A THEOREM ON HOLOMORPHIC EXTENSION OF CR-FUNCTIONS

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We prove the holomorphic extendability on a domain $D \subset C^n$, $n \geq 2$, of the continuous CR-functions on a relatively open connected subset of $\partial D$, provided the complementary subset of $\partial D$ is $\mathcal{O}(\overline{D})$-convex.

Introduction. Let $D$ be a relatively compact open domain in $C^n$, $n \geq 2$, with boundary $\partial D$, and $K$ a compact subset of $\partial D$. We require $D$ and $K$ to be such that $\partial D \setminus K$ is a real hypersurface of class $C^1$ in $C^n \setminus K$.

The purpose of this paper is to give a sufficient condition on $D$ and $K$ guaranteeing the holomorphic extendability on all of $D$ of the CR-functions on $\partial D \setminus K$. Our theorem, which states the condition, improves and generalizes previous results in this direction obtained in Lupaccioli-Tomassini [6] and in Tomassini [10].

Let $\mathcal{O}(\overline{D})$ be the algebra of complex-valued functions on $\overline{D}$ each of which is holomorphic on an open neighborhood of $\overline{D}$, and $\hat{K}_D$ the $\mathcal{O}(\overline{D})$-hull of $K$. i.e.,

$$\hat{K}_D = \bigcap_{\varphi \in \mathcal{O}(\overline{D})} \{ z \in \overline{D} ; |\varphi(z)| \leq \max_{K} |\varphi| \}.$$

Our main result is the following theorem on holomorphic extension of CR-functions.

THEOREM 1. Assume that $K$ is $\mathcal{O}(\overline{D})$-convex, i.e., $\hat{K}_D = K$, and $\partial D \setminus K$ is connected. Then every continuous CR-function $f$ on $\partial D \setminus K$ has a unique extension $F$ continuous on $\overline{D} \setminus K$ and holomorphic on $D$.

A seemingly more general theorem is the following one.

THEOREM 2. Assume that $\partial D \setminus \hat{K}_D$ is a connected real hypersurface of class $C^1$ in $C^n \setminus \hat{K}_D$. Then every continuous CR-function $f$ on $\partial D \setminus \hat{K}_D$ has a unique extension $F$ continuous on $\overline{D} \setminus \hat{K}_D$ and holomorphic on $D \setminus \hat{K}_D$.

\textsuperscript{1}Added in proof. Recently Edgar Lee Stout kindly informed me of his paper [12], where the same condition is already recognized to be sufficient, when $D$ is a domain of holomorphy, for a parallel extendability's property in the setting of holomorphic functions.
However, if we set \( D' = D \setminus \hat{K}_D \) and \( K' = D' \cap \hat{K}_D \), it is an easy matter to see that Theorem 2 is equivalent to Theorem 1 with \( D' \) and \( K' \) in place of \( D \) and \( K \).

Before going into the proof of Theorem 1, let us discuss a nontrivial situation where it applies.

Observe that, since plainly

\[
\hat{K}_D = \bigcap_{U \supseteq D} \hat{K}_U,
\]

where \( U \) ranges over the open neighbourhoods of \( D \), it suffices, in order that \( \hat{K}_D = K \), that, for some \( U \), \( \hat{K}_U \cap \overline{D} = K \), i.e. \( \hat{K}_U \) does not meet \( \overline{D} \setminus K \). Suppose, then, that the following holds: there is an upper semicontinuous plurisubharmonic function \( \rho \) on a Stein open neighbourhood \( U \) of \( \overline{D} \), so that \( K \subset \{ \rho = 0 \} \) and \( \overline{D} \setminus K \subset \{ \rho > 0 \} \). Since \( \hat{K}_U \) coincides with \( \hat{K}_U^p \), the hull of \( K \) with respect to the plurisubharmonic functions on \( U \) (cf. Hörmander [5], p. 91), it follows that \( \hat{K}_U \) is contained in \( \{ \rho \leq 0 \} \), and hence \( \hat{K}_U \cap \overline{D} = K \). In the case \( \rho \) is pluriharmonic, \( U \) may be required to be simply connected, instead that Stein; for \( \rho \) has then a unique pluriharmonic extension \( \tilde{\rho} \) to the envelope of holomorphy \( \tilde{U} \) of \( U \), and hence \( \hat{K}_U \subset \hat{K}_U = \hat{K}_U^p \subset \{ \tilde{\rho} \leq 0 \} \).

1. Preliminary facts. (a) We denote by \( \omega(\xi) \) the Martinelli form relative to a point \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \), that is

\[
\omega(\xi) = C_n \frac{dz_1 \wedge \cdots \wedge dz_n}{|z - \xi|^{2n}} \\
\wedge \sum_{a=1}^{n} (-1)^{a-1}(\bar{\xi}_a - \bar{\xi}_a) d\bar{z}_1 \wedge \cdots \wedge \hat{\alpha} \cdots \wedge d\bar{z}_n
\]

(\( C_n = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n \)).

