CROSSED PRODUCT AND HEREDITARY ORDERS

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Let $\Lambda$ be the crossed product order $(O_L/O_K, G, \rho)$ where $L/K$ is a finite Galois extension of local fields with Galois group $G$, and $\rho$ is a factor set with values in $O_L^*$. Let $\Lambda_0 = \Lambda$, and let $\Lambda_{i+1}$ be the left order $O_I(\text{rad } \Lambda_i)$ of rad $\Lambda_i$. The chain of orders $\Lambda_0, \Lambda_1, \ldots, \Lambda_s$ ends with a hereditary order $\Lambda_s$. We prove that $\Lambda_s$ is the unique minimal hereditary order in $A = K\Lambda$ containing $\Lambda$, that $\Lambda_s$ has $e/m$ simple modules, each of dimension $f$ over the residue class field $\bar{K}$ of $O_K$, and that $s = d - (e - 1)$. Here $d, e, f$ are the different exponent, ramification index, and inertial degree of $L/K$, and $m$ is the Schur index of $A$.

1. Introduction. Let $O_K$ be a complete discrete valuation ring having field of fractions $K$ and finite residue class field $\bar{K}$. Let $L$ be a finite Galois extension of $K$, with Galois group $G$, and let $O_L$ be the valuation ring in $L$. Let $\rho$ be a factor set on $G \times G$ with values in the units of $O_L$. We are interested in the crossed product order $\Lambda = (O_L/O_K, G, \rho)$ contained in the simple algebra $A = (L/K, G, \rho)$. If $\rho$ is trivial, Auslander-Goldman [1] showed that $\Lambda$ is a maximal order in $A$ if and only if $L/K$ is unramified, and Auslander-Rim [2] showed that $\Lambda$ is hereditary if and only if $L/K$ is tamely ramified. Williamson [8] extended the Auslander-Rim result to the case that $\rho$ is any factor set. We are interested in the wild case. Benz-Zassenhaus [3] showed that $\Lambda$ is contained in a unique minimal hereditary order in $A$.

We set $\Lambda_0 = \Lambda$, and define inductively

$$\Lambda_{j+1} = \{x \in A : x \text{ rad } \Lambda_j \subseteq \text{ rad } \Lambda_j\} = O_I(\text{rad } \Lambda_j).$$

Then we have the sequence of orders

$$\Lambda_0 \subseteq \Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_s = \Lambda_{s+1}$$

for some integer $s$. Since $\Lambda_s = O_I(\text{rad } \Lambda_s)$, it follows that $\Lambda_s$ is hereditary ([6, 39.11, 39.14]). From the theory of hereditary orders (see [6, 39.14]) $\Lambda_s$ may be described as follows: if $A \cong M_n(D)$, the ring of $n \times n$ matrices over a division ring $D$, and if $\Delta$ is the unique maximal order in $D$, then $\Lambda_s$ is the set of block matrices, with entries in $\Delta$, where there are $r$ diagonal blocks of size $n_i \times n_i$, and blocks above the diagonal have
entries in \( \text{rad} \Delta \). The positive integer \( r \) is called the type number of \( \Lambda_s \), and is also equal to the number of simple \( \Lambda_s \)-modules. Our main result is the following.

**Theorem.** (1) \( \Lambda_s \) is the unique minimal hereditary \( O_K \)-order in \( A \) containing \( \Lambda \).

(2) \( \text{rad} \Lambda_s = P_L \Lambda_s \), where \( P_L \) denotes the maximal ideal of \( O_L \).

(3) \( r = e/m \), where \( e \) is the ramification index of \( L/K \) and \( m \) is the Schur index of \( A \).

(4) \( n_1 = n_2 = \cdots = n_r = f \), the inertial degree of \( L/K \).

(5) \( s = d - (e - 1) \), where \( d \) is the exponent \( P_L^d = \mathcal{O} \) of the different of \( L/K \).

We prove this by first considering the split case (when \( p = 1 \)), and then taking an unramified extension \( K' \) of \( K \) which splits \( A \), and considering \( A \otimes_K K' \) which is a crossed product \( (L'/K', G, 1) \), where \( L' = L \otimes_K K' \). Then \( L' \) is not in general a field, but a Galois algebra over \( K' \), and we find it convenient to prove the Theorem when \( L \) is a Galois algebra over \( K \) to begin with; we take \( O_L \) to be the integral closure of \( O_K \) in \( L \), we replace \( P_L \) by \( \text{rad} O_L \), and we give suitable definitions of \( d, e, \) and \( f \) in §2. We deal with the split case in §3, and the general case in §4. We find generators for the hereditary order \( \Lambda_s \) in §5, in the totally ramified split case. In §6 we show how our results yield those of Auslander-Goldman-Rim-Williamson, as well as some others.

We cite Reiner [6] as a general reference.

2. **Galois algebras.** Let \( L \) be a commutative Galois algebra over \( K \), with finite Galois group \( G \), by which we mean that \( L \) is a commutative separable \( K \)-algebra with \( G \) a group of automorphisms of \( L \) fixing \( K \) such that the fixed subalgebra \( L^G = K \) and \( |G| = \dim_K L \). Let \( O_L \) be the integral closure of \( O_K \) in \( L \). Let \( E \) denote the set of primitive idempotents of \( L \). Then for \( \varepsilon \in E \), the integral closure \( O_{L\varepsilon} \) of \( O_K \) in the field \( L\varepsilon \) is a complete discrete valuation ring, and \( O_{L\varepsilon} = O_{L\varepsilon} \). Since \( L^G = K \), \( G \) acts transitively on \( E \).

