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FITTING STRUCTURES

KARSTEN JOHNSEN AND HARMUT LAUE

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Motivated by papers of H. Fitting, the problem arises whether there exists a ring which contains a given ring and a semigroup acting on each other. This problem is solved in the affirmative by the construction of a “universal envelopment”. Furthermore, the situation investigated gives rise to a generalized wreath product which is used for a description of certain automorphism groups.

0. Introduction. The endomorphisms of an abelian group form a ring in a natural and well-known way, whereas in the case of a nonabelian group one has no general “addition” of endomorphisms. One easily proves that the “sum” of two endomorphisms α, β of a group G ,

$$\alpha + \beta: g \mapsto g^\alpha g^\beta \quad \text{for all } g \in G,$$

is an endomorphism of G if and only if $[G^\alpha, G^\beta] = 1$. Hence the endomorphisms that can be added to any endomorphism are exactly the homomorphisms of G into its center $Z(G)$. In this sense, the ring $\text{Hom}(G, Z(G))$ is a “pleasant” substructure of $\text{End}(G)$. Long ago, Fitting [2] described the structure of $\text{End}_G(G)$ which, though not a ring, still has numerous ring properties; if we put $H := \text{End}_G(G)$, $S := \text{Hom}(G, Z(G))$, then e.g.

- (1) $(s_1 h) s_2 = s_1 (h s_2) \quad \text{for all } h \in H, s_1, s_2 \in S,$
- (2) $(h_1 + s) h_2 = h_1 h_2 + s h_2, \quad h_1 (h_2 + s) = h_1 h_2 + h_1 s$
for all $h_1, h_2 \in H, s \in S,$
- (3) $(h + s_1) s_2 = h s_2 + s_1 s_2, \quad s_1 (h + s_2) = s_1 h + s_1 s_2$
for all $h \in H, s_1, s_2 \in S.$

Keeping these rules as axioms, we introduce so-called *Fitting structures* in the first chapter of this paper and show that firstly there does exist a ring R containing H and S such that (1), (2), (3) are special cases of its associative and distributive laws, and that secondly any ring with this property (if—which is a non-essential restriction—it is generated by H) is a homomorphic image of R . Emanating naturally from Fitting’s notion of “Bereich” [2], the problem of the existence of enveloping rings for Fitting structures, which has been solved for $H = \text{End}_G(G)$ by Fitting

in a special way, thus finds a general positive answer. Chapter 2 shows that even if there is not given an addition of elements of H and S , such an addition can be defined after a certain enlargement of H , and that this process of making H and S into a Fitting structure is essentially uniquely determined by the actions of H on S . In Chapter 3, we introduce to each Fitting structure a *generalized wreath product* containing the usual wreath product of (semi-)groups as a special case, and use this concept to give a simple description of certain automorphism groups, applying a result of another paper by Fitting [1].

1. Fitting structures. For every ring¹ S , we put

$$\text{End}_\Lambda(S) := \{ \alpha \mid \alpha \in \text{End}(S, +), (s_1 s_2)^\alpha = s_1 (s_2^\alpha) \text{ for all } s_1, s_2 \in S \},$$

$$\text{End}_P(S) := \{ \alpha \mid \alpha \in \text{End}(S, +), (s_1 s_2)^\alpha = (s_1^\alpha) s_2 \text{ for all } s_1, s_2 \in S \}.$$

Obviously, $\text{End}_\Lambda(S)$ and $\text{End}_P(S)$ are subrings of $\text{End}(S, +)$.

1.1. DEFINITION. Let H be a semigroup, S a ring, φ a homomorphism of H into the multiplicative semigroup of $\text{End}_\Lambda(S)$, ψ an antihomomorphism of H into the multiplicative semigroup of $\text{End}_P(S)$ such that H^φ and H^ψ commute elementwise. Let σ be a homomorphism of $(S, +)$ into the symmetric group \mathfrak{S}_H on H such that $h^{s^\sigma} = h$ implies $s = 0$ for all $h \in H$, $s \in S$ (i.e., $(S, +)$ “acts freely” on H).

If $h \in H$, $s \in S$, we write sh for s^{h^φ} , hs for s^{h^ψ} , $h + s$ for h^{s^σ} . The 5-tuple $(H, S, \varphi, \psi, \sigma)$ is called a *Fitting structure* if (1), (2), (3) hold.²

By definition, we have

$$(4) \quad (sh_1)h_2 = s(h_1h_2), \quad h_1(h_2s) = (h_1h_2)s$$

for all $h_1, h_2 \in H$, $s \in S$,

$$(5) \quad (h + s_1) + s_2 = h + (s_1 + s_2) \quad \text{for all } h \in H, s_1, s_2 \in S,$$

$$(6) \quad (s_1s_2)h = s_1(s_2h), \quad h(s_1s_2) = (hs_1)s_2$$

for all $h \in H$, $s_1, s_2 \in S$,

$$(7) \quad (h_1s)h_2 = h_1(sh_2) \quad \text{for all } h_1, h_2 \in H, s \in S$$

$$(8) \quad h + s = h \Leftrightarrow s = 0 \quad \text{for all } h \in H, s \in S.$$

¹All rings in this paper are associative, but do not necessarily have an identity element.

²If H is a “Bereich” in the sense of Fitting [2], let S be the set of all elements of H which can be added to any element of H . Then S is a ring, and we get a Fitting structure with the additional property that S is contained in H , and H has an identity element. Any further possibilities to add elements of H (which might exist in Fitting’s “Bereich”) are treated as non-existent in our Fitting structures.

If H has an identity element 1, then (2) and (8) imply $1^\varphi = \text{id} = 1^\psi$.

Fitting structures $\mathcal{F} = (H, S, \varphi, \psi, \sigma)$, $\mathcal{F}' = (H', S', \varphi', \psi', \sigma')$ are called *isomorphic* if there are isomorphisms α of H onto H' , β of S onto S' with the properties $\beta\sigma' = \sigma\bar{\alpha}$, $\alpha\varphi' = \varphi\bar{\beta}$, $\alpha\psi' = \psi\bar{\beta}$, where $\bar{\alpha}$ is the isomorphism of \mathfrak{S}_H onto $\mathfrak{S}_{H'}$ induced by α (such that $\pi^{\bar{\alpha}} = \alpha^{-1}\pi\alpha$ for all $\pi \in \mathfrak{S}_H$), and $\bar{\beta}$ is the isomorphism of $\text{End}(S)$ onto $\text{End}(S')$ induced by β (such that $\zeta^{\bar{\beta}} = \beta^{-1}\zeta\beta$ for all $\zeta \in \text{End}(S)$).

\mathcal{F}' is called a *Fitting substructure* of \mathcal{F} if H' is a subsemigroup of H , S' is a subring of S , and $\varphi' = \varphi|_{H'}$, $\psi' = \psi|_{H'}$, $\sigma' = \sigma|_{S'}$. If H' is a subsemigroup of H , S' a subring of S , then $(H', S', \varphi|_{H'}, \psi|_{H'}, \sigma|_{S'})$ is a Fitting substructure of \mathcal{F} if and only if $S'H' \subseteq S'$, $H'S' \subseteq S'$, and $H' + S' = H'$.