Given a holomorphic function \( \varphi \) on an open set \( U \subset \mathbb{C}^n \) and a point \( \xi \in U \), we denote by \( L_\xi(\varphi) \) the level set of \( \varphi \) through \( \xi \), that is

\[
L_\xi(\varphi) = \{ z \in U; \varphi(z) = \varphi(\xi) \}.
\]

It is known that for any \( \varphi \in \mathcal{O}(U) \) there exist holomorphic maps \( h = (h_1, \ldots, h_n) \in \mathcal{O}^n(U \times U) \) such that, for each \( (z, \xi) \in U \times U \),

\[
(*) \quad \varphi(z) - \varphi(\xi) = \sum_{a=1}^{n} h_a(z, \xi)(z_a - \xi_a)
\]

(cf. Harvey [3], Lemma 2.3). Then we set:

\[
(1.1) \quad \mathcal{O}_\varphi^n(U \times U) = \{ h \in \mathcal{O}^n(U \times U); (*) \text{ holds} \}.
\]

Any \( h \in \mathcal{O}_\varphi^n(U \times U) \) allows one to define canonically, for \( \xi \in U \), a \( \bar{\partial} \)-primitive of \( \omega(\xi) \) on \( U \setminus L_\xi(\varphi) \), that is \( (n, n-2) \)-form \( \Phi_h(\xi) \) on
\( U \setminus L_\zeta(\varphi) \) such that
\[
\omega(\zeta) = \overline{\partial} \Phi_h(\zeta) = d\Phi_h(\zeta).
\]

As a matter of fact, consider, for every \( \alpha = 1, \ldots, n \), the following \((n, n-2)\)-form on \( \mathbb{C}^n \setminus \{z_\alpha = \zeta_\alpha\} = \mathbb{C}^n \setminus L_\zeta(z_\alpha) \):
\[
\Omega_\alpha(\zeta) = \frac{(-1)^{\alpha+n}}{n-1} C_n \frac{dz_1 \wedge \cdots \wedge dz_n}{(z_\alpha - \zeta_\alpha)|z - \zeta|^{2n-2}}
\]
\[\wedge \left[ \sum_{\beta=1}^{\alpha-1} (-1)^\beta \left( \bar{z}_\beta - \bar{\zeta}_\beta \right) d\bar{z}_1 \wedge \cdots \hat{\beta} \cdots \hat{\alpha} \cdots \wedge d\bar{z}_n \right. \]
\[+ \sum_{\beta=\alpha+1}^n (-1)^{\beta-1} \left( \bar{z}_\beta - \bar{\zeta}_\beta \right) d\bar{z}_1 \wedge \cdots \hat{\beta} \cdots \hat{\alpha} \cdots \wedge d\bar{z}_n \bigg].
\]

One verifies that, on \( \mathbb{C}^n \setminus L_\zeta(z_\alpha) \), \( \omega(\zeta) = \overline{\partial} \Omega_\alpha(\zeta) \).
Then set
\[
(1.2) \quad \Phi_h(\zeta) = \frac{1}{\varphi(z) - \varphi(\zeta)} \sum_{\alpha=1}^n h_\alpha(z, \zeta)(z_\alpha - \zeta_\alpha) \Omega_\alpha(\zeta).
\]

It is plain that \( \Phi_h(\zeta) \) is indeed a real analytic \( \overline{\partial} \)-primitive of \( \omega(\zeta) \) on \( U \setminus L_\zeta(\varphi) \).

Such \( \overline{\partial} \)-primitives of the Martinelli form will play a fundamental role in the proof of our extension theorem. Now we derive the properties of them that will be needed.

