 1$. Assume that $d = \gcd(b, m)$ is greater than 1. Then $d$ divides

$$m = (b + 1)\left(\frac{p(b - 1)^2}{q} + b\right)$$

but is relatively prime to $b + 1$ and $b - 1$. Thus $d$ divides $p$, contradicting the assumption that $\gcd(b, p) = 1$.

Since $\gcd(b, m) = 1$, it follows that, for $0 \leq e < f \leq n - 1$, $\{(-b)^e, (-b)^f\}$ is a packing set if and only if $\{1, (-b)^{f-e}\}$ is. Thus it suffices to show that for $1 \leq e \leq n - 1$, $S(k)$ 2-packs $C(m)$ with packing set $\{1, (-b)^e\}$.

For $1 \leq e \leq n - 1$ let $x_e = \alpha_e + (1 - \alpha_e)b$ and $y_e = (1 - \alpha_e) + \alpha_e b$. Note that $x_e$ and $y_e$ are positive and that

$$x_e = b + \frac{p}{q}(1 - b)$$

is an integer since $b \equiv 1 \pmod{q_e}$. Also, $x_e + y_e = b + 1$, so $y_e$ is an integer.

We will show inductively that $m$ divides $x_e + y_e(-b)^e$. Consider $e = 1$. We have $x_1 = b$ and $y_1 = 1$, hence $x_1 + y_1(-b)^1 = 0$, which is divisible by $m$. This checks the assertion for $e = 1$.

Suppose the result holds for some $e < n - 1$. It may be shown by algebra that

$$x_{e+1} + y_{e+1}(-b)^{e+1} = \frac{1 - (-b)^e}{1 + b}m + \alpha_{e+1}(1 - b)(x_e + y_e(-b)^e).$$

Note that $[1 - (-b)^e]/(1 + b)$ is an integer. Writing $\alpha_{e+1} = p_{e+1}/q_{e+1}$, we see that $\alpha_{e+1}(1 - b) = (p_{e+1}/q_{e+1})(1 - b)$ is an integer since $q_{e+1}$ divides $b - 1$. Since $m$ divides $x_e + y_e(-b)^e$ it follows that $m$ divides $x_{e+1} + y_{e+1}(-b)^{e+1}$ and the induction is complete.

Since

$$0 < k = \frac{m}{b+1} - 1 < \frac{m}{b+1} = \frac{m}{x_e + y_e},$$

we may apply Lemma 2.1 with \( a, b, x, \) and \( y \) replaced by 1, \((-b)^e, x_e,\) and \( y_e \) respectively. That lemma implies that \( S(k) \) 2-packs \( C(m) \) with packing set \( \{1, (-b)^e\} \).

That

\[
\lim_{b \to \infty} \frac{m^2}{k^3} = \frac{1}{\alpha}
\]

is clear.

Note that the conditions \( b \equiv 1 \pmod{L} \) and \( \gcd(b, p) = 1 \) are satisfied for infinitely many \( b \); just let \( b \equiv 1 \pmod{pL} \). In fact, it can be shown by induction that \( \gcd(p, L) = 1 \) and therefore for any integer \( a \) the simultaneous congruences \( b \equiv a \pmod{p} \) and \( b \equiv 1 \pmod{L} \) are solvable. Choosing \( a \) relatively prime to \( p \) forces \( b \) to be relatively prime to \( p \).

**Lemma 2.3.** Let \( n \geq 3 \) be an integer and let \( \alpha < 1 \) be a positive rational number. Define \( \alpha_1 = 0 \) and \( \alpha_{r+1} = \alpha/(1 - \alpha_r) \) for \( r \geq 1 \). Suppose \( 0 \leq \alpha_r \leq 1 \) for \( 1 \leq r \leq n - 1 \). Then for each positive integer \( k \) there is an integer \( m(k) \) such that \( S(k) \) \( n \)-packs \( C(m(k)) \) and

\[
\lim_{k \to \infty} \frac{(m(k))^2}{k^3} = \frac{1}{\alpha}.
\]

**Proof.** Let \( k \) be a positive integer. Let \( k' \) and \( k'' \) be consecutive terms in the sequence of \( k \)'s produced in Lemma 2.2, \( k' < k \leq k'' \). Let \( m' \) and \( m'' \) be the corresponding values in the sequence of \( m \)'s. Then \( S(k) \) \( n \)-packs \( C(m'') \) and

\[
\frac{(m'')^2}{k^3} = \left( \frac{k''}{k} \right)^3 \frac{(m'')^2}{(k'')^3}.
\]

by the construction in Lemma 2.2, \( \lim_{k \to \infty} (k''/k') = 1 \) and \( \lim_{k \to \infty} (m'')^2/(k'')^3 = 1/\alpha \). Letting \( m(k) = m'' \), the proof is complete.