1.2. DEFINITION. Let \mathcal{F} be a Fitting structure, R a ring, $\tilde{}$ a homomorphism of H into the multiplicative semigroup of R , and $\bar{}$ a homomorphism of S onto an ideal of R . The triple $(R, \tilde{}, \bar{})$ is called an *envelopment* of \mathcal{F} if

$$(9) \quad \tilde{h} + \bar{s} = \widetilde{h + s} \quad \text{for all } h \in H, s \in S$$

holds.

The envelopment $(R, \tilde{}, \bar{})$ is called *faithful* if $\tilde{}$ and $\bar{}$ are injective.

From (9), we conclude

$$(10) \quad \tilde{h}\bar{s} = \overline{hs}, \bar{s}\tilde{h} = \overline{sh} \quad \text{for all } h \in H, s \in S,$$

since

$$\widetilde{h^2} + \bar{s}\tilde{h} = \widetilde{h^2 + sh} = \widetilde{(h + s)h} = \widetilde{h + sh} = (\tilde{h} + \bar{s})\tilde{h} = \widetilde{h^2} + \bar{s}\tilde{h},$$

and the second part of (10) is proved similarly.

1.3. DEFINITION. Let \mathcal{F} be a Fitting structure and $(R, \tilde{}, \bar{})$, $(R', \tilde{'}, \bar{'})$ envelopments of \mathcal{F} . Then a mapping χ is called a *homomorphism* of $(R, \tilde{}, \bar{})$ into $(R', \tilde{'}, \bar{'})$ if χ is a homomorphism of R into R' such that $\tilde{h}^\chi = \tilde{h}'$, $\bar{s}^\chi = \bar{s}'$ for all $h \in H, s \in S$. If χ is an isomorphism of R onto R' , we call our envelopments *isomorphic*. An envelopment \mathcal{U} of \mathcal{F} is called *universal* if for any envelopment \mathcal{V} of \mathcal{F} there is a homomorphism of \mathcal{U} into \mathcal{V} .

Universal envelopments of isomorphic Fitting structures are isomorphic. We now prove the following existence theorem:

1.4. THEOREM. *Every Fitting structure has a faithful universal envelopment.*

Proof. Let $\mathcal{F} = (H, S, \varphi, \psi, \sigma)$ be the Fitting structure given. We put $T := \mathbb{Z}H$ and write $\hat{+}$, $\hat{\cdot}$ for the standard addition and multiplication in T . (Then $h_1 \hat{\cdot} h_2 = h_1 h_2$ for all $h_1, h_2 \in H$.) For $h \in H$, $s \in S$ define

$$\delta(h, s) := (h + s) \hat{\cdot} h$$

and let K be the additive subgroup of T generated by $\{\delta(h_1, s) \hat{\cdot} \delta(h_2, s) \mid h_1, h_2 \in H, s \in S\}$. Then

$$(11) \quad h_1 \hat{\cdot} \delta(h_2, s) \hat{\cdot} \delta(h_2, h_1 s), \delta(h_2, s) \hat{\cdot} h_1 \hat{\cdot} \delta(h_2, s h_1) \in K$$

for all $h_1, h_2 \in H, s \in S$,

as

$$\begin{aligned} h_1 \hat{\cdot} (h_2, s) \hat{\cdot} \delta(h_2, h_1 s) &= h_1 \hat{\cdot} ((h_2 + s) \hat{\cdot} h_2) \hat{\cdot} \delta(h_2, h_1 s) \\ &= h_1 (h_2 + s) \hat{\cdot} h_1 h_2 \hat{\cdot} \delta(h_2, h_1 s) \\ &= \delta(h_1 h_2, h_1 s) \hat{\cdot} \delta(h_2, h_1 s) \in K, \end{aligned}$$

the second part of (11) being proved analogously. Obviously, (11) yields

$$(12) \quad h_1 \hat{\cdot} (\delta(h_2, s) \hat{\cdot} \delta(h_3, s)), (\delta(h_2, s) \hat{\cdot} \delta(h_3, s)) \hat{\cdot} h_1 \in K$$

for all $h_1, h_2, h_3 \in H, s \in S$.

Thus K is an ideal of T , and, by definition of K ,

$$(13) \quad \delta(h_1, s) \hat{+} K = \delta(h_2, s) \hat{+} K \quad \text{for all } h_1, h_2 \in H, s \in S.$$

We now define

$$R := T/K,$$

$$\tilde{\cdot} : H \rightarrow R, \quad h \mapsto h \hat{+} K,$$

and for an arbitrary $h \in H$

$$\bar{\cdot} : S \rightarrow R, \quad s \mapsto \delta(h, s) \hat{+} K.$$

(By (13), $\bar{\cdot}$ is independent of the choice of h .) Obviously, $\tilde{\cdot}$ is a homomorphism of H into the multiplicative semigroup of R , and

$$\begin{aligned} \widetilde{h + s} &= (h + s) \hat{+} K = h \hat{+} (h + s) \hat{\cdot} h \hat{+} K \\ &= (h \hat{+} K) \hat{+} (\delta(h, s) \hat{+} K) = \tilde{h} \hat{+} \bar{s} \quad \text{for all } h \in H, s \in S, \end{aligned}$$

whence (9) holds. We want to show that $\bar{\cdot}$ is a ring homomorphism of S into an ideal of R , and start with

$$(14) \quad \delta(h, s_1 + s_2) \hat{\cdot} \delta(h, s_1) \hat{\cdot} \delta(h, s_2) \in K \quad \text{for all } h \in H, s_1, s_2 \in S.$$

For

$$\begin{aligned} \delta(h, s_1 + s_2) \hat{\cdot} \delta(h, s_1) \hat{\cdot} \delta(h, s_2) &= (h + s_1 + s_2) \hat{\cdot} h \hat{\cdot} (h + s_1) \hat{\cdot} h \hat{\cdot} \delta(h, s_2) \\ &= \delta(h + s_1, s_2) \hat{\cdot} \delta(h, s_2) \in K. \end{aligned}$$

Furthermore,

$$(15) \quad \delta(h, s_1 s_2) \hat{=} \delta(h, s_1) \hat{+} \delta(h, s_2) \in K \quad \text{for all } h \in H, s_1, s_2 \in S,$$

since

$$\begin{aligned} \delta(h, s_1 s_2) &\hat{=} \delta(h, s_1) \hat{+} \delta(h, s_2) \\ &= \delta(h, s_1 s_2) \hat{+} ((h + s_1) \hat{+} h) \hat{+} ((h + s_2) \hat{+} h) \\ &= \delta(h, s_1 s_2) \hat{+} (h + s_1)(h + s_2) \hat{+} (h + s_1)h \hat{+} h(h + s_2) \hat{+} h^2 \\ &= \delta(h, s_1 s_2) \hat{+} \delta((h + s_1)h, (h + s_1)s_2) \hat{+} \delta(h^2, hs_2) \in K, \end{aligned}$$

by (13) and (14).