Let there be given open sets \( U, U' \subset \mathbb{C}^n \) such that \( U \cap U' \neq \emptyset \), functions \( \varphi \in \mathcal{O}(U) \), \( \varphi' \in \mathcal{O}(U') \) and maps \( h \in \mathcal{O}_\varphi(U \times U) \), \( h' \in \mathcal{O}_{\varphi'}(U' \times U') \), and let \( \zeta \) be a point in \( U \cap U' \). Suppose first that \( n \geq 3 \), and consider, for every \( \alpha, \beta = 1, \ldots, n \) with \( \alpha \neq \beta \), the \((n, n-3)\)-form \( \Lambda_{\alpha,\beta}(\zeta) \) on \( \mathbb{C}^n \setminus (L_\zeta(z_\alpha) \cup L_\zeta(z_\beta)) \) defined as follows: for \( \alpha < \beta \)
\[
\Lambda_{\alpha,\beta}(\zeta) = \frac{(-1)^{n+\alpha+\beta}}{(n-1)(n-2)} C_n \frac{dz_1 \wedge \cdots \wedge dz_n}{(z_\alpha - \zeta_\alpha)(z_\beta - \zeta_\beta)|z - \zeta|^{2n-4}}
\]
\[\wedge \left[ \sum_{\gamma=1}^{\alpha-1} (-1)^\gamma \left( \bar{z}_\gamma - \bar{\zeta}_\gamma \right) d\bar{z}_1 \wedge \cdots \hat{\gamma} \cdots \hat{\alpha} \cdots \hat{\beta} \cdots \wedge d\bar{z}_n \right. \]
\[+ \sum_{\gamma=\alpha+1}^{\beta-1} (-1)^{\gamma-1} \left( \bar{z}_\gamma - \bar{\zeta}_\gamma \right) d\bar{z}_1 \wedge \cdots \hat{\gamma} \cdots \hat{\alpha} \cdots \hat{\beta} \cdots \wedge d\bar{z}_n \bigg] \]
\[+ \sum_{\gamma=\beta+1}^n (-1)^\gamma \left( \bar{z}_\gamma - \bar{\zeta}_\gamma \right) d\bar{z}_1 \wedge \cdots \hat{\gamma} \cdots \hat{\beta} \cdots \hat{\alpha} \cdots \wedge d\bar{z}_n \bigg].
\]

\(^2\)The forms \( \Omega_\alpha(\zeta) \) were considered first by Martinelli [7], to give a proof of Hartogs' theorem.
and for $\alpha > \beta \Lambda_{\alpha,\beta}(\zeta) = -\Lambda_{\beta,\alpha}(\zeta)$. One can verify that $\Omega_{\alpha}(\zeta) - \Omega_{\beta}(\zeta) = \overline{\partial} \Lambda_{\alpha,\beta}(\zeta)$. Then, consider the following $(n, n - 3)$-form on $(U \setminus L_{\zeta}(\varphi)) \cap (U' \setminus L_{\zeta}(\varphi'))$

$$X_{h,h'}(\zeta) = \frac{1}{(\varphi(z) - \varphi'(\zeta))(\varphi'(z) - \varphi'(\zeta))} \sum_{1 \leq \alpha < \beta \leq n} (h_{\alpha} h_{\beta}' - h_{\beta} h_{\alpha}') (z_{\alpha} - \zeta_{\alpha}) (z_{\beta} - \zeta_{\beta}) \Lambda_{\alpha,\beta}(\zeta).$$

It is easily seen that, on $(U \setminus L_{\zeta}(\varphi)) \cap (U' \setminus L_{\zeta}(\varphi'))$, 

$$(1.3) \quad \Phi_{h}(\zeta) - \Phi_{h'}(\zeta) = \overline{\partial} X_{h,h'}(\zeta).$$

In case $n = 2$ we simply have:

$$\Omega_{1}(\zeta) - \Omega_{2}(\zeta) = -\frac{1}{(2 \pi i)^2} \frac{dz_{1} \wedge dz_{2}}{(z_{1} - \zeta_{1})(z_{2} - \zeta_{2})},$$

and hence we find, on $(U \setminus L_{\zeta}(\varphi)) \cap (U' \setminus L_{\zeta}(\varphi'))$:

$$(1.4) \quad \Phi_{h}(\zeta) - \Phi_{h'}(\zeta) = -\frac{1}{(2 \pi i)^2} \frac{(h_{1} h_{2}' - h_{2} h_{1}') dz_{1} \wedge dz_{2}}{(\varphi(z) - \varphi'(\zeta))(\varphi'(z) - \varphi'(\zeta))}.$$

Next, we observe that all the above differential forms depend in a real analytic fashion also on the point $\zeta$, so that we may perform any derivative of these with respect to the parameters $\text{Re} \, \xi_{\alpha}, \text{Im} \, \zeta_{\alpha}, \alpha = 1, \ldots, n$ (by taking the derivative of each coefficient). In particular we may consider the forms $\partial \omega / \partial \bar{\xi}_{\alpha}$, $\partial \Omega_{\beta} / \partial \bar{\xi}_{\alpha}$, etc., obtained by applying the Wirtinger operator $\partial / \partial \bar{\xi}_{\alpha}$. We first note that, for every $\alpha = 1, \ldots, n$, the $(n, n - 2)$-form $\partial \Omega_{\alpha} / \partial \bar{\xi}_{\alpha}$ satisfies

$$\frac{\partial \Omega_{\alpha}}{\partial \bar{\xi}_{\alpha}}(\zeta) = (n - 1) \frac{z_{\alpha} - \zeta_{\alpha}}{|z - \zeta|^{2}} \Omega_{\alpha}(\zeta),$$

and hence is defined (and real analytic) on $C^{n} \setminus \zeta$, instead that only on $C^{n} \setminus L_{\zeta}(z_{\alpha})$ as $\Omega_{\alpha}(\zeta)$. It follows that, on $C^{n} \setminus \zeta$,

$$(1.5) \quad \frac{\partial \omega}{\partial \bar{\xi}_{\alpha}}(\zeta) = \overline{\partial} \left[ \frac{\partial \Omega_{\alpha}}{\partial \bar{\xi}_{\alpha}}(\zeta) \right] \quad (\alpha = 1, \ldots, n).$$