**Lemma 2.4.** Let \( \alpha > 1/4, \alpha_1 = 0, \) and \( \alpha_{r+1} = \alpha/(1 - \alpha_r) \). Let \( \theta = \cos^{-1}(1/(2\sqrt{\alpha})) \). Then for any positive integer \( r < \pi/\theta \),

\[
\alpha_r = \sqrt{\alpha} \frac{\sin(r - 1)\theta}{\sin r\theta} = 1 - \sqrt{\alpha} \frac{\sin(r + 1)\theta}{\sin r\theta}.
\]

The inductive proof is omitted.

**Lemma 2.5.** Let \( n \geq 3, \ 1/4 < \alpha \leq \frac{1}{4}\sec^2(\pi/n) \). Define \( \alpha_r \) as in Lemma 2.4. Then \( 0 < \alpha_r < 1 \) for \( 2 \leq r \leq n - 2 \) and \( 0 < \alpha_{n-1} \leq 1 \).
Proof. We have
\[ 1 > \frac{1}{\sqrt{2 \alpha}} \geq \cos \frac{\pi}{n}. \]
Thus \( \theta = \cos^{-1}(1/(2\sqrt{\alpha})) \) is less than or equal to \( \pi/n \), or equivalently, \( n \leq \pi/\theta \). By Lemma 2.4, \( \alpha_r > 0 \) for \( r = 2, 3, \ldots, n - 1 \) and \( \alpha_r < 1 \) for \( 2 \leq r \leq n - 2 \). Moreover \( \alpha_{n-1} \leq 1 \), with equality holding only if \( \alpha = \frac{1}{4} \sec^2(\pi/n) \).

**Theorem 2.** For any integer \( n \geq 3 \)
\[ \lim_{k \to \infty} \frac{g(k, n)}{k^{3/2}} \leq 2 \cos \frac{\pi}{n}. \]

Proof. Let \( \epsilon > 0 \). Pick a rational number \( \alpha > 1/4 \) such that
\[ 4 \cos^2 \frac{\pi}{n} + \frac{\epsilon}{2} > \frac{1}{\alpha} \geq 4 \cos^2 \frac{\pi}{n}. \]
Define \( \alpha_r \) as above. Then, by Lemmas 2.3 and 2.5, for \( k \) suitably large,
\[ \frac{g(k, n)^2}{k^3} < \frac{1}{\alpha} + \frac{\epsilon}{2} < 4 \cos^2 \frac{\pi}{n} + \epsilon. \]
Hence
\[ \lim_{k \to \infty} \frac{g(k, n)}{k^{3/2}} \leq 2 \cos \frac{\pi}{n}, \]
as claimed.

We illustrate the construction for \( n = 3, 4, 6, \) and then 5. The first three cases coincide with the constructions given above.

For \( n = 3, \frac{1}{4} \sec^2(\pi/n) = 1 \), a rational number which we may take as \( \alpha \). We then have \( \alpha_1 = 0, \alpha_2 = 1 \), so \( p = L = 1 \). Thus \( b \) may be any integer \( > 1 \),
\[ m = (b + 1)((b - 1)^2 + b) = (b + 1)(b^2 - b + 1) = b^3 + 1 \]
and
\[ k = m/(b + 1) - 1 = b^2 - b. \]

