DERIVATIONS WITH INVERTIBLE VALUES IN RINGS WITH INVOLUTION

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Let $R$ be a semiprime 2-torsion free ring with involution $*$ and let $S = \{ x \in R \mid x = x^* \}$ be the set of symmetric elements. We prove that if $R$ has a derivation $d$, non-zero on $S$, such that for all $s \in S$ either $d(s) = 0$ or $d(s)$ is invertible, then $R$ must be one of the following: (1) a division ring, (2) $2 \times 2$ matrices over a division ring, (3) the direct sum of a division ring and its opposite with exchange involution, (4) the direct sum of $2 \times 2$ matrices over a division ring and its opposite with exchange involution, (5) $4 \times 4$ matrices over a field with symplectic involution.

Recently Bergen, Herstein and Lanski studied the structure of a ring $R$ with a derivation $d \neq 0$ such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible. They proved that, except for a special case which occurs when $2R = 0$, such a ring must be either a division ring $D$ or the ring $D_2$ of $2 \times 2$ matrices over a division ring.

In this paper we address ourselves to a similar problem in the setting of rings with involution, namely: let $R$ be a 2-torsion free semiprime ring with involution and let $S$ be the set of symmetric elements. If $d \neq 0$ is a derivation of $R$ such that the non-zero elements of $d(S)$ are invertible, what can we conclude about $R$?

We shall prove that $R$ must be rather special. In fact we shall show the following:

**Theorem.** Let $R$ be a 2-torsion free semiprime ring with involution. Let $d$ be a derivation of $R$ such that $d(S) \neq 0$ and the non-zero elements of $d(S)$ are invertible in $R$. Then $R$ is either:

1. a division ring $D$, or
2. $D_2$, the ring of $2 \times 2$ matrices over $D$, or
3. $D \oplus D^{\text{op}}$, the direct sum of a division ring and its opposite relative to the exchange involution, or
4. $D_2 \oplus D_2^{\text{op}}$ with the exchange involution, or
5. $F_4$, the ring of $4 \times 4$ matrices over a field $F$ with symplectic involution.

In case $R = F_4$ with * symplectic we shall prove that $d$ is inner. As Herstein has pointed out, an easy example of such a ring is given by
taking $F$ to be a field in which $-1$ is not a square and $d$ the inner derivation in $F_4$ induced by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ where $I$ is the identity matrix in $F_2$.

Now, if $R = D \oplus D^{\text{op}}$ or $R = D_2 \oplus D_2^{\text{op}}$ then $S \cong D$ or $S \cong D_2$ respectively. Thus both cases come naturally from [1].

We remark that if $d(S) = 0$ then $d(\bar{S}) = 0$, where $\bar{S}$ is the subring generated by $S$; hence, if $R$ is semiprime, by [3, Theorem 2.1.5] either $S$ lies in the center of $R$ (and $R$ satisfies the standard identity of degree 4) or $d(J) = 0$ for some non-zero ideal $J$ of $R$.

Let $R$ be a ring with involution; we denote by $Z$ the center of $R$ and by $S$ and $K$ the sets of symmetric and skew elements of $R$ respectively. Throughout this paper, unless otherwise stated, $R$ will be a 2-torsion free semiprime ring with an involution $*$ and $d$ will be a derivation of $R$ such that $d(S) \neq 0$ and the non-zero elements of $d(S)$ are invertible.

We begin with the following

**Lemma 1.** If $I = I^*$ is a non-zero ideal of $R$ then $d(I \cap S) \neq 0$.

**Proof.** Suppose, by contradiction, that $d(I \cap S) = 0$ and let $t \in S$ be such that $d(t) \neq 0$. For all $s \in I \cap S$ the elements $sts$ and $st + ts$ lie in $I \cap S$, hence

$$0 = d(sts) = sd(t)s$$

$$0 = d(st + ts) = sd(t) + d(t)s$$

Multiplying the second equality from the left by $s$, we obtain $s^2d(t) = 0$. Now, from our basic hypothesis on $R$, $d(t)$ is invertible; hence $s^2 = 0$, for all $s \in I \cap S$.

Now let $x \in R$, $s \in I \cap S$. Then the element $sx + x^*s$ lies in $I \cap S$ and, so, it must be square-zero. Therefore, since $s^2 = 0$,

$$0 = (sx + x^*s)sx = (sx)^3,$$

that is, every element in the right ideal $sR$ is nilpotent of index $\leq 3$. By Levitski's Theorem [2, Lemma 1] we must have $sR = 0$ and, so, $s = 0$. This proves that $I \cap S = 0$.