Similarly, if $n \geq 3$, for every $\alpha, \beta = 1, \ldots, n$ with $\alpha \neq \beta$, the $(n, n - 3)$-form $\partial \Lambda_{\alpha,\beta} / \partial \bar{\xi}_{\alpha}$ satisfies

$$\frac{\partial \Lambda_{\alpha,\beta}}{\partial \bar{\xi}_{\alpha}}(\zeta) = (n - 2) \frac{z_{\alpha} - \zeta_{\alpha}}{|z - \zeta|^{2}} \Lambda_{\alpha,\beta}(\zeta),$$
and hence is defined on $\mathbb{C}^n \setminus L_\zeta(z_\beta)$, instead that only on $\mathbb{C}^n \setminus (L_\zeta(z_\alpha) \cup L_\zeta(z_\beta))$ as $\Lambda_{\alpha,\beta}(\zeta)$. It follows that, on $\mathbb{C}^n \setminus L_\zeta(z_\beta)$,

$$\frac{\partial \Omega_\alpha}{\partial \bar{\xi}_\alpha}(\zeta) - \frac{\partial \Omega_\beta}{\partial \bar{\xi}_\alpha}(\zeta) = \bar{\partial} \left[ \frac{\partial \Lambda_{\alpha,\beta}}{\partial \bar{\xi}_\alpha}(\zeta) \right].$$

If $n = 2$ we simply have, for $\alpha = 1, 2$:

$$\frac{\partial \Omega_1}{\partial \bar{\xi}_\alpha}(\zeta) - \frac{\partial \Omega_2}{\partial \bar{\xi}_\alpha}(\zeta) = 0.$$

Now, let there be given an open set $U \subset \mathbb{C}^n$, a function $\varphi \in \mathcal{O}(U)$ and a map $h \in \mathcal{O}_\varphi^n(U \times U)$, and let $\xi$ be a point in $U$. In case $n \geq 3$ consider, for every $\alpha = 1, \ldots, n$, the following $(n, n - 3)$-form on $\mathbb{C}^n$:

$$\Psi_h^\alpha(\zeta) = \frac{1}{\varphi(z) - \varphi(\zeta)} \sum_{\beta=1}^{n} h_\beta(z_\beta - \zeta_\beta) \frac{\partial \Lambda_{\alpha,\beta}}{\partial \bar{\xi}_\alpha}(\zeta).$$

Then we find, on $U \setminus L_\zeta(\varphi)$:

$$(1.6) \quad \frac{\partial \Phi_h}{\partial \bar{\xi}_\alpha}(\zeta) = \frac{\partial \Omega_\alpha}{\partial \bar{\xi}_\alpha}(\zeta) - \bar{\partial} \Psi_h^\alpha(\zeta) \quad (\alpha = 1, \ldots, n).$$

On the other hand, if $n = 2$, we have:

$$(1.7) \quad \frac{\partial \Phi_h}{\partial \bar{\xi}_\alpha}(\zeta) = \frac{\partial \Omega_\alpha}{\partial \bar{\xi}_\alpha}(\zeta) \quad (\alpha = 1, 2).$$

(b) It is well known that, given an oriented real hypersurface $\Sigma$ of class $C^1$ in $\mathbb{C}^n$ (without boundary, not necessarily closed) and a complex-valued function $f$ in $L^1_{loc}(\Sigma)$, one may say that $f$ is a CR-function on $\Sigma$ in case it satisfies the tangential Cauchy-Riemann equation in the weak form, that is

$$(1.8) \quad \int_{\Sigma} f \bar{\partial} \lambda = 0,$$

for every $(n, n - 2)$-form $\lambda$ of class $C^1$ on an open neighbourhood of $\Sigma$, such that $\Sigma \cap \text{Supp}(\lambda)$ is compact. However we need for our purposes a sharper characterization of continuous CR-functions on $\Sigma$ than (1.8) is. This is provided by the following proposition.

**Proposition 1.9.** Let $f$ be a complex-valued continuous function on $\Sigma$. Then $f$ is a CR-function if and only if it satisfies

$$(1.10) \quad \int_{\partial \Sigma} f \bar{\partial} \mu = \int_{\partial \Sigma} f \mu,$$
for every singular \((n + q)\)-chain \(c_{n+q}\) of \(\Sigma\) of class \(C^1\) and every \((n, q - 1)\)-form \(\mu\) of class \(C^1\) on an open neighbourhood of \(\Sigma\) \((1 \leq q \leq n - 1)\).