As (14) and (15) show, $\bar{}$ is a ring homomorphism, and by (12), \bar{S} is an ideal of R . Therefore, $(R, \bar{}, \bar{})$ is an envelopment of \mathcal{F} . We need some preliminaries to show that $\bar{}$ and $\bar{}$ are injective:

Let D be the additive subgroup of T generated by $\{\delta(h, s) | h \in H, s \in S\}$. Then $K \leq D$, and $D/K = \bar{S}$. Since

$$(16) \quad \hat{=} \delta(h, s) = \delta(h + s, -s) \quad \text{for all } h \in H, s \in S,$$

every element of D has the form $\sum_j \delta(h_j, s_j)$ for appropriate $h_j \in H, s_j \in S$. We claim:

$$(17) \quad \sum_{j=1}^k \delta(h_j, s_j) = 0 \Rightarrow \sum_{j=1}^k s_j = 0$$

for all $h_1, \dots, h_k \in H, s_1, \dots, s_k \in S$.

Suppose $\sum_{j=1}^k \delta(h_j, s_j) = 0$. Then we have $\sum_{j=1}^k (h_j + s_j) = \sum_{j=1}^k h_j$. Since $(T, \hat{+})$ is free over H , there is a permutation π of $\{1, \dots, k\}$ such that $h_j + s_j = h_{j\pi}$ for all $j \in \{1, \dots, k\}$. For $i \in \{1, \dots, k\}$, let f_i be the smallest positive integer such that $i\pi^{f_i} = i$. Then $\{i, i\pi, \dots, i\pi^{f_i-1}\}$ is the orbit of i under π , and $h_i + s_i + s_{i\pi} + \dots + s_{i\pi^{l-1}} = h_{i\pi^l}$ for $0 \leq l < f_i$, hence in particular $h_i + s_i + s_{i\pi} + \dots + s_{i\pi^{f_i-1}} = h_i$, and $s_i + s_{i\pi} + \dots + s_{i\pi^{f_i-1}} = 0$ by (8). Now if X denotes a full set of representatives of the orbits of π in $\{1, \dots, k\}$,

$$\sum_{j=1}^k s_j = \sum_{i \in X} (s_i + s_{i\pi} + \dots + s_{i\pi^{f_i-1}}) = 0,$$

proving (17).

As a consequence, we have

$$(18) \quad \sum_{j=1}^k \delta(h_j, s_j) = \sum_{j=1}^{k'} \delta(h'_j, s'_j) \Rightarrow \sum_{j=1}^k s_j = \sum_{j=1}^{k'} s'_j$$

for all $h_1, \dots, h_k, h'_1, \dots, h'_{k'} \in H, s_1, \dots, s_k, s'_1, \dots, s'_{k'} \in S$.

Therefore, $\rho: \sum_{j=1}^k \delta(h_j, s_j) \mapsto \sum_{j=1}^k s_j$ defines a mapping of D into S which obviously is an additive homomorphism. We claim:

$$(19) \quad K = \ker \rho.$$

By (16) and the definition of K , we have $K \subseteq \ker \rho$. One generally has $\delta(h_j, s_j) \equiv \delta(h_1 + s_1 + \cdots + s_{j-1}, s_j) \pmod{K}$ for $h_1, \dots, h_j \in H$, $s_1, \dots, s_j \in S$. If now $\sum_{j=1}^k \delta(h_j, s_j) \in \ker \rho$, then $\sum_{j=1}^k s_j = 0$, and consequently

$$\begin{aligned} \sum_{j=1}^k \delta(h_j, s_j) &\equiv \sum_{j=1}^k \delta(h_1 + s_1 + \cdots + s_{j-1}, s_j) \pmod{K} \\ &= ((h_1 + s_1) \hat{+} h_1) \hat{+} ((h_1 + s_1 + s_2) \hat{+} (h_1 + s_1)) \\ &\quad \hat{+} \cdots \hat{+} ((h_1 + s_1 + \cdots + s_k) \hat{+} (h_1 + s_1 + \cdots + s_{k-1})) \\ &= \hat{+} h_1 \hat{+} (h_1 + s_1 + \cdots + s_k) = 0, \end{aligned}$$

i.e., $\sum_{j=1}^k \delta(h_j, s_j) \in K$.

If $h \in H$ and $s \in S \setminus \{0\}$, then $\delta(h, s) \notin K$ by (19), and this means

$$(20) \quad \tilde{} \text{ is injective.}$$

We now want to show that $\tilde{}$ is injective which we shall conclude from

$$(21) \quad h \hat{+} h' \in D \Rightarrow h = h' + s \quad \text{with } s \in S, \text{ for all } h, h' \in H.$$

We reformulate (21) in the following form to make it accessible to an induction argument:

Suppose $h, h' \in H$ and $r \in \mathbb{N}$. If there are $h_1, \dots, h_r \in H$, $s_1, \dots, s_r \in S$ such that $h \hat{+} h' = \sum_{j=1}^r \delta(h_j, s_j)$, then there is an element $s \in S$ such that $h = h' + s$.

If $r = 1$, then $h = h_1 + s_1 = h' + s_1$, as $(T, \hat{+})$ is free over H . Now suppose $r > 1$ and (22) is true for $r - 1$ instead of r . Since $h \hat{+} h' = \sum_{j=1}^r \delta(h_j, s_j)$, we may assume $h = h_1 + s_1$, $h' = h_r$. Furthermore, $h_1 = h_i + s_i$ with $i \in \{2, \dots, r\}$. This yields

$$\begin{aligned} h \hat{+} h' &= (h_i + s_i + s_1) \hat{+} (h_i + s_i) \hat{+} \sum_{j=2}^r \delta(h_j, s_j) \\ &= (h_i + s_i + s_1) \hat{+} h_i \hat{+} \sum_{\substack{j=2 \\ j \neq i}}^r \delta(h_j, s_j) \\ &= \delta(h_i, s_i + s_1) \hat{+} \sum_{\substack{j=2 \\ j \neq i}}^r \delta(h_j, s_j), \end{aligned}$$

and an application of the induction hypothesis yields our claim. This proves (22) and the equivalent assertion (21) by means of which we conclude

(23) $\tilde{}$ is injective:

For if $h, h' \in H$ and $h \hat{=} h' \in K$, then a fortiori $h \hat{=} h' \in D$ and therefore $h = h' + s$ with $s \in S$. But then $h \hat{=} h' = \delta(h', s)$, and (20) implies $s = 0$. Therefore we have $h = h'$, and the proof of (23) is complete.

It remains to show that the envelopment $(R, \tilde{}, \bar{})$ is universal. To this end let $(R', \tilde{}, \bar{})$ be an envelopment of \mathcal{F} . Since $(T, \hat{+})$ is free over H ,

$$\chi_0: T \rightarrow R', \quad \sum_j z_j h_j \mapsto \sum_j z_j \tilde{h}_j \quad (z_j \in \mathbb{Z})$$

defines a ring homomorphism. We show

(24) $K \subseteq \ker \chi_0$.