**Lemma 2.1.** Let \( I \) be a non-zero \( O_L \)-submodule of \( L \) which is \( G \)-invariant. Then \( I = (\text{rad} O_L^i) \) for some integer \( i \).

**Proof.** For any primitive idempotent \( \varepsilon \) of \( L \), \( I\varepsilon \) is a non-zero \( O_{L\varepsilon} \)-submodule of \( L\varepsilon \), and therefore \( I\varepsilon = (\text{rad} O_{L\varepsilon})^i \varepsilon \) for some \( i \varepsilon \in \mathbb{Z} \), since \( O_{L\varepsilon} \) is a discrete valuation ring. Because \( G \) acts transitively on \( E \)
and $I$ is $G$-invariant, it follows that $i_{\varepsilon} = i$ is independent of $\varepsilon$. Then

$$I = \sum_{\varepsilon \in E} I_{\varepsilon} = \sum_{\varepsilon \in E} (\text{rad } O_{L_{\varepsilon}})^i = \sum_{\varepsilon \in E} (\text{rad } O_L)^i \varepsilon = (\text{rad } O_L)^i$$

as desired.

First, let $I = P_K O_L$. Then $P_K O_L = (\text{rad } O_L)^e$ for some integer $e$, and we call $e$ the ramification index of $L/K$.

Next, let $\text{tr}_{L/K}: L \to K$ be the trace map, and let

$$\tilde{O}_L = \{ x \in L: \text{tr}_{L/K}(xO_L) \subseteq O_K \}$$

be the complementary module to $O_L$ under the trace. Since

$$\text{tr}_{L/K}(x) = \sum_{g \in G} g(x), \quad x \in L,$$

it follows that $\tilde{O}_L$ is a $G$-invariant $O_L$-submodule of $L$, so $\tilde{O}_L = (\text{rad } O_L)^{-d}$ for some integer $d$. We call $d$ the different exponent of $L/K$ (and $(\text{rad } O_L)^d$ the different $\mathfrak{D}_{L/K}$ of $L/K$).

Define the inertial degree $f$ of $L/K$ to be $\dim_K(\text{rad } O_L/\text{rad } O_{L^e})$.

Let $\rho: G \times G \to O_L^*$ be a factor set on $G$ with values in the units of $O_L$. The crossed product algebra $A = (L/K, G, \rho)$ is the free left $L$-module with basis $u_g, g \in G$, with multiplication given by

$$xu_g \cdot yu_h = xg(y)\rho(g, h)u_{gh}, \quad x, y \in L, \ g, h \in G.$$  

The order $\Lambda = (O_L/O_K, G, \rho)$ is the $O_L$-submodule of $A$ spanned by $u_g, g \in G$. We assume that $\rho(g, 1) = \rho(1, g) = 1$, so that $O_L$ may be identified inside $\Lambda$ as $\{ xu_i : x \in O_L \}$.

**Lemma 2.2.** (1) $L$ has a normal $K$-basis with respect to $G$.

(2) $A$ is a central simple $K$-algebra, and $A$ is isomorphic to a full matrix ring over $K$ if and only if the class of $\rho$ in $H^2(G, L^*)$ is 1.

(3) The reduced trace $\text{tr}_d: A \to K$ is given by

$$\text{tr}_d \left( \sum_{g \in G} a_g u_g \right) = \text{tr}_{L/K}(a_1).$$

**Proof.** These results are well known if $L$ is a field, and the proofs are essentially the same if $L$ is a Galois algebra. We omit the details.

3. **The split case.** In this section we assume that $L/K$ is a Galois algebra, and we prove the theorem in the case that $\rho = 1$, with $P_L$ replaced by $\text{rad } O_L$, and with $d, e, f$ defined as in §2. Since $\rho = 1$, then
Let $V$ be a simple $A$-module. The structure theory for hereditary orders ([6, 39.18]) provides a $\Lambda_s$-submodule $M$ contained in $V$ with the following properties:

(a) $r$ is the unique positive integer such that $(\text{rad } \Lambda_s)^r M = P_\mathbb{K} M$, (since $\text{End}_A(V) = \mathbb{K}$).

(b) $\Lambda_s = \{ x \in A : x(\text{rad } \Lambda_s)^i M \subseteq (\text{rad } \Lambda_s)^{i+1} M, 0 \leq i < r \}$.

(c) $\text{rad } \Lambda_s = \{ x \in A : x(\text{rad } \Lambda_s)^i M \subseteq (\text{rad } \Lambda_s)^{i+1} M, 0 \leq i < r \}$.

(d) $(\text{rad } \Lambda_s)^{i-1} M/(\text{rad } \Lambda_s)^i M, 1 \leq i \leq r$, are a full set of simple $\Lambda_s$-modules.