**Proof.** This proposition asserts that (1.8) and (1.10) are equivalent for a continuous \(f\) (which would be quite immediate if \(f\) were of class \(C^1\)). We shall prove only that (1.8) implies (1.10), the converse being trivial.

For every differential form \(\mu\) of class \(C^1\) on an open neighbourhood \(V\) of \(\Sigma\), we denote by \(\mu|_{\Sigma}\) the restriction of \(\mu\) to \(\Sigma\) (i.e. the pull-back of \(\mu\) by the inclusion map \(\Sigma \hookrightarrow V\)). Then \(\mu|_{\Sigma}\) is a continuous regular form on \(\Sigma\).

Consider the continuous \(n\)-form on \(\Sigma\)
\[
\mu = f(dz_1 \wedge \cdots \wedge dz_n)|_{\Sigma}.
\]
We claim that (1.10) is equivalent to the following assertion:
\[
(*) \quad \text{\(u\) is regular on \(\Sigma\) and \(du = 0\).}
\]
As a matter of fact, taking in particular \(q = 1\) and \(\mu = dz_1 \wedge \cdots \wedge dz_n\), (1.10) gives:
\[
0 = \int_{\partial c_{n+1}} f dz_1 \wedge \cdots \wedge dz_n = \int_{\partial c_{n+1}} u,
\]
for every singular \((n + 1)\)-chain \(c_{n+1}\) of \(\Sigma\) of class \(C^1\); and this is just as to say that \((*)\) holds. Conversely, assume that \((*)\) holds. Any \((n, q - 1)\)-form \(\mu\) as in the statement can be written as \(\mu = dz_1 \wedge \cdots \wedge dz_n \wedge \tilde{\mu}\), where \(\tilde{\mu}\) is a \((0, q - 1)\)-form of class \(C^1\) on an open neighbourhood of \(\Sigma\). Then \(u \wedge \tilde{\mu}|_{\Sigma}\) is a continuous regular \((n + q - 1)\)-form on \(\Sigma\) and, since \(du = 0\), \(d(\tilde{\mu}|_{\Sigma}) = (d\tilde{\mu})|_{\Sigma}\), we have:
\[
d\left( u \wedge \tilde{\mu}|_{\Sigma} \right) = (-1)^n u \wedge (d\tilde{\mu})|_{\Sigma} = f (d\mu)|_{\Sigma} = f (\partial\mu)|_{\Sigma}.
\]
It follows that
\[
\int_{c_{n+q}} f \partial\mu = \int_{\partial c_{n+q}} u \wedge \tilde{\mu}|_{\Sigma} = \int_{\partial c_{n+q}} f\mu,
\]
that is, (1.1) holds. Next, we claim that \((*)\) is equivalent to:
\[
(**) \quad \text{\(u\) is weakly closed on \(\Sigma\), that is} \quad \int_{\Sigma} u \wedge dv = 0
\]
for every \((n - 2)\)-form \(v\) on \(\Sigma\) of class \(C^1\) and with compact support.

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3 The same result is proved in Lupacciolu-Tomassini [6] under the additional assumption that \(f\) is locally Lipschitz, but the argument used there does not work without that assumption.

4 For the definition and basic properties of continuous regular forms we refer to Whitney [11] pp. 103–108. We denote, as usual, by \(d\) the differential acting on such forms (defined by means of Stokes' formula), as the ordinary exterior differential.
This latter equivalence is a straightforward consequence of the following general facts about continuous differential forms on a manifold of class $C^1$:

(i) The differential acting on continuous regular forms may be understood in the strong sense. This means that, if $\eta, \theta$ are continuous forms, then $\eta, \theta$ are regular and $d\eta = \theta$ in the sense of regular forms if and only if there exists a sequence $\{\eta_s\}_{s=1}^{\infty}$ of forms of class $C^1$ such that $\eta_s \to \eta$ and $d\eta_s \to \theta$ as $s \to \infty$, both uniformly on compact sets (cf. Whitney [11]);

(ii) The differential in the strong sense coincides with the differential in the weak sense. This means that, if $\eta, \theta$ are continuous forms, then $d\eta = \theta$ in the strong sense if and only if \( \int \eta \wedge d\xi = (-1)^{\deg \eta} \int \theta \wedge \xi \), for every form $\xi$ of class $C^1$ and with compact support (cf. Friedrichs [2], or Fichera [1]).