For \( n = 4, \frac{1}{4} \sec^2(\pi/n) = 1/2 \), a rational number which we may take as \( \alpha \). Then we have \( \alpha_1 = 0, \alpha_2 = 1/2, \alpha_3 = 1 \), so \( p = 1 \) and \( L = 2 \). Thus \( b \) must be odd. Moreover,
\[ m = (b + 1)(\frac{1}{2}(b - 1)^2 + b) = (b + 1)(b^2 + 1)/2 \]
and \( k = (b^2 - 1)/2. \)
For \( n = 6 \), \( \frac{1}{4} \sec^2(\pi/n) = 1/3 \), which we may take as \( \alpha \). We have 
\[ \alpha_1 = 0, \quad \alpha_2 = 1/3, \quad \alpha_3 = 1/2, \quad \alpha_4 = 2/3, \quad \alpha_5 = 1, \quad \text{so} \quad p = 1 \quad \text{and} \quad L = 6. \]
Hence \( b \equiv 1 \pmod{6} \),
\[ m = (b + 1)(b^2 + b + 1)/3 \quad \text{and} \quad k = (b^2 + b - 2)/3. \]
In each of these cases \( \frac{1}{4} \sec^2(\pi/n) \) is rational and so can be used as \( \alpha \). For other values of \( n \) this is not possible. Since
\[ \cos^2\frac{\pi}{n} = \frac{1 + \cos(2\pi/n)}{2}, \]
we see that \( (1/4) \sec^2(\pi/n) \) is rational if and only if \( \cos(2\pi/n) \) is. But \( \cos(2\pi/n) \), for \( n \geq 3 \), generates a field of degree \( \varphi(n)/2 \) over the rational field, so is rational only when \( n = 3, 4, \) or 6.

For other values of \( n \), we must let \( \alpha \) be a rational number less than \( \frac{1}{4} \sec^2(\pi/n) \). For example, consider the case \( n = 5 \). We have \( \frac{1}{4} \sec^2(\pi/5) = (3 - \sqrt{5})/2 \). We may choose any rational number less than \( (3 - \sqrt{5})/2 \approx 0.382 \) but as close to it as we please to serve as \( \alpha \), say \( \alpha = 3/8 \). With this choice we have \( \alpha_1 = 0, \quad \alpha_2 = 3/8, \quad \alpha_3 = 3/5, \) and \( \alpha_4 = 15/16. \) Thus \( p = 3 \) and \( L = 80 \), so we choose \( b \equiv 1 \) or \( 161 \pmod{240} \). We have \( m = (b + 1)(3b^2 + 2b + 3)/8 \), \( k = (3b^2 + 2b - 5)/8 \), and \( \lim m^2/k^3 = 8/3 \). Choosing \( b = 241 \) gives a 5-packing with \( m^2/k^3 \approx 2.682 \).

By choosing rational numbers closer to \( \frac{1}{4} \sec^2(\pi/5) \) but less than it, we may produce 5-packings of \( S(k) \) with \( m^2/k^3 \) as close as we please to \( 4 \cos^2(\pi/5) = (3 + \sqrt{5})/2 \).

3. A lower bound on \( g(k, n) \). We next develop a sequence of lemmas that will give a lower bound on \( g(k, n) \) for \( n \geq 3 \). The approach makes use of the smallest positive integers \( x \) and \( y \) in diagrams of the type shown in Fig. 1.1. Let \( t \) be the largest of the sums \( x + y \) for all pairs \( g \) and \( h \) in the packing sets that will be considered. On the one hand, it will be shown that \( m \leq \frac{1}{4}t^3 \sec^2(\pi/n) \), so \( t \geq (4m)^{1/3}\cos^2(\pi/n) \). On the other hand, it will be shown that \( m \geq (k + 1)t - t^2/4 \) and from this that \( t \leq 2(k + 1) - 2\sqrt{(k + 1)^2 - m} \). Combining the two inequalities for \( t \) yields an inequality linking \( k \) and \( m \) from which Theorem 3 will follow.

**Lemma 3.1.** If \( m < (k + 1)^2 \) and \( S(k) \) 2-packs an abelian group \( G \) of order \( m \) with packing set \( \{ \alpha, \beta \} \), then there are integers \( x \) and \( y \) such that \( 1 \leq x, \quad y \leq k, \quad x\alpha + y\beta = 0, \) and \( m \geq (k + 1)(x + y) - xy \).

**Proof.** Consider the \( (k + 1)^2 \) elements \( x\alpha + y\beta \) in \( G \) with \( 0 \leq X, \quad Y \leq k \). Since \( |G| < (k + 1)^2 \), some two of these must be equal; say \( X\alpha + Y\beta = X'\alpha + Y'\beta \) with \( X \geq X' \). Then \( (X - X')\alpha = (Y' - Y)\beta \),
where $0 \leq X - X' \leq k$ and $-k \leq Y' - Y \leq k$. However, since $\{\alpha, \beta\}$ is a packing set for $S(k)$, we must have $1 \leq X - X' \leq k$ and $-k \leq Y' - Y \leq -1$. In other words, $(X - X')\alpha + (Y - Y')\beta = 0$ with $1 \leq X - X' \leq k$ and $1 \leq Y - Y' \leq k$. Pick integers $x$ and $y$ so that $(x, y)$ is as close as possible to $(0, 0)$ such that $x\alpha + y\beta = 0$, $1 \leq x \leq k$, and $1 \leq y \leq k$. We will show that $m \geq (k + 1)(x + y) - xy$.

