A NOTE ON ORDERINGS ON ALGEBRAIC VARIETIES

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It was proven in [A–G–R] that if \( V \subseteq \mathbb{R}^n \) is a surface and \( \alpha \) a total ordering in its coordinate polynomial ring, \( \alpha \) can be described by a half branch (i.e., there exists \( \gamma(0, \varepsilon) \to V \), analytic, such that for every \( f \in \mathbb{R}[V] \), \( \operatorname{sgn}_\alpha f = \operatorname{sgn} f(\gamma(t)) \) for \( t \) small enough). Here we prove (in any dimension) that the orderings with maximum rank valuation can be described in this way. Furthermore, if the ordering is centered at a regular point we show that the curve can be extended \( C^\infty \) to \( t = 0 \).

1. (1.0) Let \( V \) be an algebraic variety over \( \mathbb{R} \) and \( \alpha \) an ordering in \( K = \mathbb{R}(V) \). If \( \alpha \) is described by a half-branch \( \gamma: (0, \varepsilon) \to V \), no non-zero polynomial vanishes over \( \gamma(t) \) for \( t \) small enough. Consequently, if \( V' \) is birationally equivalent to \( V \) (i.e., \( \mathbb{R}(V') = \mathbb{R}(V) \)), \( \alpha \cap \mathbb{R}[V] \) is also described by a curve in \( V' \).

(1.1) PROPOSITION. Let \( V \) be an algebraic variety over \( \mathbb{R} \) and \( n = \dim V \). If \( \mathbb{R}[V] \) is an integral extension of \( \mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[\bar{x}] \) and \( \alpha \) an ordering on \( \mathbb{R}[V] \) such that \( \beta = \alpha \cap \mathbb{R}[\bar{x}] \) can be described by a half-branch, then the same holds true for \( \alpha \).

Proof. By our previous remark (1.0) we can suppose \( V \) is a hypersurface. Thus \( \mathbb{R}[V] = \mathbb{R}[\bar{x}, x_{n+1}](P) \) where \( P \in \mathbb{R}[\bar{x}][x_{n+1}] \) is a monic polynomial in \( x_{n+1} \). Let \( \delta \) be the discriminant of \( P \) and \( \pi: V \to \mathbb{R}^n \) the projection on the first \( n \)-coordinates. Then the restriction

\[
\pi_1: V \setminus \pi^{-1}(\delta = 0) \to \mathbb{R}^n \setminus \{ \delta = 0 \}
\]

has finite fibers with constant cardinal over every connected component. Moreover, by the implicit function theorem, \( \pi_1 \) is an analytic diffeomorphism from every connected component of \( V \setminus \pi^{-1}(\delta = 0) \) onto someone of \( \mathbb{R}^n \setminus \{ \delta = 0 \} \).

Let \( \gamma: (0, \varepsilon) \to \mathbb{R}^n \) be the curve describing \( \beta \). The connected components \( C_1, \ldots, C_p \) of \( \mathbb{R}^n \setminus \{ \delta = 0 \} \) are open semi-algebraic sets, and we can write

\[
C_i = \bigcup_{j=1}^q \{ f_{ij1} > 0, \ldots, f_{ijr} > 0 \}, \quad f_{ij} \in \mathbb{R}[\bar{x}].
\]
As \( \gamma \) describes the ordering in \( \mathbb{R}[x] \) and the \( C_i \)'s are pairwise disjoint, for \( t \) small enough, \( f_{i/t} (\gamma (t)) \) does not change the sign and \( \gamma (t) \) is contained in a unique \( C_{i_0} \). We put \( C = C_{i_0} \).

Let \( D_1, \ldots, D_s \) (we shall see below that \( s \) is not zero) be the connected components of \( V \setminus \pi^{-1} \{ \delta = 0 \} \) diffeomorphic to \( C \) via \( \pi \). We claim that

\[
   s = \text{number of extensions of } \beta \text{ to } R(V).
\]

By construction \( s \) is the number of roots of \( P(x, x_{n+1}) \) for every \( x \in C \). On the other hand, the number of extensions of \( \beta \) to \( R(V) \) coincides with the number of roots of \( P \in \mathbb{R}(x)[x_{n+1}] \) in a real closure of \( (\mathbb{R}(x), \beta) \) (see [Pr] 3.12). We shall prove now the latter is also the number of real roots of \( P(x, x_{n+1}) \) for \( x \in C \).