Now we show that (1.8) implies (**), which will conclude the proof. We shall use the following fact: there exists an open neighbourhood $W$ of $\Sigma$ in $C^n$ and a retraction $r: W \to \Sigma$ of class $C^1$ (which means that $r(z) = z$ for each $z \in \Sigma$). This is a special case of a standard theorem in Differential Topology (cf. Munkres [8], p. 51, or Whitney [11], p. 121). If $v$ is any $(n-2)$-form on $\Sigma$ of class $C^1$ and with compact support, consider its pull-back $r^*v$ to $W$. $r^*v$ is a continuous regular $(n-2)$-form on $W$, and hence we can find a sequence $\{\eta_s\}_{s=1}^{\infty}$ of $(n-2)$-forms of class $C^1$ on $W$ such that

$$
\lim_{s \to \infty} \eta_s = r^*v, \quad \lim_{s \to \infty} d\eta_s = r^*dv,
$$

both uniformly on compact subsets of $W$. Moreover, since $\Sigma \cap \text{Supp}(r^*v) = \text{Supp}(v)$ is compact, we can arrange that so too is $\Sigma \cap \text{Supp}(\eta_s)$, for every $s$. It follows that

$$
\int_{\Sigma} u \wedge dv = \lim_{s \to \infty} \int_{\Sigma} u \wedge (d\eta_s)|_{\Sigma}
$$

\begin{align*}
&= \lim_{s \to \infty} \int_{\Sigma} f dz_1 \wedge \cdots \wedge dz_n \wedge d\eta_s \\
&= (-1)^n \lim_{s \to \infty} \int_{\Sigma} f \bar{\partial} (dz_1 \wedge \cdots \wedge dz_n \wedge \eta_s),
\end{align*}

and hence (1.8) implies $\int_{\Sigma} u \wedge dv = 0$. 

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5 Clearly, the interest of this fact is in the "if", the "only if" being trivial.

6 If $\Sigma$ were of class $C^2$, we could use the more elementary "tubular neighbourhood theorem".
2. Proof of Theorem 1. Let $V$ be an open neighbourhood of $K$ in $\mathbb{C}^n$ and $\sigma: \mathbb{C}^n \to \mathbb{R}$ a $C^\infty$ function such that $0 \leq \sigma(z) \leq 1$ for all $z$, $\sigma(z) = 1$ for $z \in K$, $\text{Supp}(\sigma)$ is compact and contained in $V$. For a generic small $\varepsilon > 0$, set $D_\varepsilon = D \cap \{1 - \sigma > \varepsilon\}$, $\Gamma_\varepsilon = \partial D \cap \{1 - \sigma = \varepsilon\}$ and $K_\varepsilon = \overline{D} \cap \{1 - \sigma = \varepsilon\}$. Then $D_\varepsilon$ is a subdomain of $D$, $\partial D_\varepsilon = \Gamma_\varepsilon \cup K_\varepsilon$, $\Gamma_\varepsilon$ and $K_\varepsilon$ are compact real hypersurfaces with boundary, of class $C^1$, such that $\Gamma_\varepsilon \cap K_\varepsilon = \partial \Gamma_\varepsilon = \partial K_\varepsilon$, and $\Gamma_\varepsilon$ is connected. Clearly, $D$ is exhaustible by an increasing sequence of subdomains of this sort, $\{D_s\}_{s=1}^\infty$, say, so that

$$\partial D_s = \Gamma_s \cup K_s \quad (s = 1, 2, \ldots),$$

with obvious meaning of $\Gamma_s$, $K_s$, and

$$D = \bigcup_{s=1}^\infty D_s, \quad \partial D \setminus K = \bigcup_{s=1}^\infty \Gamma_s.$$

We assume that the sequence $\{D_s\}_{s=1}^\infty$ has been chosen once for all.

Now, let $U$ be an open neighbourhood of $\overline{D}$ and let $\varphi \in \mathcal{O}(U)$. For every positive integer $s$ we set:

$$U_s(\varphi) = \left\{ \xi \in U; \left| \varphi(\xi) \right| > \max_{\overline{D} \setminus D_s} |\varphi| \right\}.$$

Then $U_s(\varphi)$ is an open subset of $U \setminus \overline{D} \setminus D_s$ such that, if $\xi \in U_s(\varphi)$, the level set $L_\xi(\varphi)$ of $\varphi$ through $\xi$ is all contained in $U_s(\varphi)$. Moreover we set:

$$U(\varphi) = \left\{ \xi \in U; \left| \varphi(\xi) \right| > \max_K |\varphi| \right\}.$$

Since $\{\overline{D} \setminus D_s\}_{s=1}^\infty$ is a decreasing sequence of compact neighbourhoods of $K$ in $\overline{D}$ such that $K = \cap_{s=1}^\infty \overline{D} \setminus D_s$, it follows that $U_1(\varphi) \subset U_2(\varphi) \cdots$, and

$$U(\varphi) = \bigcup_{s=1}^\infty U_s(\varphi). \quad (2.1)$$

Moreover, since $\hat{K}_\overline{D} = \cap_{U \supseteq \overline{D}} \hat{K}_U$ (where $U$ ranges over the open neighbourhoods of $\overline{D}$), the assumption of Theorem 1 implies:

$$\overline{D} \setminus K \subset \bigcup_{U \supseteq \overline{D}} \bigcup_{\varphi \in \mathcal{O}(U)} U(\varphi). \quad (2.2)$$