Let $h_1, \dots, h_k \in H, s_1, \dots, s_k \in S$ and $\sum_{j=1}^k \delta(h_j, s_j) \in K$. Then

$$\begin{aligned} \left(\sum_{j=1}^k \delta(h_j, s_j) \right)^{\chi_0} &= \sum_{j=1}^k ((h_j + s_j)^{\chi_0} - h_j^{\chi_0}) \\ &= \sum_{j=1}^k (\widetilde{h_j + s_j} - \tilde{h}_j) = \sum_{j=1}^k (\tilde{h}_j + \bar{s}_j - \tilde{h}_j) \\ &= \sum_{j=1}^k \bar{s}_j = \overline{\sum_{j=1}^k s_j} = 0, \end{aligned}$$

by (9) and (19).

By (24),

$$\chi: R \rightarrow R', \quad \sum_j z_j h_j \hat{+} K \mapsto \sum_j z_j \tilde{h}_j \quad (z_j \in \mathbb{Z})$$

defines a homomorphism, and for all $h \in H, s \in S$ we have

$$\begin{aligned} \tilde{h}^x &= (h \hat{+} K)^x = \tilde{h}, \\ \bar{s}^x &= (\delta(h, s) \hat{+} K)^x = \widetilde{h + s} - \tilde{h} = \tilde{h} + \bar{s} - \tilde{h} = \bar{s}, \end{aligned}$$

by (9). This completes the proof of our theorem.

2. Fitting pre-structures.

2.1. DEFINITION. Let H be a semigroup, S a ring, φ a homomorphism of H into the multiplicative semigroup of $\text{End}_\Lambda(S)$, ψ an antihomomorphism of H into the multiplicative semigroup of $\text{End}_p(S)$ such

that H^φ and H^ψ commute elementwise. (We use the notations introduced in 1.1.) The 4-tuple $\mathcal{F}_0 := (H, S, \varphi, \psi)$ is called a *Fitting pre-structure* if condition (1) holds.

Let $\mathcal{F} = (H^*, S^*, \varphi^*, \psi^*, \sigma^*)$ be a Fitting structure with $S = S^*$, and μ a monomorphism of H into H^* . The pair (\mathcal{F}, μ) is called a *continuation* of \mathcal{F}_0 if

$$(25) \quad \varphi = \mu\varphi^*, \quad \psi = \mu\psi^*$$

holds.

Isomorphisms of Fitting pre-structures and Fitting sub-pre-structures are defined in complete analogy to the corresponding notions for Fitting structures, the conditions on σ, σ' being omitted.

If (\mathcal{F}_1, μ_1) and (\mathcal{F}_2, μ_2) with $\mathcal{F}_1 = (H^*, S^*, \varphi^*, \psi^*, \sigma^*)$, $\mathcal{F}_2 = (H^{**}, S^{**}, \varphi^{**}, \psi^{**}, \sigma^{**})$ are continuations of \mathcal{F}_0 , then a *homomorphism* of (\mathcal{F}_1, μ_1) into (\mathcal{F}_2, μ_2) is defined to be a homomorphism ω of H^* into H^{**} with the property

$$(26) \quad \mu_2 = \mu_1\omega.$$

If ω is a bijection of H^* onto H^{**} , our continuations are called *isomorphic*. A continuation \mathcal{F} of \mathcal{F}_0 is called *universal* if for any continuation \mathcal{F}' of \mathcal{F}_0 there is a homomorphism of \mathcal{F} into \mathcal{F}' .

Universal continuations of isomorphic Fitting pre-structures are isomorphic. We now prove the following existence theorem:

2.2. THEOREM. *Every Fitting pre-structure has a universal continuation.*

Proof. Let $\mathcal{F}_0 = (H, S, \varphi, \psi)$ be the Fitting pre-structure given. We put $H^{\mathcal{F}_0} := H \times S$ and define

$$(h_1, s_1)(h_2, s_2) := (h_1h_2, s_1h_2 + h_1s_2 + s_1s_2)$$

for all $h_1, h_2 \in H$, $s_1, s_2 \in S$. One readily verifies that $H^{\mathcal{F}_0}$ is a semi-group and the mapping

$$\mu: H \rightarrow H^{\mathcal{F}_0}, \quad h \mapsto (h, 0)$$

is a monomorphism. We call $H^{\mathcal{F}_0}$ the *continuation semigroup* of \mathcal{F}_0 . We put $S^* := S$, and define for all $h \in H$, $s \in S$

$$(h, s)^{\varphi^*}: S \rightarrow S, \quad r \mapsto rh + rs$$

$$(h, s)^{\psi^*}: S \rightarrow S, \quad r \mapsto hr + sr.$$

Then $(h, s)^{\varphi^*} \in \text{End}_\Lambda(S)$, $(h, s)^{\psi^*} \in \text{End}_P(S)$, since

$$(r_1r)h + (r_1r)s = r_1(rh) + r_1(rs) = r_1(rh + rs),$$

$$h(rr_1) + s(rr_1) = (hr)r_1 + (sr)r_1 = (hr + sr)r_1$$

for all $h \in H$, $r, r_1, s \in S$. We show

(27) φ^* is a homomorphism of $H^{\mathcal{F}_0}$ into $\text{End}_\Delta(S)$, ψ^* is an antihomomorphism of $H^{\mathcal{F}_0}$ into $\text{End}_P(S)$, and (25) holds.

We confine ourselves to the assertions about φ, φ^* and leave the proof of the assertions about ψ, ψ^* to the reader. For all $h, h_1, h_2 \in H$, $r, s_1, s_2 \in S$ we have

$$\begin{aligned} r((h_1, s_1)(h_2, s_2))^{\varphi^*} &= r(h_1 h_2, s_1 h_2 + h_1 s_2 + s_1 s_2)^{\varphi^*} \\ &= (rh_1 + rs_1)h_2 + (rh_1 + rs_1)s_2 = r^{(h_1, s_1)\varphi^*}(h_2, s_2)^{\varphi^*} \end{aligned}$$

and

$$r^{h\varphi} = rh + r \cdot 0 = r^{(h, 0)\varphi^*} = r^{h\mu\varphi^*}.$$

In the following we write (as in 1.1) $r(h, s)$ for $r^{(h, s)\varphi^*}$, $(h, s)r$ for $r^{(h, s)\psi^*}$ and verify

(28) H^{φ^*} and H^{ψ^*} commute elementwise,

as for all $h_1, h_2 \in H$, $r, s_1, s_2 \in S$

$$\begin{aligned} ((h_1, s_1)r)(h_2, s_2) &= (h_1 r + s_1 r)(h_2, s_2) \\ &= (h_1 r + s_1 r)h_2 + (h_1 r + s_1 r)s_2 \\ &= h_1(rh_2 + rs_2) + s_1(rh_2 + rs_2) \\ &= (h_1, s_1)(rh_2 + rs_2) = (h_1, s_1)(r(h_2, s_2)). \end{aligned}$$

A similar standard calculation yields

(29) $(r_1(h, s))r_2 = r_1((h, s)r_2)$ for all $h \in H$, $r_1, r_2, s \in S$,

i.e., condition (1) is satisfied.