Let \( S = \{ P_0, \ldots, P_l \} \mathbb{R}(x)[x_{n+1}] \) be the standard Sturm sequence of

\[
   P(x, x_{n+1}) = x_{n+1}^m + a_1 x_{n+1}^{m-1} + \cdots + a_m, \quad M = 1 + m + \sum_{i=1}^{m} a_i^2
\]

and \( \Delta \) the product of all numerators and denominators of the non-zero coefficients of the polynomials in \( x_{n+1} \) used in the construction of \( S \). In this situation, by Artin's specialization theorem there exists \( x_0 \in \mathbb{R}^n \) such that

\[
   (a) \ f_{i_0/h}(x_0) > 0, \ \Delta(x_0) \neq 0, \text{ some } j = 1, \ldots, q, \text{ all } h = 1, \ldots, r
\]

\[
   (b) \ \text{sgn}_P P_k(\pm M) = \text{sgn}_R P_k(x_0, \pm M(x_0)), \quad k = 0, \ldots, l.
\]

By (a), \( x_0 \in C \) and \( S_{x_0} = \{ P_1(x_0), \ldots, P_l(x_0) \} \) is the standard Sturm sequence of \( P(x_0, x_{n+1}) \). By (b) the number of sign changes of \( S_{x_0} \) and \( S \) coincides. Then the claim is proven.

Now, let us denote by \( \gamma_k = (\pi_{|D_k})^{-1} \circ \gamma, \ k = 1, \ldots, l \) the liftings of \( \gamma \). Then it is easy to prove:

\[
   (a') \text{ If } f \in R[V] \setminus \{ 0 \}, \ f(\gamma_k(t)) \neq 0 \text{ and its sign does not change for } t \text{ small enough. Consequently every } \gamma_k \text{ defines an ordering that we call } \alpha_k.
\]

\[
   (b') \text{ If } k \neq k', \ \alpha_k \neq \alpha_{k'}.
\]

From the remarks above, \( \alpha \) must be equal to some \( \alpha_k \), hence it is described by the corresponding \( \alpha_k \).

2. (2.0) Let \( K \) and \( \Delta \) be ordered fields and \( p: K \to \Delta, \ \infty \) a place such that for \( x \) positive, \( p(x) \) is not negative. Then we define a signed place \( \hat{p}: K \to \Delta \cup \{ +\infty, \infty \} = \Delta, \ \pm \infty \) in the following way:

\[
   \hat{p}(x) = p(x) \text{ if } p(x) \neq \infty; \quad \hat{p}(x) = \text{sign}(x) \cdot \infty \text{ if } p(x) = \infty.
\]

Now assume \( K \) is the function field of a real algebraic variety \( V \), and \( \alpha \) an ordering in \( K \). A point \( O \in V \) is the center of \( \alpha \) in \( V \) if the real valued canonical place \( p_\alpha \) associated to \( \alpha \) (see [B] Chap. VII) is finite over
$R[V]$ and the ideal of $O$ is the center of $p_\alpha$ in $R[V]$. In that case, every function positive at $O$ is positive in $\alpha$, and if $\alpha$ is described by $\gamma$, then $\lim_{t \to 0} \gamma(t) = O$.

We are interested in the case when the rank of $p_\alpha$ is maximum (i.e., it coincides with the dimension of $V$). In this situation the decomposition of $p_\alpha$ in rank 1 places is

$$
K = K \to K_{n-1}, \infty \to \cdots \to R, \infty,
$$

where $K_j$ is a function field over $R$ of dimension $j$. Then it is possible to define uniquely orderings in $K_j$ ($j = 1, \ldots, r$) such that, considering $\alpha$ in $K$, all places verify the compatibility conditions. Thus we consider the associated signed places $\hat{\theta}_j: K_j \to K_{j-1}, \pm \infty$ (see [B] Chap. VIII), to get a decomposition of $\hat{p}_\alpha$ in rank 1 signed places.

(2.1) PROPOSITION. If $p_\alpha$ has a maximum rank, $\alpha$ can be described by a half-branch.

Proof. The proof goes by induction. If $n = 1$, by 1.1 and 1.0 we can suppose $K = R(x)$, $\alpha$ centered at $x = 0$, and $x >_\alpha 0$. Then, there is a unique ordering with this property (i.e., making $x$ infinitesimal with respect to $R$ and positive), and it is described by the curve $\gamma(t) = t$.

In the general situation we can choose $\xi_1, \ldots, \xi_{n-1}, \xi_n$ in $K$ such that $\theta_{n-1}(\xi_1), \ldots, \theta_{n-1}(\xi_n) \in K_{n-1}$ and:

(i) $\theta_{n-1}(\xi_1), \ldots, \theta_{n-1}(\xi_{n-1})$ are algebraically independent.