For $x \in I$, $x + x^* \in I \cap S$; hence $x = -x^*$ and $x^2 \in I \cap S = 0$. This $I$ is a nilideal of index $\leq 2$. This forces $I = 0$, a contradiction. $\square$

At this stage we are able to prove our result in case $R$ is not simple; in fact we have

**Proposition 1.** If $R$ is not a simple ring then either $R \cong D \oplus D^{\text{op}}$, $D$ a division ring, or $R \cong D_2 \oplus D_2^{\text{op}}$ and $*$ is the exchange involution.
Proof. Let \( I \neq R \) be an ideal of \( R \) such that \( I = I^* \).

Since \( d(I^2 \cap S) \subset d(I^2) \subset I \), Lemma 1 shows that \( I^2 = 0 \) and the semiprimeness of \( R \) forces \( I = 0 \). We have proved that \( R \) does not contain proper \(*\)-ideals.

If \( R \) is not simple, then there exists a proper ideal \( I \neq I^* \). Since \( I + I^* \) is a non-zero \(*\)-ideal of \( R \), \( I + I^* = R \). Also \( I \cap I^* \neq R \) is a \(*\)-ideal of \( R \), hence \( I \cap I^* = 0 \). Thus we have that \( R = I \oplus I^* \). Moreover since \( I^2 \neq I^{*2} \) we also get \( R = I^2 \oplus I^{*2} \) and, so, \( I = I^2 \) and \( I^* = I^{*2} \); hence, they are both invariant under \( d \). Clearly \( S = \{ x + x^* | x \in I \} \) and so \( d(x) \) and \( d(x^*) \) are both 0 or both units in \( I \) and \( I^* \) respectively.

By [1, Theorem 1] \( I \), and hence also \( I^* \), is either a division ring \( D \) or \( D_2 \). If \( d(I) = 0 \), then \( d(I^*) \neq 0 \) and the argument above leads to the same conclusion. Clearly the involution in \( R \) is the exchange involution. \( \square \)

If \( R \) is a prime ring we denote by \( C \) the extended centroid of \( R \) and by \( Q = RC \) the central closure of \( R \) (see [3, pg. 22]). The next lemma holds for arbitrary rings with involution, with a derivation \( d \neq 0 \).

**Lemma 2.** Let \( R \) be a prime ring with involution, with a derivation \( d \neq 0 \). Let \( x \in R \) be such that for all \( s \in S \)

\[
xsx*d(R)xsx* = 0.
\]

Then either \( x*d(R)x = 0 \) or \( Q = RC \) has a minimal right ideal.

**Proof.** For \( y \in R \) let \( u = x^*d(y)x \). Then if \( s \in S \), \( ususu = ususu^* = 0 \); now, if \( r \in R \), \( su^*r^* + rus \in S \) and, so,

\[
0 = vsu(su^*r^* + rus)u(su^*r^* + rus)u = usurusrusu.
\]

This says that every element in the right ideal \( usuR \) is nilpotent of index \( \leq 3 \). By Levitski’s theorem [2, Lemma 1.1], \( usuR = 0 \) and so \( usu = 0 \) for all \( s \in S \). By [5, Lemma 3], if \( u \neq 0 \), \( Q = RC \) has a minimal right ideal. \( \square \)

In light of Proposition 1 we now make a first reduction: from now on, unless otherwise stated, we will always assume that \( R \) is a simple ring with 1. In this case clearly \( R \) coincides with its own central closure.

The next lemmas give us some information about the nature of the symmetric elements in the kernel of \( d \).

**Lemma 3.** Let \( a \in S \). If for all \( s \in S \) we have that \( asa = \lambda a \), for some \( \lambda = \lambda(s) \in z \), then \( R \) has a minimal right ideal.
Proof. Let \( x \in R \). Then \( ax + x^*a = \lambda a \), for some \( \lambda \in Z \), that is \( ax^*a = \lambda a - axa \). Let \( \mu \in Z \) be such that \( a(xax + x^*ax^*)a = \mu a \). Playing these off against each other we get

\[
0 = axaxa + ax^*ax^*a - \mu a = 2axaxa - 2\lambda axa + (\lambda^2 - \mu)a.
\]

Therefore \( 2(ax)^3 - 2\lambda(ax)^2 + (\lambda^2 - \mu)ax = 0 \) and, since \( \text{char } R \neq 2 \), \( ax \) is algebraic over \( Z \) of degree at most 3. This proves that \( aR \) is algebraic over \( Z \) of bounded degree. Thus \( aR \) satisfies a polynomial identity; hence \( R \) satisfies a generalized polynomial identity. Since \( R \) coincides with its own central closure, by a theorem of Martindale [3, Theorem 1.3.2.] \( R \) has a minimal right ideal.