For all $r \in S$ we put

$$r^{\sigma^*}: H^{\mathcal{F}_0} \rightarrow H^{\mathcal{F}_0}, \quad (h, s) \mapsto (h, r + s).$$

Then r^{σ^*} is a permutation of $H^{\mathcal{F}_0}$. As before, we write $(h, s) + r$ for $(h, s)r^{\sigma^*}$ and observe

(30) σ^* is a homomorphism of $(S, +)$ into $\mathfrak{S}_{H^{\mathcal{F}_0}}$

(31) $(h, s) + r = (h, s) \Leftrightarrow r = 0$, for all $h \in H$, $r, s \in S$.

We put $\mathcal{F} := (H^{\mathcal{F}_0}, S^*, \varphi^*, \psi^*, \sigma^*)$. In order to show that \mathcal{F} is a Fitting structure it remains to check conditions (2) and (3) which here turn into

(32) $((h_1, s_1) + r)(h_2, s_2) = (h_1, s_1)(h_2, s_2) + r(h_2, s_2),$
 $(h_1, s_1)((h_2, s_2) + r) = (h_1, s_1)(h_2, s_2) + (h_1, s_1)r$

for all $h_1, h_2 \in H$, $r, s_1, s_2 \in S$,

$$(33) \quad \begin{aligned} ((h, s) + r_1)r_2 &= (h, s)r_2 + r_1r_2, \\ r_1((h, s) + r_2) &= r_1(h, s) + r_1r_2, \end{aligned}$$

for all $h \in H, s, r_1, r_2 \in S$,

both being immediate consequences of our definitions. Now (27) and (30) show that (\mathcal{F}, μ) is a continuation of \mathcal{F}_0 , and we claim:

$$(34) \quad \mathcal{F} \text{ is universal.}$$

For if (\mathcal{F}', μ') with $\mathcal{F}' = (H^{**}, S^{**}, \varphi^{**}, \psi^{**}, \sigma^{**})$ is a continuation of \mathcal{F}_0 , we put

$$\omega: H^{\mathcal{F}_0} \rightarrow H^{**}, \quad (h, s) \mapsto h^{\mu'} + s$$

and calculate for $h_1, h_2 \in H, s_1, s_2 \in S$ by means of (25):

$$\begin{aligned} ((h_1, s_1)(h_2, s_2))^\omega &= (h_1h_2, s_2^{h_1^*} + s_1^{h_2^*} + s_1s_2)^\omega \\ &= (h_1h_2)^{\mu'} + s_2^{h_1^*} + s_1^{h_2^*} + s_1s_2 \\ &= h_1^{\mu'}h_2^{\mu'} + s_2^{h_1^{\psi^{**}}} + s_1^{h_2^{\varphi^{**}}} + s_1s_2 \\ &= (h_1^{\mu'} + s_1)(h_2^{\mu'} + s_2) = (h_1, s_1)^\omega (h_2, s_2)^\omega, \end{aligned}$$

whence ω is a homomorphism. Since for all $h \in H$ we have $h^{\mu\omega} = (h, 0)^\omega = h^{\mu'}$, we put $\mu' = \mu\omega$ so that (26) holds. Thus the proof of our theorem is complete.

We add some remarks on the continuation semigroup $H^{\mathcal{F}_0}$ of a Fitting pre-structure \mathcal{F}_0 . For any ring S ,

$$s_1 \circ s_2 := s_1 + s_2 + s_1s_2 \quad (s_1, s_2 \in S)$$

defines an associative composition with identity element 0. As is well known, S is a radical ring if and only if (S, \circ) is a group. We have:

$$(35) \quad \text{If } H \text{ has an identity element } 1, \text{ then}$$

$$\nu: S \rightarrow H^{\mathcal{F}_0}, \quad s \mapsto (1, s)$$

is a monomorphism of (S, \circ) into $H^{\mathcal{F}_0}$.

$$(36) \quad \text{If } H \text{ has a zero element } 0, \text{ then}$$

$$\lambda: S \rightarrow H^{\mathcal{F}_0}, \quad s \mapsto (0, s)$$

is a monomorphism of (S, \cdot) into $H^{\mathcal{F}_0}$.

$$(37) \quad \text{An element } (h_0, s_0) \in H^{\mathcal{F}_0} \text{ is an identity element of } H^{\mathcal{F}_0} \text{ if and only if } h_0 \text{ is an identity element of } H, \\ s_0H = 0 = Hs_0 \text{ and } (h_0 + s_0)^\varphi = \text{id}_S = (h_0 + s_0)^\psi,$$

since (h_0, s_0) is an identity element of $H^{\mathcal{F}_0}$ if and only if $h_0h = h = hh_0$ and $h_0s + s_0h + s_0s = s = hs_0 + sh_0 + s_0s$ for all $h \in H, s \in S$.

We obviously have

- (38) If $H^{\mathcal{F}_0}$ is a group, then so is H .
- (39) If H has an identity element 1 such that $1^\varphi = \text{id}_S = 1^\psi$, then $(1, 0)$ is an identity element of $H^{\mathcal{F}_0}$, and $H^{\mathcal{F}_0}$ is a group if and only if H is a group and S is a radical ring. In this case $H^{\mathcal{F}_0}$ is a semidirect product of H and (S, \circ) .

Herein the statement about $(1, 0)$ follows from (37). Now let H be a group and S a radical ring. If for $h \in H$, $s \in S$ the \circ -inverse element of sh^{-1} is denoted by $(sh^{-1})^-$, we have

$$(h, s)(h^{-1}, h^{-1}(sh^{-1})^-) = (1, 1(sh^{-1})^- + sh^{-1} + sh^{-1}(sh^{-1})^-) = (1, 0).$$

Therefore $H^{\mathcal{F}_0}$ is a group. As to the converse, observing (38), it suffices to show that S is a radical ring. But if $s \in S$ and $(h_1, s_1) \in H^{\mathcal{F}_0}$ is the inverse of $(1, s)$, then

$$(1, 0) = (1, s)(h_1, s_1) = (h_1, 1 \cdot s_1 + sh_1 + ss_1),$$

$$(1, 0) = (h_1, s_1)(1, s) = (h_1, h_1s + s_1 \cdot 1 + s_1s),$$

hence $h_1 = 1$ and $s_1 \circ s = 0 = s \circ s_1$. Thus s is \circ -invertible. Let finally μ be the embedding of H into $H^{\mathcal{F}_0}$ as in the proof of 2.2 and ν as in (35). Then S^ν is a normal subgroup of $H^{\mathcal{F}_0}$ and isomorphic to (S, \circ) , H^μ is a subgroup of $H^{\mathcal{F}_0}$ which is isomorphic to H such that $S^\nu \cap H^\mu = \{(1, 0)\}$, and for all $h \in H$, $s \in S$ we have

$$(h, s) = (1, sh^{-1})(h, 0) = (sh^{-1})^\nu h^\mu,$$

whence $H^{\mathcal{F}_0} = S^\nu H^\mu$.