(ii) $\xi_1, \ldots, \xi_n$ are algebraically independent

(iii) $p_\alpha(\xi_i) = 0$ ($i = 1, \ldots, n$).

Since $K$ is the quotient field of the integral closure of $B = R[\xi_1, \ldots, \xi_{n-1}, \xi_n]$ we can suppose $K = q \cdot f(B)$ by 1.1. Then the kernel of $\theta_{n-1}: B \to K_{n-1}$ is an height one prime ideal and hence it is generated by some $F \in B$. The field $K_{n-1}$ is the function field of the hypersurface $\{ F = 0 \}$. Moreover we may assume $F >_\alpha 0$.

Let us consider, according to 2.0, the ordering $\beta$ associated to $r = \theta_0 \circ \cdots \circ \theta_{n-2}$ in $K_{n-1}$. Then $p_\beta = r$ and $\beta$ is centered at $0 = (0, \ldots, 0)$ which belongs to the hypersurface. Consequently, for every $f \in B$ we have:

(2.1.1) if $f(0) = p_\alpha(f) \neq 0$, then $\text{sgn}_\alpha f = \text{sgn} f(0)$

if $\theta_{n-1}(f) \neq 0$, $\text{sgn}_\alpha f = \text{sgn}_\beta \tilde{f}$, where $\tilde{f}$ is $f + (F)$

if $\theta_{n-1}(f) = 0$ and $f = u \cdot F'$ with g.c.d $(u, F) = 1$,

then $\text{sgn}_\alpha f = \text{sgn}_\alpha u = \text{sgn}_\beta \tilde{u}$.
Now we need a lemma:

(2.2) LEMMA. Let \( H = \{ F(x) = 0 \} \) be a real irreducible hypersurface in \( \mathbb{R}^n \) and \( \beta \) a rank \((n - 1)\) ordering in \( H \) (i.e., in \( \mathbb{R}[x]/(F) \)) centered at the point \( 0 \) and described by \( \gamma: (0, \varepsilon) \rightarrow H \). Then, there is not more than one ordering \( \alpha \) in \( \mathbb{R}[x] \) making \( F \) infinitesimal and positive, and inducing \( \beta \) in \( \mathbb{R}[x]/(F) \). Moreover \( \alpha \) can be described by a half-branch.

Proof. The first claim is an easy consequence of 2.1.1.

Next, as \( p_\beta \) has rank \( n - 1 \), \( p_\beta \) is discrete and its value group is isomorphic to \( \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \), lexicographically ordered. Let \( h \in \mathbb{R}[x]/(F) \) have value \((a_1, \ldots, a_{n-1})\) with \( a_1 \geq 1 \) (notice that this is possible because the valuation ring of \( p_\beta \) contains \( \mathbb{R}[x]/(F) \)), and put \( \psi(t) = h(\gamma(t)) \). Since \( p_\beta(h) = 0 \), \( h(0) = 0 \) and \( \lim_{t \rightarrow 0} \psi(t) = 0 \), \( \psi \) is analytic in \((0, \varepsilon)\). Now we define the analytic curve:

\[
\gamma^*: (0, \varepsilon) \rightarrow \mathbb{R}^n: t \mapsto \left( \gamma_i(t) + c_i e^{-1/\psi(t)^2} \right) \quad i = 1, \ldots, n
\]

where the \( c_i \)'s will be determined later.

Thus, the result follows from the statements (a) and (b) below.

(a) For any \( c_i \)'s, if \( G \in \mathbb{R}[x] \) is positive along \( \gamma \), so is along \( \gamma^* \).

(b) There is \((c_1, \ldots, c_n) \in \mathbb{R}^n\) such that \( F(\gamma^*(t)) > 0 \) for \( t \) small enough.

To prove (a) we first write:

\[
(2.2.1) \quad G(\gamma^*(t)) = G(\gamma(t)) + m(t) e^{-1/\psi(t)^2}
\]

where \( m(t) \) is a polynomial in \( \gamma_1(t), \ldots, \gamma_n(t) \) and \( e^{-1/\psi(t)^2} \). On the other hand, looking at the value of \( h \), for large \( m \in \mathbb{N} \) we know that \( \frac{h^m}{G} (G = G + (F) \in \mathbb{R}[x]/(F)) \) is infinitesimal in \( \beta \) w.r.t. \( \mathbb{R} \) and so, \( 1 - \frac{h^m}{G} > 0 \). Since \( G \) is positive in \( \beta \), taking an even \( m \) we have \( G > \beta h^m > 0 \). Hence \( G(\gamma(t)) > \beta \psi(t)^m > 0 \) for small \( t \) enough, what implies \( \lim_{t \rightarrow 0} e^{-1/\psi(t)^2}/G(\gamma(t)) = 0 \). Thus, we get (a) after dividing in 2.2.1 by \( G(\gamma(t)) \) and taking the limit when \( t \rightarrow 0 \).