(e) $n_i = \dim_{\mathbb{K}} (\text{rad } \Lambda_s)^{i-1} M/(\text{rad } \Lambda_s)^i M, 1 \leq i \leq r$.

The algebra $A$ acts on $L$, via

$$(\sum x_g u_g) \cdot y = \sum x_g g(y), \quad \sum x_g u_g \in A, \ y \in L,$$

and acts irreducibly on $L$, so we may take $L$ to be $V$. The non-zero $\Lambda$-submodules of $L$ are $O_L$-submodules of $L$ which are $G$-stable, so they are precisely $(\text{rad } O_L)^i, i \in \mathbb{Z}$, by Lemma 2.1. We denote the $\Lambda$-module $(\text{rad } O_L)^i$ by $M_i$.

**Lemma 3.1.** For each integer $j \geq 0$,

1. $M_i$ is a $\Lambda_j$-module, $i \in \mathbb{Z}$, and every non-zero $\Lambda_j$-submodule of $V$ is $M_i$ for some $i$.
2. $(\text{rad } \Lambda_j) M_i = M_{i+1}$.

**Proof.** If (1) holds for some $j$, then $(\text{rad } \Lambda_j) M_i \subseteq M_i$, by Nakayama’s Lemma, so $(\text{rad } \Lambda_j) M_i \subseteq M_{i+1}$, since $M_{i+1}$ is the unique maximal $\Lambda_j$-submodule of $M_i$. But $\text{rad } O_L \subseteq \text{rad } \Lambda_j$, since $(\text{rad } O_L) M_i \subseteq M_i$ for each $i$, and $(\text{rad } O_L) M_i = M_{i+1}$, so $(\text{rad } \Lambda_j) M_i = M_{i+1}$, proving (2). For (1), we use induction on $j$, having noted that it holds for $\Lambda_0$. Then for $j + 1$,

$$\Lambda_{j+1} M_i = \Lambda_{j+1} (\text{rad } \Lambda_j) M_{i-1} \quad \text{(by (2) for } j)$$

$$\subseteq (\text{rad } \Lambda_j) M_{i-1} \quad \text{(by definition of } \Lambda_{i+1})$$

$$= M_i$$

so $M_i$ is a $\Lambda_{j+1}$-module, $i \in \mathbb{Z}$. Since any $\Lambda_{j+1}$-module is also a $\Lambda$-module, the proof is complete.

**Lemma 3.2.** (1) $\Lambda_s = \{ x \in A : xM_i \subseteq M_i, i \in \mathbb{Z} \}$.

(2) $\text{rad } \Lambda_s = \{ x \in A : xM_i \subseteq M_{i+1}, i \in \mathbb{Z} \}$.

(3) $\text{rad } \Lambda_s = (\text{rad } O_L) \Lambda_s = \Lambda_s (\text{rad } O_L)$. 
Proof. The structure of $\Lambda_s$ is given in terms of a $\Lambda_s$-submodule $M$ contained in $V$. From Lemma 3.1, any $\Lambda_s$-submodule of $V$ must be $M_k$ for some integer $k$. We have, from (b) and Lemma 3.1,

$$\Lambda_s = \{ x \in A: xM_{k+i} \subseteq M_{k+i}, 0 \leq i < r \}.$$ 

From (a), $M_{k+r} = (\text{rad} \Lambda_s)^r M_k = P_k M_k$, and since $P_k$ is a principal ideal of $O_\kappa$, then $M_{k+r} \cong M_k$ as $\Lambda_s$-modules. Then for $i \in \mathbb{Z}$,

$$(\text{rad} \Lambda_s)^i M_{k+r} = M_{i+k+r} \cong (\text{rad} \Lambda_s)^i M_k = M_{i+k}$$

so $M_{i+r} \cong M_i$ as $\Lambda_s$-modules, $i \in \mathbb{Z}$. Thus

$$\Lambda_s = \{ x \in A: xM_i \subseteq M_i, i \in \mathbb{Z} \},$$

proving (1), and (2) follows from (1). Since $\text{rad} O_\kappa \subseteq \text{rad} \Lambda_s$ and

$$(\text{rad} \Lambda)^i M_i = M_{i+1} = (\text{rad} \Lambda_s) M_i, i \in \mathbb{Z},$$

(3) follows from (2).

Parts (1)–(4) of the Theorem are now straightforward in this case. If $\Gamma$ is a hereditary order in $A$ containing $\Lambda$, then applying the structure theory to $\Gamma$, there is a $\Gamma$-submodule $M$ of $V$ such that

$$\Gamma = \{ x \in A: x(\text{rad} \Gamma)^i M \subseteq (\text{rad} \Gamma)^i M, 1 \leq i \leq \text{type number of } \Gamma \}.$$ 