Next, for every $U, \varphi, s$ as above and $h \in \mathcal{O}_\varphi^n(U \times U)$ (cf. (1.1)), consider the complex-valued function $F^s_h$ on $U_s(\varphi) \setminus \partial D$ given by

$$F^s_h(\xi) = \int_{\Gamma_s} f(\omega(\xi)) - \int_{\partial \Gamma_s} f(\Phi_h(\xi)),$$

where $\omega(\xi)$ and $\Phi_h(\xi)$ are the complex-valued functions (2.1) and (2.2). Since $\varphi \in \mathcal{O}(U)$, the identity $\int f(\varphi(\xi)) = \int f(\varphi(\xi))$ holds for every $\xi \in U_s(\varphi)$, and

$$F^s_h(\xi) = \int_{\Gamma_s} f(\omega(\xi)) - \int_{\partial \Gamma_s} f(\Phi_h(\xi)),$$
where $\Phi_h(\zeta)$ is the $\bar{\partial}$-primitive (1.2) of the Martinelli form $\omega(\zeta)$, $\Gamma_s$ is oriented as a part of $\partial D$ and $\partial \Gamma_s$ as the boundary of $\Gamma_s$. Since, for $\zeta \in U_s(\varphi)$ and $z \in \partial \Gamma_s$, $|\varphi(\zeta)| > |\varphi(z)|$ (because $\partial \Gamma_s \subset D \setminus D_s$), the singular set $L_s(\varphi)$ of $\Phi_h(\zeta)$ does not meet $\partial \Gamma_s$, so that $F^s_{\Gamma_s}$ is indeed defined, and real analytic, on $U_s(\varphi) \setminus \Gamma_s = U_s(\varphi) \backslash \partial D$.

**Proposition 2.4.** Suppose there exists at least a function $F$ as in the statement of Theorem 1. Then, for every $U, \varphi, h, s$ as above,

$$F = F^s_{\Gamma_s} \text{ on } D \cap U_s(\varphi).$$

As a consequence, on account of (2.1) and (2.2), if such a $F$ actually exists, it is necessarily unique.

**Proof.** Clearly $D \cap U_s(\varphi) \subset D_s$, and, by assumption, $F \in C^0(\overline{D_s}) \cap \mathcal{O}(D_s)$ and $F = f$ on $\Gamma_s$. Therefore, since, by the Martinelli formula, for $\zeta \in D_s$, we have:

$$F(\zeta) = \int_{\Gamma_s} f \omega(\zeta) + \int_{K_s} F \omega(\zeta),$$

we are required to show that, for $\zeta \in D \cap U_s(\varphi)$, we also have:

$$(*) \quad \int_{K_s} F \omega(\zeta) = -\int_{\partial \Gamma_s} f \Phi_h(\zeta).$$

Since $F$ is continuous on $\overline{D \setminus K}$ and holomorphic on $D$, the forms $F \omega(\zeta)$, $F \Phi_h(\zeta)$ are both continuous on $(\overline{D \setminus K}) \setminus L_{2s}(\varphi)$, real analytic on $D \setminus L_s(\varphi)$, and on $D \setminus L_s(\varphi)$ satisfy $F \omega(\zeta) = d(F \Phi_h(\zeta))$. Moreover, since $\zeta \in U_s(\varphi)$, it follows that $K_s \subset (\overline{D \setminus K}) \setminus L_s(\varphi)$. Then consider the restrictions $(F \omega(\zeta))|_{K_s}$, $(F \Phi_h(\zeta))|_{K_s}$; these are continuous on $K_s$, regular on $K_s \setminus \partial K_s$ and on $K_s \setminus \partial K_s$ satisfy $(F \omega(\zeta))|_{K_s} = d[(F \Phi_h(\zeta))|_{K_s}]$. Hence Stokes' theorem for regular forms on a manifold with boundary (cf. Whitney [11], p. 109) implies:

$$\int_{K_s} F \omega(\zeta) = \int_{\partial K_s} F \Phi_h(\zeta).$$

Finally, since $\partial K_s = -\partial \Gamma_s$ (= $\partial \Gamma_s$ with the opposite orientation), ($*$) follows.