In order to give examples of Fitting structures, it is sufficient, by 2.2, to construct Fitting pre-structures:

2.3. EXAMPLE. Let M be a set and A a subset of M which is an abelian group with respect to some composition $+$. Then the set

$$S(M, A) := \{s | s: M \rightarrow A, s|_A \in \text{End}(A)\}$$

is a ring with respect to the compositions

$$\begin{aligned} s_1 + s_2: & M \rightarrow A \\ & m \mapsto m^{s_1} + m^{s_2} \\ s_1 s_2: & M \rightarrow A \\ & m \mapsto r(m^{s_1})^{s_2}, \end{aligned}$$

and the set

$$H(M, A) := \{ h | h: M \rightarrow M, h|_A \in \text{End}(A) \}$$

is a semigroup with respect to the composition

$$\begin{aligned} h_1 h_2: M &\rightarrow M \\ m &\mapsto (m^{h_1})^{h_2}. \end{aligned}$$

For all $h \in H(M, A)$, $s \in S(M, A)$ let s^{h^Φ} (resp. s^{h^Ψ}) be the usual composition (of mappings) sh (resp. hs).

Then $(H(M, A), S(M, A), \Phi, \Psi)$ is a Fitting pre-structure.

We show that all “well-behaved” Fitting pre-structures can be subsumed under this type of example:

2.4. THEOREM. *Let H be a semigroup with identity element 1, $\mathcal{F} = (H, S, \varphi, \psi)$ a Fitting pre-structure such that $1^\psi = \text{id}_S$. Then there are M, A as in 2.3 such that \mathcal{F} is isomorphic to a Fitting sub-pre-structure of $(H(M, A), S(M, A), \Phi, \Psi)$.*

Proof. W.l.o.g. we may assume $H \cap S = \emptyset$. Then we put $M := H \cup S$, $(A, +) := (S, +)$. For all $h \in H$, $s \in S$ we define

$$\begin{aligned} h^\alpha: M &\rightarrow M, & m &\mapsto mh = \begin{cases} mh & \text{for } m \in H \\ m^{h^\varphi} & \text{for } m \in S \end{cases} \\ s^\beta: M &\rightarrow A, & m &\mapsto ms = \begin{cases} s^{m^\psi} & \text{for } m \in H \\ ms & \text{for } m \in S. \end{cases} \end{aligned}$$

Obviously, α is a homomorphism of H into $H(M, A)$, and β is a homomorphism of S into $S(M, A)$. For $h \in \ker \alpha$ we have $1 = 1^{h^\alpha} = 1 \cdot h = h$; thus α is injective. Similarly, $s \in \ker \beta$ implies $0 = 1^{s^\beta} = 1 \cdot s = s$, hence β is injective, too. For all $m \in M$, $h \in H$, $s \in S$ we have

$$\begin{aligned} m^{h^\alpha s^\beta} &= (mh)s = m(hs) = m^{(hs)^\beta}, \\ m^{s^\beta h^\alpha} &= (ms)h = m(sh) = m^{(sh)^\beta}, \\ m^{(s^\beta)^{(h^\alpha)^\Phi}} &= m^{s^\beta h^\alpha} = (ms)h = m(sh) \\ &= m^{(sh)^\beta} = m^{s^{(h^\varphi)^\beta}} = m^{(s^\beta)^{\beta^{-1}h^\varphi\beta}} = m^{(s^\beta)^{(h^\varphi)^\beta}}, \\ m^{(s^\beta)^{(h^\alpha)^\Psi}} &= m^{h^\alpha s^\beta} = (mh)s = m(hs) \\ &= m^{(hs)^\beta} = m^{s^{(h^\psi)^\beta}} = m^{(s^\beta)^{\beta^{-1}h^\psi\beta}} = m^{(s^\beta)^{(h^\psi)^\beta}}. \end{aligned}$$

Hence $H^\alpha S^\beta \subseteq S^\beta$, $S^\beta H^\alpha \subseteq S^\beta$, $\alpha\Phi = \varphi\bar{\beta}$, $\alpha\Psi = \psi\bar{\beta}$. Therefore $(H^\alpha, S^\beta, \Phi|_{H^\alpha}, \Psi|_{H^\alpha})$ is a Fitting sub-pre-structure of $(H(M, A), S(M, A), \Phi, \Psi)$ and isomorphic to (H, S, φ, ψ) . The hypothesis that H has an identity element 1 such that $1^\psi = \text{id}_S$ has only been used to prove that α and β are injective. As is easily seen, for that purpose even weaker hypotheses on H are sufficient.

Let G be a group and A a characteristic abelian normal subgroup of G . Then $\mathcal{F} := (H(G, A), S(G, A), \Phi, \Psi)$ is a Fitting pre-structure. If we put $H := \text{Aut}(G)$, $S := \text{Hom}(G, A)$, then we obviously have $HS \subseteq S$, $SH \subseteq S$, whence (H, S, φ, ψ) with $\varphi = \Phi|_H$, $\psi = \Psi|_H$ is a Fitting sub-pre-structure of \mathcal{F} . If we put for $h \in H(G, A)$, $s \in S(G, A)$

$$h^{s^2}: G \rightarrow G, \quad g \mapsto g^h g^s,$$

then $s^2 \in \mathfrak{S}_{H(G, A)}$, and Σ is a homomorphism of $(S(G, A), +)$ into $\mathfrak{S}_{H(G, A)}$ such that $h^{s^2} = h \Leftrightarrow s = 0$ for all $h \in H(G, A)$, $s \in S(G, A)$. It is easy to see that (2) and (3) hold; thus $(H(G, A), S(G, A), \Phi, \Psi, \Sigma)$ is a Fitting structure. We put $\sigma := \Sigma|_H$. In general, $(H, S, \varphi, \psi, \sigma)$ need not be a Fitting structure. But we have:

2.5. THEOREM. *Let G be a group which has no nontrivial direct abelian factor. Assume $Z(G)$ is finite. Then $(\text{Aut}(G), \text{Hom}(G, Z(G)), \varphi, \psi, \sigma)$ is a Fitting structure.*

(Here φ, ψ, σ have the meaning introduced above.)

Proof. By our preparatory considerations it suffices to show:

$$(40) \quad \alpha^{\zeta^\sigma} \in \text{Aut}(G) \quad \text{for all } \alpha \in \text{Aut}(G), \zeta \in \text{Hom}(G, Z(G)).$$

Since $\alpha^{\zeta^\sigma} \in \text{Aut}(G)$ if and only if $\text{id}_G^{(\alpha^{-1}\zeta)^\sigma} \in \text{Aut}(G)$, for our proof of (40) we may assume $\alpha = \text{id}_G$. Surely, $\text{id}_G^{\zeta^\sigma}$ is a homomorphism. By our hypotheses on G and $Z(G)$, we have (see [1]) $\zeta^n = 0$ for an appropriate $n \in \mathbb{N}$. Since

$$\text{id}_G^{\zeta^\sigma} \cdot \text{id}_G^{(-\zeta + \zeta^2 - \dots \pm \zeta^{n-1})^\sigma} = \text{id}_G = \text{id}_G^{(-\zeta + \zeta^2 - \dots \pm \zeta^{n-1})^\sigma} \cdot \text{id}_G^{\zeta^\sigma},$$

$\text{id}_G^{\zeta^\sigma}$ is bijective, proving (40).