**Lemma 4.** Suppose \( R \) does not contain minimal right ideals. If \( a \in S \) is such that \( d(a) = 0 \) then either \( a \) is invertible or \( ad(R)a = 0 \).

Proof. Suppose \( a \neq 0 \) and \( a \) is not invertible. Since \( d(a) = 0 \) then, for all \( s \in S \), \( d(asa) = ad(s)a \) and it is not invertible. Hence \( ad(s)a = 0 \).

Now let \( x \in R \). Then \( ad(x + x^*)a = 0 \) implies \( ad(x)a = -ad(x^*)a \). Therefore for all \( s \in S \), recalling that \( d(a) = ad(s)a = 0 \) we get

\[
asad(x)a = ad(sax)a = -ad(x^*sa)a = ad(x)asa.
\]

We have proved that for all \( x \in R \), \( s \in S \),

\[(1) \quad asad(x)a = ad(x)asa\]

Since \( d(a) = 0 \), \( d(aR) \subseteq aR \); moreover if \( \rho_R(a) \) is the left annihilator of \( a \) in \( R \), \( d(\rho_R(a)) \subseteq \rho_R(a) \); this says that \( d \) induces a derivation (which we will still denote by \( d \)) in the prime ring \( R_1 = aR/\rho_R(a) \cap aR \). Moreover, for \( s \in S \), if \( as \) is the image of \( as \) in \( R_1 \), from (1) we get

\[
\overline{as}d(\overline{ax}) = d(\overline{ax})\overline{as}, \quad \text{for all } \overline{ax} \in R_1.
\]

By [4] since \( \text{char } R \neq 2 \) either \( d = 0 \) in \( R_1 \) or \( \overline{as} \in Z(R_1) \), the center of \( R_1 \). That is, either \( ad(R)a = 0 \) or \( axaxa = axasa \) for all \( x \in R \).

If \( ad(R)a = 0 \) we are done; therefore we may assume that \( axaxa = axasa \), for all \( x \in R \), \( s \in S \). But then, by [3, Lemma 1.3.2.], \( asa = \lambda a \), for some \( \lambda \in Z \) and, by Lemma 3, \( R \) has a minimal right ideal, a contradiction.

We remark that since \( R \) is simple with 1 then it must be a primitive ring. Now, through a repeated application of the density theorem we will be able to prove that \( R \) is artinian.
**Proposition 2.** *R is a simple artinian ring.*

*Proof.* Since *R* is primitive it is a dense ring of linear transformations on a vector space *V* over a division ring *D*. By [3, Lemma 1.1.2.] to prove that *R* is artinian it is enough to prove that *R* has a minimal right ideal or equivalently that *R* contains a non-zero transformation of finite rank. Suppose, by contradiction, that this is not the case.

Let *s* ∈ *S* be such that *d*(*)s*) ≠ 0 and suppose that there exist linearly independent vectors *v*, *w* ∈ *V* such that

\[ v_0 = w_0 = 0. \]

Since *d*(*)s*) is invertible, the vectors *v*_0*(*s*) and *w*_0*(*s*) are linearly independent over *D*. Moreover, since *R* doesn’t contain non-zero transformations of finite rank, there exists a vector *u* ∈ *V* such that *u*_0*(*s*) *D* + *w*_0*(*s*) *D*, i.e., *u*, *v*_0*(*s*), *w*_0*(*s*) are linearly independent over *D*.

By the density of the action of *R* on *V*, there exists *x* ∈ *R* such that

\[ u*x* ≠ 0 \]
\[ vD(s)x = 0 \]
\[ wD(s)x ≠ 0. \]

Let *t* ∈ *S*. Since *vD*(*)s*) *x* = *v*_0*(*s*) *x* = 0 then *vD*(*)s* *t* *x* *s* *s* = 0; hence, since *s* *t* *x* *s* ∈ *S* and *d*(*)s* *t* *x* *s*) is not invertible, we must have *d*(*)s* *t* *x* *s*) = 0. Moreover *s*, and so *s* *t* *x* *s*, is not invertible. Since *R* has no minimal right ideals, by applying Lemma 4 to the element *s* *t* *x* *s*, we get

\[ s* t x* s (R) s* t x* s x = 0, \]

for all *t* ∈ *S*. Hence Lemma 2 implies

\[ x*sd(R)xRx = 0. \]

Now let *y*, *z* ∈ *R*. Since *x* *s* *d*(*y*) *s* *x* = 0 we have

\[ 0 = x*sd(y)szx = x*syd(sz)xz. \]