For (b), we take the Taylor expansion of \( F \) at \( \gamma(t) \) and compute it at \( \gamma^*(t) \):

\[
(2.2.2) \quad F(\gamma^*(t)) = \sum_{i=1}^{n} \frac{\partial F(\gamma(t))}{\partial x_i} c_i e^{-1/\psi(t)^2} + \sum_{i,j} \frac{\partial^2 F(\gamma(t))}{\partial x_i \partial x_j} c_i c_j e^{-2/\psi(t)^2} + \ldots
\]
As \( \frac{\partial F}{\partial x_i} \in (F) \) for some \( i \), we have \( c_i = \text{sgn}_\beta(\partial F/\partial x_i) \) \( (= \pm 1 \neq 0) \)
and we take \( c_j = 0 \) for \( j \neq i \). Then, \( \beta \) being described by \( \gamma \):

\[
H(t) = \sum_{i=1}^{n} \frac{\partial F(\gamma(t))}{\partial x_i} c_i > 0, \quad \text{for small } t.
\]

Again we have \( \lim_{t \to 0} e^{-1/\dot{\gamma}(t)^2}/H(t) = 0 \). Then, dividing in 2.2.2 by \( H(t) \), we find \( F(\gamma^*(t))/H(t) > 0 \), hence \( F(\gamma^*(t)) > 0 \), for small \( t \).

(2.3) REMARK. Looking at the class of the curve \( \gamma \) at 0, we see that if \( O \in \text{Reg } H \), and \( \gamma \) can be extended \( C^\infty \) to \( t = 0 \), the same holds true for \( \gamma^* \).

(2.4) REMARK. Notice that 2.2 and 2.3 hold also true if we replace \( \mathbb{R}^n \) by an algebraic variety \( V \) with \( O \in \text{Reg } V \). In fact the same proof applies, by taking a regular system of parameters at \( O \) in the place of \( x_1, \ldots, x_n \).

(2.5) Application. As an example of the constructibility of the proof of 2.1 we determine the curves describing the rank 2 orderings in \( \mathbb{R}^2 \) (see [A-G-R]).

Firstly, after changes \( x \to \pm (x \pm a)^{\pm 1} \), \( y \to \pm (y \pm b)^{\pm 1} \), we can suppose \((0,0)\) is the center of the ordering \( \alpha \) and \( x > \alpha 0, y > \alpha 0 \). Assume the divisor \( w \) which specializes \( p_\alpha \) is centered in \( \mathbb{R}[x, y] \) at \( F(x, y) = 0 \), and \( x = t^n, y = a_1 t^{n_1} + \cdots (n \leq n_1), t > 0 \), is a primitive parametrization of the half-branch describing the corresponding ordering in \( \mathbb{R}[x, y]/(F) \). According to the above parametrization and looking at the proof of 2.2, we may choose \( h(x) = x, c_1 = 0 \) and \( c_2 = \pm 1 \) in the proof of 2.2, and we get a half-branch describing \( \alpha \) of the form:

\[
\gamma(t) = (t^n, \pm e^{-1/t^{2n}} + a_1 t^{n_1} + \cdots)
\]

Now assume that the prime divisor \( w \) is centered at the maximal ideal, \((x, y)\). Let us call \( v \) the valuation corresponding to \( p_\alpha \). Following Abhyankar [A], after a finite number of quadratic transforms along \( w \) we get the previous situation. In fact, we call \( A_0 = \mathbb{R}[x, y] \) and, if \( v(x) \leq v(y) \) (so \( w(x) \leq w(y) \)) we put: \( r_0 = p_\alpha(y/x), y_1 = (y - r_0 x)/x, x_1 = x \) and \( A_1 = A_0[x_1, y_1] \). Repeating this procedure we end at \( A_s = A_{s-1}[x_s, y_s] = \mathbb{R}[x_s, y_s] \) such that, the center of \( w \) in \( A_s \) is 1-dimensional, and \( w \) is centered at \((x_{s-1}, y_{s-1})\) in \( A_{s-1} \). We have, say,