Since $\Lambda \subseteq \Gamma$, $M$ is a $\Lambda$-module, so $M = M_j$ for some integer $j$. Also, since $(\text{rad} O_\kappa) M_i \subseteq M_i$, $i \in \mathbb{Z}$, then $\text{rad} O_\kappa \subseteq \text{rad} \Gamma$, and then $(\text{rad} \Gamma)^i M_j = M_{j+i}, i \in \mathbb{Z}$. It follows from Lemma 3.2 that $\Lambda_s \subseteq \Gamma$, proving (1) of the theorem. Part (2) is contained in Lemma 3.2. For (3), we know from (a) that $r$ is the integer such that $(\text{rad} \Lambda)^r M_k = P_k M_k$. But

$$P_k M_k = P_k O_\kappa M_k = (\text{rad} O_\kappa)^e M_k = M_{k+e}$$

so $r = e$. (Note that $m = 1$ here.) For (4),

$$(\text{rad} \Lambda)^{r-1} M_k / (\text{rad} \Lambda)^r M_k = M_{k+e} / (\text{rad} \Lambda)^r M_k$$

and as $K$-modules $(\text{rad} O_\kappa)^{k+e} / (\text{rad} O_\kappa)^{k+1} \cong O_\kappa / \text{rad} O_\kappa$ so

$$n_i = \dim_K O_\kappa / \text{rad} O_\kappa = f, 1 \leq i \leq r.$$ 

In order to prove (5), we use the following result.

**Lemma 3.3.** Suppose that $a$ is an integer $\geq 0$ such that $(\text{rad} \Lambda)^a$ is the largest left $\Lambda_s$-ideal contained in $\Lambda$. Then $s = a$.

**Proof.** If $a = 0$, then $\Lambda_s \subseteq \Lambda$, so $\Lambda_s = \Lambda$, and $s = 0$. Assuming that $a > 0$, we show that $(\text{rad} \Lambda)^{a-1}$ is the largest left $\Lambda_s$-ideal contained in $\Lambda_1$. First,

$$(\text{rad} \Lambda)^{a-1} \text{rad} \Lambda \subseteq (\text{rad} \Lambda)^{a-1} \text{rad} \Lambda_s = (\text{rad} \Lambda)^a.$$
Now \((\text{rad } \Lambda_s)^a \subseteq \Lambda\) by hypothesis, and \(\Lambda_s \cap \Lambda \subseteq \text{rad } \Lambda\), by Lemma 3.2. Thus \((\text{rad } \Lambda_s)^a \subseteq \text{rad } \Lambda\). Then \((\text{rad } \Lambda_s)^{a-1}(\text{rad } \Lambda) \subseteq \text{rad } \Lambda\), so \((\text{rad } \Lambda_s)^{a-1} \subseteq \Lambda_1\).

Next, if \(L\) is a left \(\Lambda_s\)-ideal contained in \(\Lambda_1\), then \(L \text{rad } \Lambda \subseteq \text{rad } \Lambda\), so \(L \text{ rad } \Lambda \subseteq (\text{rad } \Lambda_s)^a\). Then

\[
L \text{ rad } \Lambda_s = L(\text{rad } \Lambda) \Lambda_s \subseteq (\text{rad } \Lambda_s)^a.
\]

Since \(\text{rad } \Lambda_s\) is invertible, \(L \subseteq (\text{rad } \Lambda_s)^{a-1}\) as desired.

Now by induction, the length of the chain \(\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_s\) is \(a - 1\), so \(s = a\), and the proof is complete.

Let \(\text{trd} : A \rightarrow K\) be the reduced trace, and for an \(O_K\)-submodule \(L\) of \(A\) with \(KL = A\), let

\[
\tilde{L} = \{ x \in A : \text{trd}(xL) \subseteq O_K \}
\]

be the complementary module.

**Lemma 3.4.** Let \(\Gamma\) be any hereditary \(O_K\)-order contained in the split simple algebra \(A = M_n(K)\). Then

\[
\tilde{\Gamma} = P_K^{-1} \text{rad } \Gamma.
\]

**Proof.** Suppose that \(\Gamma\) has type number \(r\), invariants \(n_1, \ldots, n_r\), and \(\Gamma\) consists of block matrices as mentioned in section 1. Let \(\pi_K\) be a prime element of \(O_K\). For integers \(i, j, 1 \leq i, j \leq n\), let \(Y_{ij}\) denote the matrix whose \(i, j\)-entry is \(\pi_K\) if the \(i, j\)-position is above the diagonal of blocks of \(\Gamma\), or \(1\) otherwise, and all of whose other entries are \(0\) (so \(Y_{ij} \in \Gamma\)). Let \(y_{ij}\) denote the non-zero entry of \(Y_{ij}\). Let \(X = (x_{ij})\) be any element of \(A\). Then \(XY_{ij}\) has at most one non-zero entry on the main diagonal, namely \(x_{ij}y_{ji}\). We have \(\text{trd}(XY_{ij}) = \text{trace of matrix } XY_{ij} = x_{ij}y_{ji}\). Then \(X \in \tilde{\Gamma} \iff x_{ij}y_{ji} \in O_K\), all \(i, j \iff\) when \(X\) is partitioned according to the block partition induced by \(\Gamma\), the entries below the diagonal of blocks are in \(P_K^{-1}\), and the other entries are in \(O_K\). But such matrices are precisely those in \(P_K^{-1} \text{rad } \Gamma\). Since the \(Y_{ij}\) give a free basis for \(\Gamma\) over \(O_K\), the result follows.

**Lemma 3.5.** Let \(w = d - (e - 1)\). Then \((\text{rad } \Lambda_s)^w\) is the largest left \(\Lambda_s\)-ideal contained in \(\Lambda\).