Consider the elements $X\alpha + Y\beta$ with $0 \leq X, Y \leq k$ and either $X < x$ or $Y < y$. There are $(k + 1)(x + y) - xy$ such elements; we claim that they are distinct.

For suppose two are equal, say $X\alpha + Y\beta = X'\alpha + Y'\beta$ with $X \geq X'$. As before, $1 \leq X - X'$, $Y - Y' \leq k$ and $(X - X')\alpha + (Y - Y')\beta = 0$. Furthermore, either $X < x$ or $Y < y$, so either $X - X' < x$ or $Y - Y' < y$. If both inequalities hold, then $(X - X', Y - Y')$ contradicts the choice of $(x, y)$. So assume, without loss of generality, that $X - X' < x$ and $Y - Y' \geq y$. Then $(x - (X - X'))\alpha = ((Y - Y') - y)\beta$; $1 \leq x - (X - X') \leq k$ and $0 \leq (Y - Y') - y \leq k - y < k$, contradicting the fact that $\{\alpha, \beta\}$ is a packing set for $S(k)$. Hence the $(k + 1)(x + y) - xy$ elements are distinct, implying that $m \geq (k + 1)(x + y) - xy$.

**Lemma 3.2.** Assume that $\{\alpha, \beta, \gamma\}$ is a packing set for $S(k)$ in a group $G$ of order $m < 2(k + 1)^{3/2}$. Then $\{\alpha, \beta, \gamma\}$ generates $G$.

**Proof.** Let $H$ be the subgroup of $G$ generated by $\{\alpha, \beta, \gamma\}$. As was shown in [St1], $(k + 1)^3 \leq |H|^2$. If $H$ is a proper subgroup of $G$, $|H| \leq |G|/2$. Thus

$$(k + 1)^3 \leq \frac{m^2}{4}$$

so $m \geq 2(k + 1)^{3/2}$. This contradiction establishes the lemma.

Let $\alpha, \beta, \gamma$ be nonzero elements in $C(p)$ for some prime $p$. Assume that $a, a', b, b', c, c'$ are integers not divisible by $p$ such that

$$a\beta + a'\gamma = b\gamma + b'\alpha = c\alpha + c'\beta = 0.$$ 

Then, in the field $GF(p)$ we have

$$\frac{a}{a'} = -\frac{\gamma}{\beta}, \quad \frac{b}{b'} = -\frac{\alpha}{\gamma}, \quad \frac{c}{c'} = -\frac{\beta}{\alpha}.$$ 

Thus, in $GF(p)$,

$$\frac{a}{a'} \frac{b}{b'} \frac{c}{c'} = -1 \quad \text{so} \quad abc + a'b'c' = 0.$$ 

That is, $p|abc + a'b'c'$. The next lemma generalizes this fact to all finite abelian groups.
Lemma 3.3. Let $G$ be a finite abelian group of order $m$. Let $\alpha$, $\beta$, and $\gamma$ generate $G$ and let $a$, $b$, $c$, $a'$, $b'$, $c'$ be integers such that

$$a\beta + a'\gamma = b\gamma + b'\alpha = c\alpha + c'\beta = 0;$$

as in Fig. 3.1.

Then

$$m|abc + a'b'c'. $$

Proof. Consider the $\mathbb{Z}$-lattice in $\mathbb{R}^3$,

$$L = \{(x, y, z)|x\alpha + y\beta + z\gamma = 0\}. $$

Since $\alpha$, $\beta$, $\gamma$ generate $G$, $\mathbb{Z}^3/L \cong G$, and thus $|Z^3/L| = m$. Let $K$ be the lattice generated by $(0, a, a'), (b', 0, b), (c, c', 0)$. The determinant

$$\begin{vmatrix}
0 & a & a' \\
b' & 0 & b \\
c & c' & 0 \\
\end{vmatrix}$$

is equal to $abc + a'b'c'$. Since $K$ is a sublattice of $L$, $|Z^3: L|$ divides $|Z^3: K|$. That is, $m$ divides $abc + a'b'c'$, which was to be proved.

We now begin the proof of Theorem 3, which will incorporate further lemmas at the appropriate points in the argument.