3. Wreath products over Fitting structures.

3.1. DEFINITION. Let $\mathcal{F} = (H, S, \varphi, \psi, \sigma)$ be a Fitting structure and $n \in \mathbb{N}$. For $\pi \in \mathfrak{S}_n$, a $(n \times n)$ -matrix $A = (a_{ij})$ is called a π -matrix over $H \cup S$ if

$$a_{ij} \in \begin{cases} H & \text{for } j = i\pi \\ S & \text{for } j \neq i\pi. \end{cases}$$

If $\pi, \pi' \in \mathfrak{S}_n$ and (a_{ij}) is a π -matrix, (a'_{ij}) is a π' -matrix over $H \cup S$, we define, using the product and sum notations introduced in 1.1,

$$(a_{ij})(a'_{ij}) := (b_{ij}) \quad \text{with } b_{ij} = \sum_{k=1}^n a_{ik}a'_{kj} \quad \text{for } 1 \leq i, j \leq n.$$

We observe:

(41) If $\pi, \pi' \in \mathfrak{S}_n$ and A is a π -matrix, A' is a π' -matrix over $H \cup S$, then AA' is a $(\pi\pi')$ -matrix over $H \cup S$.

3.2. DEFINITION. Let $\mathcal{F} = (H, S, \varphi, \psi, \sigma)$ be a Fitting structure and $n \in \mathbb{N}$. Let X be a subgroup of \mathfrak{S}_n . We put

$$H \setminus_S X := \{(A, \pi) \mid \pi \in X, A \text{ is a } \pi\text{-matrix over } H \cup S\},$$

and define for $(A, \pi), (A', \pi') \in H \setminus_S X$

$$(A, \pi)(A', \pi') := (AA', \pi\pi').$$

By (41), this is a composition in $H \setminus_S X$. We observe:

(42) $H \setminus_S X$ is a semigroup.

We call $H \setminus_S X$ the *wreath product of H and X over S* . If $H \cap S = \emptyset$ and $(A, \pi) \in H \setminus_S \mathfrak{S}_n$, then π is uniquely determined by A . In this case the elements (A, π) of the wreath product can be identified with their first components, the matrices A .

We add a few simple remarks:

(43) If $H = S$ and X is the trivial subgroup of \mathfrak{S}_n , then $H \setminus_S X$ is isomorphic to the multiplicative semigroup of the ring $(S)_n$ of all $(n \times n)$ -matrices over S .

(44) If $\mathcal{F}' = (H', S', \varphi', \psi', \sigma')$ is a Fitting substructure of \mathcal{F} and X' is a subgroup of X , then $H' \setminus_{S'} X'$ is a subsemigroup of $H \setminus_S X$.

(45) The standard wreath product $H \setminus X$ is isomorphic to $H \setminus_{S_0} X$ (writing S_0 for the trivial ring); thus it is contained in every wreath product $H \setminus_S X$ as a subsemigroup.

For $(H, \{0\}, \varphi_0, \psi_0, \sigma|_{\{0\}})$ is a Fitting substructure of $(H, S, \varphi, \psi, \sigma)$ where we write φ_0, ψ_0 for the (unique) actions of H on $\{0\}$, whence the second part of (45) is a consequence of (44).

Matrix multiplications yield actions of $H \setminus X$ on $(S)_n$: We put

$$B^{A^\varphi} := BA, \quad B^{A^\psi} = AB \quad \text{for } (A, \pi) \in H \setminus X, B \in (S)_n,$$

where these products are defined analogously to the matrix product introduced above. Then one readily verifies that $\mathcal{F}_0 := (H \setminus X, (S)_n, \hat{\varphi}, \hat{\psi})$ is a Fitting pre-structure. The mapping κ of the continuation semigroup $(H \setminus X)^{\mathcal{F}_0}$ into $H \setminus X$ such that $(A, B)^\kappa = A + B$ for all $(A, \pi) \in (H \setminus X)^{\mathcal{F}_0}$, $B \in (S)_n$ is an epimorphism. (The addition of A and B means, as usual, addition of corresponding components, using the notations of 1.1 with regard to σ .) If H has an identity element 1, then

$$\ker \kappa = \left\{ \left(\begin{array}{ccc|c} 1 + s_1 & & & 0 \\ & \ddots & & \\ & & & \\ 0 & & & 1 + s_n \end{array} \right) \middle| s_j \in S \right\} \cong \left(S \oplus \dots \oplus S, \circ \right)_n.$$

We claim

- (46) Let $\mathcal{F} = (H, S, \varphi, \psi, \sigma)$ be a Fitting structure, $n \in \mathbb{N}$ and X a subgroup of \mathfrak{S}_n . Suppose H has an identity element 1. Then $H \setminus X$ is a group if and only if H is a group and S is a radical ring.

For, if H is a group and S is a radical ring, then, by (45), $H \setminus X$ is a group and, by [3, I, 7. Th. 3], $(S)_n$ is a radical ring. Therefore, (39) yields that $(H \setminus X)^{\mathcal{F}_0}$ is a group, and so is a fortiori the semigroup $H \setminus X$, being isomorphic to $(H \setminus X)^{\mathcal{F}_0} / \ker \kappa$. Conversely, suppose $H \setminus X$ is a group. Its identity element being denoted by (I, id) , where I is the identity matrix, we have $(I + B, \text{id}) \in H \setminus X$ for every $B \in (S)_n$. If we put $C := (I + B)^{-1}$, then $BC \in (S)_n$, and

$$B \circ (-BC) = B - BC - B^2C = B - B(I + B)C = B - BI = 0,$$

$$(-BC) \circ B = -BC + B - BCB = B - BC(I + B) = B - BI = 0.$$

Therefore $(S)_n$, hence S , is a radical ring. Now let $h \in H$ and set

$$A := \begin{pmatrix} h & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

As before, we have $(A, \text{id}) \in H \setminus X$, and the entry c in the upper left corner of A^{-1} satisfies $hc = 1 = ch$. The assumption $c \in S$ would imply $1 \in S$ and, regarding the equations $1^\varphi = \text{id}_S = 1^\psi$, 1 were an identity

element of S . As S is a radical ring, this would yield $S = \{1\} \subseteq \{1\} \cdot H \subseteq S$, whence $S = \{1\} = H$, and everything would be trivial. But if $c \notin S$, then $c \in H$, and h is an invertible element of H , as desired.