**Proof.** From Lemma 3.2, we have \(\text{rad } \Lambda_s = (\text{rad } O_L) \Lambda_s\), so \((\text{rad } \Lambda_s)^w = (\text{rad } O_L)^{d-(e-1)} \Lambda_s\). From Lemma 3.4

\[
\tilde{\Lambda}_s = P_K^{-1} \text{rad } \Lambda_s = (\text{rad } O_L)^{-e}(\text{rad } O_L) \Lambda_s = (\text{rad } O_L)^{-e+1} \Lambda_s,
\]
so

\[(\text{rad } \Lambda_s)^w = (\text{rad } O_L)^d \tilde{\Lambda}_s = \left( (\text{rad } O_L)^{-d} \Lambda_s \right)^{\sim} \text{.}\]

From Lemma 2.2, \(\text{trd}(\sum x_g u_g) = \text{tr}_{L/K}(x_1)\), so

\[\tilde{\Lambda} = \mathcal{D}^{-1} \Lambda = (\text{rad } O_L)^{-d} \Lambda \subseteq (\text{rad } O_L)^{-d} \Lambda_s,\]

\[(\text{rad } \Lambda_s)^w = \left( (\text{rad } O_L)^{-d} \Lambda_s \right)^{\sim} \subseteq \tilde{\Lambda} = \Lambda,\]

so \((\text{rad } \Lambda_s)^w\) is contained in \(\Lambda\). If \(L\) is any other left \(\Lambda_s\)-ideal contained in \(\Lambda\), then \(\tilde{L}\) is a right \(\Lambda_s\)-module containing \(\tilde{\Lambda}\), so

\[\tilde{L} \supseteq \tilde{\Lambda} \Lambda_s = \mathcal{D}^{-1} \Lambda_s = (\text{rad } O_L)^{-d} \Lambda_s,\]

\[L = \tilde{L} \subseteq \left( (\text{rad } O_L)^{-d} \Lambda_s \right)^{\sim} = (\text{rad } \Lambda_s)^w,\]

completing the proof.

Now (5) of the Theorem follows from Lemmas 3.3 and 3.5.

4. The general case. In this section we continue with the assumption that \(L/K\) is a Galois algebra, and we prove the Theorem in the case that \(\rho\) is any factor set with values in \(O_K^*\). Since \(K\) is finite, there is an unramified field extension \(K'\) of \(K\) such that the algebra \(A' = A \otimes_K K'\) splits ([7, Prop. 2, p. 191]). Let \(O'\) be the integral closure of \(O_K\) in \(K'\), and let \(\Lambda' = \Lambda \otimes_{O_K} O'\).

**Lemma 4.1.** If \(\Gamma\) is an \(O_K\)-order, then

\[\text{rad}(\Gamma \otimes_{O_K} O') = (\text{rad } \Gamma) \otimes_{O_K} O'.\]

*Proof.* Denote \(O_K\) by \(O\), and \(P_K\) by \(P\). Clearly

\[(\text{rad } \Gamma) \otimes_{O} O' \subseteq \text{rad}(\Gamma \otimes_{O} O').\]

For the reverse inclusion, we have

\[(\Gamma \otimes_{O} O')/(\text{rad } \Gamma) \otimes_{O} O' \cong (\Gamma/\text{rad } \Gamma) \otimes_{O} O'.\]

Since \(P \subseteq \text{rad } \Gamma\), then \(\Gamma/\text{rad } \Gamma\) is an \(O/P\)-module, and

\[(\Gamma/\text{rad } \Gamma) \otimes_{O} O' \cong (\Gamma/\text{rad } \Gamma) \otimes_{O/P} (O'/PO').\]

Since \(K'/K\) is unramified, then \(O'/PO'\) is field, which is separable over \(K\) since \(K\) is finite. Then the semi-simple \(O/P\)-algebra \(\Gamma/\text{rad } \Gamma\) remains semi-simple after tensoring with \(O'/PO'\), so \(\Gamma \otimes_{O} O'/(\text{rad } \Gamma) \otimes_{O} O'\) is semi-simple, and the result follows.
We let $G$ act on $L' = L \otimes_K K'$ by
\[ g(x \otimes y) = g(x) \otimes y, \quad x \in L, \ y \in K', \ g \in G. \]
Then $L'$ is a Galois algebra over $K'$ with Galois group $G$. We have $O_{L'} = O_L \otimes_{O_K} O'$, and
\[ \Lambda' = \Lambda \otimes_{O_K} O' = \left( O_{L'}/O', G, \rho \right). \]

Let us show that in going from $L/K$ to $L'/K'$, the numbers $d, e, f$ are unchanged.