Theorem 3. If $n \geq 3$, $k \geq 1$, $m \geq 1$, and $S(k)$ $n$-packs an abelian group of order $m$, then

$$k + 1 \leq \left(4\cos^2\frac{\pi}{n}\right)^{-1/3} m^{2/3} + \frac{1}{4} \left(4\cos^2\frac{\pi}{n}\right)^{1/3} m^{1/3}. $$

Proof. Suppose not. Then

$$k + 1 > \left(x + \frac{1}{4x}\right)\sqrt{m} \quad \text{where} \quad x = \left(4\cos^2\frac{\pi}{n}\right)^{-1/3} m^{1/6}. $$

But $x + 1/4x \geq 1$ for $x > 0$, so $m < (k + 1)^2$. 
Let the packing set be \{g_0, \ldots, g_{n-1}\}. Let \( K = k + 1 \). By Lemma 3.1, for \( i \neq j \), there are integers \( a_{ij} \) with \( 1 \leq a_{ij} \leq k \), \( a_{ij}g_i + a_{ji}g_j = 0 \), and \( m \geq K(a_{ij} + a_{ji}) - a_{ij}a_{ji} \).

**Lemma 3.4.** Let \( m, K, a, a' \) be positive real numbers such that \( a, a' \leq K \) and \( K^2 \geq m \geq K(a + a') - aa' \). Let \( t = a + a' \). Then \( t \leq 2K - 2\sqrt{K^2 - m} \).

**Proof.** We have \( m \geq Kt - aa' \). Since \( a + a' = t \), the largest possible value of \( aa' \) is \( t^2/4 \). Hence \( m \geq Kt - t^2/4 \) so \( t^2 - 4Kt \geq -4m \). Completing the square shows that \((2K - t)^2 \geq 4K^2 - 4m \) and, since \( 2K - t \geq 0 \), \( 2K - t \geq \sqrt{4K^2 - 4m} \), from which the lemma follows.

**Proof of Theorem 3 continued.** Let \( t = \max_{0 \leq i < j \leq n-1}(a_{ij} + a_{ji}) \). By Lemma 3.4, \( t \leq 2K - 2\sqrt{K^2 - m} \).

Note that

\[
K > \left(4 \cos^2 \frac{\pi}{n}\right)^{-1/3} m^{2/3} + \frac{1}{4} \left(4 \cos^2 \frac{\pi}{n}\right)^{1/3} m^{1/3} > \left(\frac{m}{2}\right)^{2/3}
\]

so \( m < 2K^{3/2} \). By Lemma 3.2, if \( i, j, \) and \( l \) are distinct indices between 0 and \( n - 1 \), then \( \{g_i, g_j, g_l\} \) generates \( G \). By Lemma 3.3, \( m|a_{ij}a_{jl}a_{li} + a_{ji}a_{lj}a_{il}| \) so \( m \leq a_{ij}a_{jl}a_{li} + a_{ji}a_{lj}a_{il} \).

Let \( b_{ij} = a_{ij}/t \). Then we have \( b_{ij} \geq 0 \), \( b_{ij} + b_{ji} \leq 1 \), and \( m \leq t^3(b_{ij}b_{ji}b_{lj} + b_{ji}b_{lj}b_{ij}) \). The next two lemmas will allow us to derive a relationship between \( m, t, \) and \( n \) from these inequalities.

**Lemma 3.5.** Let \( n \) be an integer \( \geq 3 \). Let \( x_1, x_2, \ldots, x_{n-1} \) be real numbers, \( 0 \leq x_i \leq 1 \). Then there are distinct indices \( j \) and \( l \) such that

\[
x_j(1 - x_i) \quad \text{and} \quad x_i(1 - x_j)
\]

are both less than or equal to \( \frac{1}{4} \sec^2(\pi/n) \). This is best possible in the sense that \( \frac{1}{4} \sec^2(\pi/n) \) cannot be replaced by a smaller number.