Hence *x* *s* *R* *d*(*s* *R*) *s* *x* = 0 and, since *x* *s* ≠ 0, the primeness of *R* forces

\[ d(sxR)x = 0. \]

If *y* ∈ *R* we get

\[ 0 = d(sxy)x = d(s)yxRx + sd(xy)xz; \]

hence, since *w* = 0, 0 = *wD*(*)s* *x* *y* *s* = *wD*(*)s* *y* *x* *s* *s*. But *wD*(*)s*) ≠ 0, and, by the density of the action of *R* on *V*, *wD*(*)s*) *x* *R* = *V*; thus

\[ 0 = wD(s)xRx = Vxssx \]

implying *x* *s* = 0, a contradiction.

We have proved that for every *s* ∈ *S* with *d*(*)s*) ≠ 0, \( \dim_D \ker s \leq 1 \).

Now let *W* be a finite dimensional subspace of *V* such that \( \dim_D W > 1 \) and let \( \rho = \rho_w = \{ x \in R \mid Wx = 0 \} \); *ρ* is a right ideal of *R*. 
We claim that there exists \( s \in \rho \cap S \) such that \( s^2 \neq 0 \). In fact, suppose not and let \( x \in \rho, s \in \rho \cap S \). Then, since \((xs + sx*)^2 = S^2 = 0\); we get \( 0 = s(xs + sx*) = s(xs)^2 \), i.e., \( sp \) is a right ideal nil of bounded index. By Levitski's theorem \( sp = 0 \); hence \((\rho \cap S)p = 0 \). Now, since \( R \) has no minimal right ideals, by [3, Lemma 5.1.2.], for \( v \not\in W \), there exists \( x \in \rho \) such that \( x^* \in \rho, vx^* = 0 \) and \( v(x + x^*) = vx \not\in W + Dv \). But then, by density, there exists \( y \in \rho \) such that \( v(x + x^*)y \neq 0 \), contradicting the fact that \((x + x^*)y \in (\rho \cap S)p = 0 \). This establishes the claim.

Then set \( s \in \rho \cap S \) such that \( s^2 \neq 0 \). Since \( \rho \) is a proper right ideal of \( R \), \( s \) is not invertible; moreover, since \( \text{dim}_D \ker s \geq \text{dim} W > 1 \), \( d(s) = 0 \). Hence, by Lemma 4, \( sd(R)s = 0 \).

Now, if \( x \in \rho \) then \( sx^* + xs \in \rho \cap S \) and \( d(s) = 0 \) implies \( 0 = d(sx^* + xs) = sd(x^*) + d(x)s \). Since \( sd(x^*)s = 0 \), multiplying by \( s \) from the right we get \( d(x)s^2 = 0 \). Thus \( d(\rho)s^2 = 0 \). Now, for \( x, y \in \rho \), \( 0 = d(xy)s^2 = d(xys^2) = 0 \) and, since \( R \) is prime and \( s^2 \neq 0 \), \( d(\rho)p = 0 \). Clearly \( d(\rho) \neq 0 \); so, let \( x \in \rho \) be such that \( d(x) \neq 0 \). If \( vd(x) \not\in W \) for some \( v \in V \), then by density there exists \( r \in \rho \) such that \( vd(x)r \neq 0 \), contradicting the fact that \( d(x)r \in d(\rho)p = 0 \). Thus \( Vd(x) \subset W \) and \( d(x) \) is a transformation of finite rank, a contradiction.

We are now in a position to prove the Theorem:

**Proof of the Theorem.** By Proposition 1 and Proposition 2 we may assume that \( R \) is a simple artinian ring. Hence, \( R = D_n \), the ring of \( n \times n \) matrices over a division ring \( D \).

Suppose first that * on \( D_n \) is of transpose type and assume \( n > 2 \). Let \( e_{ij} \) be the usual matrix units. For \( i = 1, \ldots, n \) \( e_{ii} = e_{ii}^* \in S \) implies \( d(e_{ii}) = e_{ii}d(e_{ii}) + d(e_{ii})e_{ii} \). Thus, since \( \text{rank } e_{ii} = 1 \), \( \text{rank } d(e_{ii}) \leq 2 \) and, being \( n > 2 \), \( d(e_{ii}) \) cannot be invertible. Hence \( d(e_{ii}) = 0 \), \( i = 1, \ldots, n \).