Applying Lemma 4.1 to the $O_K$-order $O_L$, we have $\text{rad} O_{L'} = (\text{rad} O_L) \otimes_{O_K} O'$. Since the maximal ideal $P'$ of $O'$ is $P_K O'$, then
\[ P'O_{L'} = (P_K O_L) \otimes_{O_K} O' = (\text{rad} O_L)^e \otimes_{O_K} O' = (\text{rad} O_{L'})^e \]
so the ramification index of $L'/K'$ is still $e$. Similarly,
\[ \dim_K (O_{L'}/\text{rad} O_{L'}) = \dim_K (O_L/\text{rad} O_L) = f. \]

For the different exponent of $L'/K'$, since
\[ \text{tr}_{L'/K'}(x \otimes y) = \text{tr}_{L/K}(x) \otimes y, \quad x \in L, \ y \in K', \]
then clearly $\hat{O}_L \otimes_{O_K} O' \subseteq \hat{O}_{L'}$; since $\hat{O}_L = (\text{rad} O_L)^{-d}$, and $\text{rad} O_{L'} = (\text{rad} O_L) \otimes_{O_K} O'$, then $(\text{rad} O_{L'})^{-d} \subseteq \hat{O}_{L'}$. If $(\text{rad} O_{L'})^{-d-1} \subseteq \hat{O}_{L'}$, then $(\text{rad} O_L)^{-d-1} \subseteq \hat{O}_L$, which is not so. Therefore $\hat{O}_{L'} = (\text{rad} O_{L'})^{-d}$.

**Lemma 4.2.** If $\Gamma$ is an $O_K$-order contained in a semi-simple algebra $A$, then
\[ O_I(\text{rad} \Gamma) \otimes_{O_K} O' = O_I(\text{rad} \left( \Gamma \otimes_{O_K} O' \right)). \]

**Proof.** It is clear that the left side is contained in the right. There is an isomorphism
\[ \phi: O_I(\text{rad} \Gamma) \to \text{Hom}_\Gamma(\text{rad} \Gamma, \text{rad} \Gamma), \]
where $\text{rad} \Gamma$ is considered as a right $\Gamma$-module. Similarly, there is an isomorphism
\[ \psi: O_I(\text{rad} \Gamma') \to \text{Hom}_{\Gamma'}(\text{rad} \Gamma', \text{rad} \Gamma'), \]
where $\Gamma' = \Gamma \otimes_{O} O'$. Since $\Gamma$ is noetherian, then $\text{rad} \Gamma$ is finitely presented over $\Gamma$, so from [6, 2.37] we have an isomorphism
\[ \sigma: \text{Hom}_\Gamma(\text{rad} \Gamma, \text{rad} \Gamma) \otimes_{O} O' \to \text{Hom}_{\Gamma \otimes_{O} O'}(\text{rad} \Gamma \otimes_{O} O', \text{rad} \Gamma \otimes_{O} O') \]
\[ = \text{Hom}_{\Gamma'}(\text{rad} \Gamma', \text{rad} \Gamma'). \]
from Lemma 4.1. The map
\[ \psi^{-1}\sigma(\phi \otimes 1) : O_t(\text{rad } \Gamma) \otimes_{O_K} O' \to O_t(\text{rad } \Gamma') \]
is the identity, and the result is proved.

**Lemma 4.3.** Let \( \Lambda = (O_L/O_K, G, \rho) \) be a crossed product order in \( A = (L/K, G, \rho) \) and suppose that \( A \) splits over \( K \). Then \( \Lambda \cong (O_L/O_K, G, 1) \).

**Proof.** Since the algebra \( A \) is split over \( K \), the class of \( \rho \) in \( H^2(G, L^*) \) is 1. We shall show that the map \( H^2(G, O_L^*) \to H^2(G, L^*) \) is one-to-one, and then the class of \( \rho \) in \( H^2(G, O_L^*) \) will be 1, and the result will follow.

Let \( E \) be the set of primitive idempotents of \( L \) and let \( M = \bigoplus_{\varepsilon \in E} Z\varepsilon \) be the free \( Z \)-module with basis \( E \); \( G \) acts on \( M \) via its action on \( E \). For \( \varepsilon \) in \( E \), let \( v_\varepsilon \) be the normalized valuation on the field \( L\varepsilon \), and define \( v : L^* \to M \) by
\[ v(x) = \sum_{\varepsilon \in E} v_\varepsilon(x\varepsilon)\varepsilon, \quad x \in L^*. \]
Then we get an exact sequence of \( G \)-modules
\[ o \to O_L^* \to L^* \xrightarrow{v} M \to o, \]
giving rise to the exact sequence
\[ H^1(G, M) \to H^2(G, O_L^*) \to H^2(G, L^*). \]
Since \( M \) is a permutation module, \( M \) is isomorphic to the induced module \( \text{Ind}_{H}^{G}(Z) = ZG \otimes_{ZH} Z \), where \( H \) is the stabilizer of an idempotent in \( E \), and \( H \subseteq H, Z) = 0 \), since \( H \) is finite. Then \( H^2(G, O_L^*) \to H^2(G, L^*) \) is one-to-one, as desired.