**Proof.** Let \( \alpha = \frac{1}{4} \sec^2(\pi/n) \), \( \alpha_1 = 0 \) and \( \alpha_{i+1} = \alpha/(1 - \alpha_i) \). By Lemmas 2.4 and 2.5, \( 0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} = 1 \), and the interval \([0, 1]\) is partitioned into \( n - 2 \) sections, \([\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \ldots, [\alpha_{n-2}, \alpha_{n-1}] \). Hence some section, say \([\alpha_p, \alpha_{p+1}]\), contains a pair \( x_j \) and \( x_l \), \( l \neq j \). We then have

\[
x_j(1 - x_l) \leq \alpha_{p+1}(1 - \alpha_p) = \alpha
\]
and

\[ x_i(1 - x_j) \leq \alpha_{p+1}(1 - \alpha_p) = \alpha. \]

To show that this result is best possible, consider the sequence

\[ x_i = \alpha_i, \ i = 1, 2, \ldots, n - 1. \]

Note that \( x_i + 1(1 - x_i) = \alpha. \) Thus, if \( j > i, \)
\( x_j(1 - x_i) \geq \alpha. \) Hence, if \( j \neq l \) at least one of \( x_j(1 - x_i) \)
and \( x_l(1 - x_j) \) is \( \geq \alpha = \frac{1}{4}\sec^2(\pi/n). \)

**Lemma 3.6.** Let \( n \) be an integer \( \geq 3. \) For \( 0 \leq i, j \leq r - 1, \) let \( b_{ij} \) be nonnegative real numbers such that \( b_{ij} + b_{ji} \leq 1. \) Then for some \( j \) and \( l, 0 < j < l < r - 1, \)

\[ b_{0j}b_{jl}b_{10} + b_{j0}b_{lj}b_{0l} \leq \frac{1}{4}\sec^2\left(\frac{\pi}{n}\right). \]

**Proof.** Let \( x_i = b_{0i}, \ i = 1, 2, \ldots, n - 1. \) By Lemma 3.5, there are distinct indices \( j \) and \( l \) such that \( x_j(1 - x_i) \) and \( x_l(1 - x_j) \) are both \( \leq \frac{1}{4}\sec^2(\pi/n). \) Then

\[ b_{0j}b_{jl}b_{10} + b_{j0}b_{lj}b_{0l} \leq (b_{jl} + b_{lj})\max(b_{0j}b_{10}, b_{j0}b_{0l}) \]

\[ \leq 1 \cdot \max(b_{0j}(1 - b_{0l}), b_{0l}(1 - b_{0j})) \leq \frac{1}{4}\sec^2(\pi/n). \]

**Proof of Theorem 3 continued.** By Lemma 3.6 we have \( m \leq (t^3/4)\sec^2(\pi/n) \) so \( t \geq (4\cos^2(\pi/n))^{1/3}m^{1/3}. \) Combining this with the inequality \( t \leq 2K - 2\sqrt{K^2 - m} \) proved above, we obtain \( C \leq 2K - 2\sqrt{K^2 - m}, \) where \( C = (4\cos^2(\pi/n))^{1/3}m^{1/3}. \) Hence \( 2\sqrt{K^2 - m} \leq 2K - C. \) Squaring and simplifying gives \( K \leq m/C + C/4 \) from which Theorem 3 follows.

For \( n \geq 3, \) Theorem 3 implies that

\[ \lim_{k \to \infty} \frac{g(k, n)}{k^{3/2}} \geq 2\cos\frac{\pi}{n}. \]

Combining this with Theorem 2 completes the proof of Theorem 1.

**4. Some questions.** For \( n = 3, 4 \) and \( 6 \) a stronger version of Theorem 3 holds, namely \( k + 1 \leq (4\cos^2(\pi/n))^{-1/3}m^{2/3}. \) The case \( n = 3 \) is treated in [St1] and the case \( n = 4 \) by Hickerson through a method that does not seem to generalize to larger values of \( n. \) These facts suggest two questions.

Let \( n \geq 3 \) and \( k \geq 1. \) Is \( g(k, n)/(k + 1)^{3/2} \geq 2\cos(\pi/n)? \)

For \( n \geq 3 \) what is the exact value of \( g(k, n)? \)

The cases \( n = 3, 4, \) and \( 6 \) also suggest the following question:

Let \( g'(k, n) \) be the smallest value of \( m \) for which \( S(k) \) \( n \)-packs \( C(m) \)
with a packing set which is a multiplicative subgroup of the ring of
integers mod $m$. What is \( \lim_{k \to \infty} \left( g'(k, n)/k^{3/2} \right) \)? Even for $n = 5$ the answer is not known.

See [St2] for further information about $g(k, n)$ and a discussion of related problems.

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Received June 14, 1984.

UNIVERSITY OF CALIFORNIA
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Pacific Journal of Mathematics
Vol. 122, No. 1       January, 1986

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