Now, if \( i \neq j \), for a suitable \( 0 \neq c \in D \), \( e_{ij} + ce_{ji} = e_{ij} + e_{ij}^* \in S \). Thus

\[
d(e_{ij} + ce_{ji}) = d(e_{ii}(e_{ij} + ce_{ji}) + (e_{ij} + ce_{ji})e_{ii})
\]

\[
= e_{ii}d(e_{ij} + ce_{ji}) + d(e_{ij} + ce_{ji})e_{ii};
\]

and so, \( \text{rank } d(e_{ij} + ce_{ji}) \leq 2 \). It follows \( d(e_{ij} + ce_{ji}) = 0 \) which implies \( 0 = d(e_{ii}(e_{ij} + ce_{ji})) = d(e_{ij}) \).

We have proved that \( d(e_{ij}) = 0 \) for \( i, j = 1, \ldots, n \). Now let \( x \in D \).
If \( i \neq j, \ S \ni xe_{ij} + (xe_{ij})^* = xe_{ij} + c_1x*c_2e_{ji} \) for suitable \( c_1, c_2 \in D \cap S \). We have:

\[
\text{rank}(d(xe_{ij} + c_1x*c_2e_{ji})) = \text{rank}(d(x)e_{ij} + d(e_1x*c_2)e_{ji}) \leq 2,
\]

hence \( d(xe_{ij} + e_1x*c_2e_{ji}) = 0 \), and, multiplying by \( e_{ji} \) from the right we get \( d(x)e_{ii} = 0 \), for all \( i = 1, \ldots, n \). Thus \( d(x) = d(xI) = \Sigma_i d(x)e_{ii} = 0 \), i.e. \( d(D) = 0 \). In short \( d = 0 \) in \( D_n \).

Now suppose that \( * \) is symplectic. In this case \( D = F \) is a field and suppose \( n > 4 \). Let \( I_1 = e_{11} + e_{22} \), \( I_2^2 = I_1 \in S \), so \( \text{rank}(d(I_1)) = \text{rank}(I_1d(I_1) + d(I_1)I_1) \leq 4 \) implies \( d(I_1) = 0 \). Now, for \( i \) odd, \( a = e_{1i} + e_{i+1,2} \in S \); hence \( d(a) = d(I_1a + aI_1) = I_1d(a) + d(a)I_1 \) has rank \( \leq 4 \). It follows \( d(a) = 0 \) and, so, for \( i \neq 1, 0 = d(I_1a) = d(e_{1i}). \) On the other hand, if \( i \) is even, \( e_{1i} - e_{i-1,2} \in S \) and by the same argument we get \( d(e_{1i}) = 0 \) for \( i \neq 2 \). Moreover by looking at \( e_{1i} + e_{i1}^* \) as above, we obtain \( d(e_{1i}) = 0 \) for \( i \neq 1, 2 \). At this stage it easily follows \( d(e_{ij}) = 0 \) for all \( i, j = 1, \ldots, n \). Since \( d(I_1) = 0 \) implies \( d(F) = 0 \), then \( d = 0 \) in \( F_n \) and we are done.

We are left with the case \( R = F_4 \) and \( * \) symplectic. We will prove that in this case \( d \) must be inner. By a well known result on finite dimensional simple algebras it is enough to prove that \( d(F) = 0 \). So, suppose by contradiction that there exists \( a \in F \) such that \( d(a) \neq 0 \) and let \( s \in S, \ s \neq 0 \), be such that \( d(s) = 0 \). Then, since \( d(a) \in F, \ d(as) = d(a)s \neq 0 \) implying \( s \) invertible. Therefore, for every \( s \in S, \ s \neq 0 \), \( d(s) = 0 \) implies \( s \) invertible.

Now, if \( I \) is the identity matrix in \( F_2 \), \( t = (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}) \in S \) and, since \( t \) is not invertible, \( d(t) \neq 0 \). Moreover it is easy to prove that \( d(t) = (\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}) \) where \( A, B \in F_2 \). Now let \( V \) be a 4-dimensional vector space over \( F \) and let \( \{ e_1, e_2, e_3, e_4 \} \) be the standard basis for \( V \). Then since \( d(t) \) is invertible, \( e_1d(t), e_2d(t) \) are linearly independent over \( F \); moreover \( e_1d(t), e_2d(t) \in \text{Span}_F\{e_3, e_4\} \).

Clearly, there exists an element \( x \in F_4 \) such that \( e_1d(t)x = e_2d(t)x = 0 \) and \( \text{span}_F\{e_1x, e_2x\} = \text{span}_F\{e_3, e_4\} \).

Now writing

\[
x = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\]

where \( X_{ij} \in F_2 \), we have that \( X_{21} \) is a unit and that \( (txx^*t)_{2,2} = X_{21}X_{21}^* \neq 0 \), a contradiction.

\[ \square \]

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