From Lemma 4.2, the chains
\[ \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_s \]
\[ \Lambda'_0 \subseteq \Lambda'_1 \subseteq \cdots \subseteq \Lambda'_s \]
have the same length, and \( \Lambda'_s \) is hereditary. Since the Theorem has been proved in the split case, and since \( \Lambda' \equiv (O_{L'}/O', G, 1) \), which follows from Lemma 4.3, we find that \( s = d - (e - 1) \). If \( \Gamma \) is a hereditary order in \( A \) containing \( \Lambda \), then \( \Gamma' = \Gamma \otimes_{O_K} O' \) is a hereditary order in \( A' \) containing \( A' \), and since \( \Lambda'_s \) is the unique minimal hereditary order in \( A' \) containing \( A' \), then \( \Lambda'_s \subseteq \Gamma' \). We may embed \( \Gamma \) in \( \Gamma' \) as \( \Gamma \otimes_{O_K} 1 \), and \( A \) in \( A' \) as \( A \otimes_{K} 1 \), and then \( \Gamma = \Gamma' \cap A \supseteq \Lambda'_s \cap A = \Lambda_s \), so \( \Lambda_s \) is the unique minimal hereditary order in \( A \) containing \( \Lambda \).
From [6, 39.14] we have
\[ \Lambda_s / \text{rad} \Lambda_s \equiv \prod_{i=1}^{r} M_{n_i}(\Delta) \]
where \( \Delta = \Delta / \text{rad} \Delta \), and \( \Delta \) is the unique maximal order in \( \text{End}_A(V) \), with \( V \) a simple \( A \)-module. Then
\[ \Lambda_s / \text{rad} \Lambda_s \cong (\Lambda_s / \text{rad} \Lambda_s) \otimes_{O_K} O' \cong (\Lambda_s / \text{rad} \Lambda_s) \otimes_{K} K' \]
\[ \cong \prod_{i=1}^{r} M_{n_i}(\Delta \otimes K') \].

Now \( \Delta \otimes_{K} K' \equiv (K')^m \), where \( m \) is the Schur index of \( A \), since \( K \) is finite ([6, 14.3]). Thus
\[ \Lambda_s / \text{rad} \Lambda_s \cong \left( \prod_{i=1}^{r} M_{n_i}(K') \right)^m \].

Therefore the type number of \( \Lambda_s / \text{rad} \Lambda_s \), known to be \( e \) from §3, is equal to \( mr \), yielding
\[ r = \frac{e}{m} \].

Each invariant \( n_i = f \), since the invariants \( n_i \) of \( \Lambda_s \) are \( f \). Therefore the proof of the theorem is complete.

5. Generators for \( \Lambda_s \) in the split case. In this section we find generators for \( \Lambda_s \) in the case that \( \rho = 1 \). To simplify the exposition, we assume that \( L \) is a field, which is totally ramified over \( K \). We let \( P_L \) be the maximal ideal of \( O_L \), and let \( v_L \) be the normalized valuation on \( L \). Let \( M_i \) denote the \( \Lambda \)-module \( P_i^L \), \( i \in \mathbb{Z} \).

**Lemma 5.1.** Let \( w = d - (e - 1) \), and let \( x \) be an element of \( L \) such that \( v_L(x) = -w \). Let \( \alpha = x \sum_{g \in \mathcal{O}} u_g \in A \). Then \( \alpha M_i \subseteq M_i \), \( i \in \mathbb{Z} \) (so \( \alpha \in \Lambda_s \), from Lemma 3.2), and unless \( i \equiv -w \) (mod \( e \)), \( \alpha M_i \subseteq M_{i+1} \), whereas if \( i \equiv -w \) (mod \( e \)), \( \alpha M_i \not\subseteq M_{i+1} \).

**Proof.** Let \( \text{tr} \) denote the trace from \( L \) to \( K \). We first compute \( \text{tr}(P_i^L) \), \( i \in \mathbb{Z} \). We have, for \( j \in \mathbb{Z} \),
\[ \text{tr}(P_i^L) \subseteq P_i^K \iff \text{tr}(P_i^L P_{e j}^{-1}) \subseteq O_K \]
\[ \iff \text{tr}(P_i^{-e j}) \subseteq O_K \iff P_i^{-e j} \subseteq \mathcal{O}^{-1} \]
\[ \iff P_i^{-e j + d} \subseteq O_L \iff i - ej + d \geq 0 \]
\[ \iff j \leq \frac{i + d}{e} \].
We have used $\mathcal{O} = P_L^d$. Thus
\[
\text{tr}(P_L^i) = P_K^{[(i+d)/e]},
\]
where \([ \ ]\) denotes greatest integer. Since $\sum u_g \cdot y = \sum g(y) = \text{tr} y$, $y \in L$, we have
\[
O_L \alpha M_i = O_L x \text{tr}(P_L^i) = x O_L P_L^{((i+d)/e)} = x P_L^{((i+d)/e)}.
\]
Write
\[
\left[ \frac{i + d}{e} \right] = \left[ \frac{i + w}{e} + \frac{d - w}{e} \right] = \left[ \frac{i + w}{e} + \frac{e - 1}{e} \right].
\]
If $(i + w)/e \notin \mathbb{Z}$, then $\left[(i + d)/e\right] > (i + w)/e$, so $e[(i + d)/e] \geq i + w$, and
\[
O_L \alpha M_i \subseteq x P_L^{i + w + 1} = P_L^{i + 1} = M_{i+1}.
\]
If $(i + w)/e \in \mathbb{Z}$, then $\left[(i + d)/e\right] = (i + w)/e$, so $e[(i + d)/e] = i + w$, and
\[
O_L \alpha M_i = x P_L^{i + w} = M_i.
\]
This completes the proof.