We finally show that our notion of a generalized wreath product is useful for the description of automorphism groups of groups:

To this end let G be a direct indecomposable nonabelian group satisfying the minimum condition on central subgroups, and let $n \in \mathbb{N}$. We put $S := \text{Hom}(G, Z(G))$. Then S is a nil ring, hence a radical ring. By 2.5, $(\text{Aut}(G), S, \varphi, \psi, \sigma)$ is a Fitting structure, and the associated wreath product $(\text{Aut}(G)) \wr_S \mathfrak{S}_n$ is a group, by (46). Since $\text{Aut}(G) \cap S = \emptyset$, we may identify its elements (A, π) with their first components, the matrices A . For $A = (\alpha_{ij}) \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n$ we define

$$(47) \quad (g_1, \dots, g_n)A := \left(\prod_{i=1}^n g_i^{\alpha_{i1}}, \dots, \prod_{i=1}^n g_i^{\alpha_{in}} \right) \quad \text{for } g_1, \dots, g_n \in G.$$

This gives a mapping α_A of $G \times \dots \times G$ (n factors) into itself which can formally be regarded as the multiplication of the row (g_1, \dots, g_n) and the matrix A . The properties of A imply:

$$(48) \quad \text{For } g_1, \dots, g_n \in G, j \in \{1, \dots, n\}, \text{ there is at most one } i \in \{1, \dots, n\} \text{ such that } g_i^{\alpha_{ij}} \notin Z(G), \text{ viz. } i = j\pi^{-1}, \text{ where } \pi \text{ is the permutation determined by } A.$$

Therefore for $g_1, \dots, g_n, h_1, \dots, h_n \in G, j \in \{1, \dots, n\}$ we have

$$\prod_{i=1}^n (g_i h_i)^{\alpha_{ij}} = \prod_{i=1}^n g_i^{\alpha_{ij}} h_i^{\alpha_{ij}} = \prod_{i=1}^n g_i^{\alpha_{ij}} \prod_{i=1}^n h_i^{\alpha_{ij}},$$

yielding

$$(49) \quad \text{For all } A \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n, \alpha_A \text{ is an endomorphism of } G \times \dots \times G.$$

For $A = (\alpha_{ij}), B = (\beta_{ij}) \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n, g_1, \dots, g_n \in G$ and $j \in \{1, \dots, n\}$, we have, by (48)

$$\prod_{i=1}^n g_i^{\sum_{k=1}^n \alpha_{ik} \beta_{kj}} = \prod_{i=1}^n \left(\prod_{k=1}^n g_i^{\alpha_{ik}} \right)^{\beta_{kj}} = \prod_{k=1}^n \left(\prod_{i=1}^n g_i^{\alpha_{ik}} \right)^{\beta_{kj}}$$

which implies

$$(50) \quad \alpha_A \alpha_B = \alpha_{AB} \quad \text{for all } A, B \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n.$$

This and the obvious statement

$$(51) \quad \alpha_I = \text{id}, \quad \text{where } I \text{ is the identity element of } (\text{Aut}(G)) \wr_S \mathfrak{S}_n$$

imply:

(52) Associating to each $A \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n$ the automorphism α_A yields a homomorphism ι of $(\text{Aut}(G)) \wr_S \mathfrak{S}_n$ into $\text{Aut}(G \times \cdots \times G)$ (n factors).

If $A \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n$ such that $\alpha_A = \text{id}_{G \times \cdots \times G}$ (n factors), then for all $i \in \{1, \dots, n\}$, $g^S \in G$

$$(1, \dots, 1, g, 1, \dots, 1) = (1, \dots, 1, g, 1, \dots, 1)A = (g^{\alpha_1}, \dots, g^{\alpha_n}),$$

where g is in the i th place. Hence $A = I$. Thus we have

(53) ι is injective.

Finally we claim

(54) ι is surjective.

To this end we define for all $j \in \{1, \dots, n\}$

$$\begin{aligned} \varepsilon_j: \quad G &\rightarrow G \times \cdots \times G \\ &g \mapsto (1, \dots, 1, g, 1, \dots, 1), \\ &\hspace{10em} (\text{where } g \text{ is in the } j\text{th place}) \\ \delta_j: \quad G \times \cdots \times G &\rightarrow G \\ &(g_1, \dots, g_n) \mapsto g_j, \end{aligned}$$

and put for all $\alpha \in \text{Aut}(G \times \cdots \times G)$

$$A_\alpha := (\alpha_{ij}) \quad \text{with } \alpha_{ij} = \varepsilon_i \alpha \delta_j \quad \text{for } i, j \in \{1, \dots, n\}.$$

Then α_{ij} is an endomorphism of G , and by [1, Satz 2] there is exactly one $\pi \in \mathfrak{S}_n$ such that, for $i \in \{1, \dots, n\}$, $\alpha_{i, i\pi} \in \text{Aut}(G)$ and $\alpha_{ij} \in S$ for $j \neq i\pi$. Hence $A_\alpha \in (\text{Aut}(G)) \wr_S \mathfrak{S}_n$. By (47), $A'_\alpha = \alpha$, proving (54).

Summarizing, we have proved:

3.3. THEOREM. *Let G be a direct indecomposable nonabelian group and suppose $Z(G)$ is finite. Let $n \in \mathbb{N}$ and put $S := \text{Hom}(G, Z(G))$. Then*

$$\text{Aut}(G \times \cdots \times G) \cong (\text{Aut}(G)) \wr_S \mathfrak{S}_n.$$

If G satisfies the additional condition $\text{Hom}(G, Z(G)) = 0$ (which is in the finite case equivalent to $(|G/G'|, |Z(G)|) = 1$), our Theorem yields via (45) the well-known statement:

$$\text{Aut}(G \times \cdots \times G) \cong (\text{Aut}(G)) \wr_n \mathfrak{S}_n.$$

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UNIVERSITÄT KIEL
OLSHAUSENSTR. 40-60
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RAMESH A. GANGOLLI
University of Washington
Seattle, WA 98195

ROBION KIRBY
University of California
Berkeley, CA 94720

C. C. MOORE
University of California
Berkeley, CA 94720

H. SAMELSON
Stanford University
Stanford, CA 94305

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

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Michael James Cambern , Near isometries of Bochner L^1 and L^∞ spaces	1
Kun Soo Chang, Gerald William Johnson and David Lee Skoug , The Feynman integral of quadratic potentials depending on two time variables	11
Robert Coleman , One-dimensional algebraic formal groups	35
Alberto Collino , The Abel-Jacobi isomorphism for the cubic fivefold	43
N. J. Dev and S. S. Khare , Finite group action and vanishing of $N_*^G[F]$	57
Harold George Diamond and Jeffrey D. Vaaler , Estimates for partial sums of continued fraction partial quotients	73
Kenneth R. Goodearl , Patch-continuity of normalized ranks of modules over one-sided Noetherian rings	83
Dean Robert Hickerson and Sherman K. Stein , Abelian groups and packing by semicrosses	95
Karsten Johnsen and Harmut Laue , Fitting structures	111
Darren Long , Discs in compression bodies	129
Joseph B. Miles , On the growth of meromorphic functions with radially distributed zeros and poles	147
Walter Volodymyr Petryshyn , Solvability of various boundary value problems for the equation $x'' = f(t, x, x', x'') - y$	169
Elżbieta Pol , The Baire-category method in some compact extension problems	197
Masami Sakai , A new class of isocompact spaces and related results	211
Thomas Richard Shemanske , Representations of ternary quadratic forms and the class number of imaginary quadratic fields	223
Tsuyoshi Uehara , On class numbers of cyclic quartic fields	251