\[
y_s = (y_{s-1} - r_{s-1} x_{s-1})/x_{s-1}
\]
and $x_s = x_{s-1}$. Hence $w(x_s) = w(x_{s-1}) > 0$ and $M_w \cap A_s = (x_s)$. Thus, according to the proof of 2.2, the half-branch $x_s = \pm e^{-1/t^2}$, $y_s = t$ describes the ordering in $A_s$. Hence, going backwards in the quadratic transformations, it follows easily that the ordering $\alpha$ can be described by a curve

$$(P(t, e^{-1/t^2}), Q(t, e^{-1/t^2}))$$

for some polynomials $P$ and $Q$.

3. (3.0) We finish this note with some considerations about the class at $t = 0$ of the $\gamma$'s describing orderings (see also [R] §3). To start with notice that any algebraically independent power series $x_1(t), \ldots, x_n(t)$, describe an ordering in $R[x]$. Then by [An] the set of such orderings is dense in the space of all orderings endowed with the Harrison Topology [H]. Moreover, the valuations associated to these orderings are discrete of rank one. Hence the orderings with maximum rank valuation, cannot be described by curves which are analytic at $t = 0$ unless the variety is a curve. So, the best result we can expect is the following:

(3.1) **Proposition.** If $V \subset R^n$ is an algebraic variety an $\alpha$ an ordering centered at $0 = (0, \ldots, 0)_{\text{Reg}}V$, with associated valuation of maximum rank, there is a half-branch describing $\alpha$ which can be extended $C^\infty$ (but not analytically) to $t = 0$. Furthermore the set of orderings of $R[V]$ described by half-branches $C^\infty$ at $t = 0$ but not by analytic ones, is dense in the space of orderings.

**Proof.** The proof goes by induction on $d = \dim V$. If $d = 1$, the valuation associated to the ordering $\alpha$ is discrete, has rank one, and the ordering is described by the unique branch of $V$ through $0$:

$$(t, u_2(t), \ldots, u_n(t))$$

where each $u_i(t)$ is analytic and the choice $t > 0$ or $t < 0$.

In the general case, set $\hat{\rho}_a = p$ and consider again

$$K = R(V) \xrightarrow{q} K_{n-1} \xrightarrow{r} R, \pm \infty, \quad p = r \circ q,$$

the decomposition of $p$ in signed places of rank one.

As we did in 2.1 we can find an (affine) algebraic variety $V_1$ and $\pi$: $V_1 \rightarrow V$ birational morphism such that the center of $q$ in $V_1$, say $H_1$, has dimension $d - 1$. By means of Hironaka's desingularization I [Hi] we may assume $V_1$ is smooth. Then by Hironaka's desingularization II (loc. cit), we find $\tilde{V}$ and $\tilde{\pi}$: $\tilde{V} \rightarrow V_1$, a proper birational map such that $\tilde{\pi}^{-1}(H_1)$ is
a normal crossing divisor. Let \( \tilde{0} \) be the center of \( p \) in \( \tilde{V} \) and \( \tilde{H} \) the center of \( q \). Since the valuation ring of \( q, R[V, T], \) dominates \( R[\tilde{V}] \) and \( \tilde{H} \) lies over \( H_1 \), we have \( K_{n-1} = a\tilde{f} \cdot \tilde{H} \) and the center of \( r \) in \( \tilde{H} \) is \( \tilde{0} \).

We call \( \beta \) the ordering in \( K_{n-1} \) corresponding to the precedent decomposition (i.e. \( \tilde{\beta} = r \)). Since \( r \) has maximum rank, by our inductive hypothesis the ordering \( \beta \cap R[\tilde{H}] \) can be described by \( \gamma : (0, \epsilon) \to \tilde{H} \), with \( \lim_{t \to 0} \gamma(t) = 0 \), and \( \gamma \) can be extended \( C^\infty \) to \( t = 0 \). Then, considering a modification \( \gamma^* \) of \( \gamma \) as we did in 2.2 and using Remarks 2.3 and 2.4, \( \alpha \) is described in \( \tilde{V} \) by \( \gamma^* \) and it can be extended \( C^\infty \) to \( t = 0 \). Finally \( \pi_1 \circ \tilde{\pi} \circ \gamma^* \) is a curve which defines the ordering \( \alpha \) and can be extended \( C^\infty \) to \( t = 0 \).

The second part comes from the first one, the above remark 3.0, and the fact that the set of orderings with maximum rank are dense (see [B], 8.4.9).

**References**


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