Let $\pi_L$ be a prime element of $O_L$. Then from Lemma 3.2, we have $\pi_L^{-1} \Lambda_s \pi_L = \Lambda_s$. Let $\alpha = x \sum u_g$ be the element of Lemma 5.1, and define
\[
\alpha_i = \pi_L^{-i} \alpha \pi_L^i, \quad 0 \leq i < e.
\]

From Lemma 5.2, it follows that $\alpha_i$ acts non-trivially on $M_{-w+i}/M_{-w+i+1}$, whereas $\alpha_i$ annihilates $M_j/M_{j+1}$ if $j \not\equiv -w + i \pmod{e}$. Thus the simple $\Lambda_s$-modules $M_0/M_1$, $M_1/M_2$, ..., $M_{e-1}/M_e$ are non-isomorphic, and hence form a complete set of simple $\Lambda_s$-modules. Recall that $\Lambda_s/\text{rad} \Lambda_s \cong \prod_{i=1}^e M_n(\overline{K})$, and each $n_i = f = 1$, since we are assuming that $L/K$ is totally ramified. Hence $\Lambda_s/\text{rad} \Lambda_s$ is commutative. Further, $r = e$, so $\dim_{\overline{K}}(\Lambda_s/\text{rad} \Lambda_s) = e$. Then the elements $\alpha_i + \text{rad} \Lambda_s$ generate $\Lambda_s/\text{rad} \Lambda_s$ as a $\overline{K}$-module, $0 \leq i < e$. Since $\text{rad} \Lambda_s = P_L \Lambda_s$, we see that $O_L \alpha_i$, $0 \leq i < e$, generate $\Lambda_s$ as an $O_K$-module. So $\pi_L/\alpha_i$, $0 \leq j < e$, $0 \leq i < e$, generate $\Lambda_s$ as an $O_K$-module.

Finally, from the formula $\text{tr}(P_L^i) = P_K^{[(i+d)/e]}$ from Lemma 5.1, if we set $i = -w$, then $i + d = e - 1$, so $\text{tr}(P_L^{-w}) = O_K$. Thus we may find $y$ in $L$ with $v_L(y) = -w$ such that $\text{tr}(y) = u$ is a unit of $O_K$. Then $x = u^{-1}y$ has $v_L(x) = -w$ and $\text{tr}(x) = 1$. Now $(\sum u_g)x(\sum u_g) = \text{tr}(x)\sum u_g = \sum u_g$, so $\alpha = x\sum u_g$ is idempotent. From the action of $\alpha$ on the simple modules $M_i/M_{i+1}$, we find that $\alpha$ is a primitive idempotent of $\Lambda_s$, and that the elements $\alpha_i + \text{rad} \Lambda_s$ are all the primitive idempotents of $\Lambda_s/\text{rad} \Lambda_s$. 

6. Complements. The results of Auslander-Goldman-Rim-Williamson mentioned in the Introduction follow easily from our Theorem. If \( p = 1 \), \( \Lambda \) is a maximal order in \( A \) if \( s = 0 \), \( r = 1 \) if \( e/m = 1 \) if \( e = 1 \), since \( m = 1 \). For any \( p \), \( \Lambda \) is hereditary if \( s = 0 \) if \( d = e - 1 \) if \( L/K \) is tamely ramified, from [7, Prop. 13, p. 67].

We also recover a result of Janusz [4], who showed that, in the tamely ramified case, \( \Lambda \) has type \( e/m \) and invariants \( f \). (See also Merklen [5].)

From the fact that \( r = e/m \), we find a way to compute the Schur index \( m \) of \( A \) as follows: the centre of \( \Lambda_s / \text{rad} \Lambda_s \) has \( e/m \) component fields (each of dimension \( m \) over \( K \)).

It may be shown that the index

\[
(\Lambda_s : \Lambda) = \pi_K^{n^2(d-(e-1))/2e}
\]

where \( n = [L : K] \). This follows from

\[
(\bar{\Lambda} : \Lambda) = (\bar{\Lambda}_s : \Lambda_s)(\Lambda_s : \Lambda)^2.
\]

Note that Lemma 3.4 (that \( \bar{\Lambda} = P_K^{-1} \text{rad} \Lambda \) if \( \Lambda \) is hereditary) also holds in the non-split case, as may be shown by tensoring with an unramified extension.

In the split case (§3), the \( \Lambda \)-lattices contained in a irreducible \( A \)-module \( V \) are linearly ordered, but this fails to be true if \( A \) is not split. However, it may be shown, in general, that the \( \Lambda \)-lattices \( M \) in \( V \) such that \( \text{End}_A(M) \) is the maximal order in \( \text{End}_A(V) \) are linearly ordered, and this can be used to prove the Theorem, just as in §3.

Note that we could have used right-orders \( \Lambda'_{j+1} = O_j(\text{rad} \Lambda'_j) \) throughout, instead of left orders, and still obtain the same answer \( s = d - (e - 1) \) for the length of the chain \( \Lambda'_0 \subseteq \cdots \subseteq \Lambda'_s \). By uniqueness of \( \Lambda_s \), we would get \( \Lambda_s = \Lambda'_s \), but we do not know whether \( \Lambda_j = \Lambda'_j \) for all \( j, 1 < j < s \).

REFERENCES


Received September 21, 1984 and in revised form March 26, 1985. Supported by NSERC.

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