The above proposition disposes of the uniqueness’ assertion in Theorem 1 and, further, implies that the proof of the existence of a holomor-

---

7In this paper we take as the canonical orientation of $\mathbb{C}^n$ and of $D$ the one given by the volume-form $(i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. 
phic continuation of \( f \) on \( D \) shall be a matter of showing that the \( F_h^s \)'s do in fact define a holomorphic function \( F \) on \( D \) such that, for each \( z^0 \in \partial D \setminus K \), \( F(\xi) \to f(z^0) \) as \( \xi \to z^0 \) in \( D \). In the first place we have:

**Proposition 2.5.** The functions \( F_h^s \)'s are each other coherent and holomorphic. Hence there is a unique holomorphic function \( F \) on

\[
\left( \bigcup_{U \supset D} \bigcup_{\varphi \in \sigma(U)} U(\varphi) \right) \setminus \partial D
\]

such that, for every \( U, \varphi, h, s \),

\[
F = F_h^s \quad \text{on } U_s(\varphi) \setminus \partial D.
\]

**Proof.** We first prove the coherence. This means that, for every \( U, \varphi, h, s \) and \( U', \varphi', h', s' \), we have:

\[
(*) \quad F_h^s = F_h^{s'} \quad \text{on } U_s(\varphi) \cap U_s'(\varphi') \setminus \partial D.
\]

We may assume that \( s \geq s' \). Then (*) will be a consequence of the following two equalities:

(i) \( F_h^s = F_h^{s'} \) on \( U_s'(\varphi') \setminus \partial D \);
(ii) \( F_h^{s'} = F_h^s \) on \( U_s(\varphi) \cap U_s'(\varphi') \setminus \partial D \)

(recall that \( U_s(\varphi) \subset U_s'(\varphi) \) and \( U_s'(\varphi') \subset U_s'(\varphi) \)). To prove (i) (in case \( s > s' \)), consider the \((2n-1)\)-chain of \( \partial D \setminus K \), of class \( C^1 \), \( c_{2n-1} = \Gamma_s - \Gamma_s' \). If \( \xi \) is any point in \( U_s'(\varphi') \setminus \partial D \), it is plain that

\[
F_h^s(\xi) - F_h^{s'}(\xi) = \int_{c_{2n-1}} f \omega(\xi) - \int_{\partial c_{2n-1}} f \Phi_h^s(\xi);
\]

moreover, since \( \text{Supp}(c_{2n-1}) \subset D_s \setminus D_s' \subset D \setminus D_s' \) and \( L_\xi(\varphi') \subset U_s'(\varphi') \subset U' \setminus D \setminus D_s' \), it follows that \( \text{Supp}(c_{2n-1}) \) is contained in \( U' \setminus L_\xi(\varphi') \), where \( \omega(\xi) \), \( \Phi_h^s(\xi) \) are both defined and satisfy \( \omega(\xi) = \overline{\partial} \Phi_h^s(\xi) \). Then, if we take a \((n, n-2)\)-form \( \mu \) of class \( C^\infty \) on all of \( C^n \) and equal to \( \Phi_h^s(\xi) \) on an open neighbourhood of \( \text{Supp}(c_{2n-1}) \), we may replace \( \omega(\xi) \), \( \Phi_h^s(\xi) \), in the right side of the above equality, respectively by \( \overline{\partial} \mu \), \( \mu \). Hence Proposition 1.9 gives at once that \( F_h^{s'}(\xi) = F_h^s(\xi) \).

Next we prove (ii). On account of (1.3), (1.4), we have, for each \( \xi \in U_s(\varphi) \cap U_s'(\varphi') \setminus \partial D \):

\[
F_h^s(\xi) - F_h^{s'}(\xi) = \left\{
\begin{array}{ll}
- \int_{\partial \Gamma_s} f \overline{\partial} X_{h,h'}(\xi) & \text{if } n \geq 3,

\frac{1}{(2\pi i)^2} \int_{\partial \Gamma_s} f(z) \frac{(h_1 h'_2 - h_2 h'_1) dz_1 \wedge dz_2}{(\varphi(z) - \varphi(\xi))(\varphi'(z) - \varphi'(\xi))} & \text{if } n = 2.
\end{array}
\right.
\]
In case \( n \geq 3 \), we may replace \( X_{h,h'}(\xi) \), in the integral on the right side, by any \((n, n - 3)\)-form \( \tilde{X} \) of class \( C^\infty \) on all of \( \mathbb{C}^n \) and equal to \( X_{h,h'}(\xi) \) on an open neighbourhood of \( \partial \Gamma_s \). Hence Proposition 1.9 (for \( q = n - 1 \), \( c_{n+q} = \Gamma_s \) and \( \mu = \overline{\partial} \tilde{X} \)) implies that \( F^s_h(\xi) = F^s_{h'}(\xi) \).

In case \( n = 2 \), we have to argue differently. Since \( \xi \in U_s(\varphi) \cap U'_s(\varphi') \) and \( \partial \Gamma_s \subset \overline{D \setminus D_s} \), it follows that, for each \( z \in \partial \Gamma_s \), \( |\varphi(\xi)| > \max_{\overline{D \setminus D_s}} |\varphi| \geq |\varphi(z)| \), and hence \( |\varphi(z)/\varphi'(\xi)| < 1 \). Similarly, \( |\varphi'(z)/\varphi'(\xi)| < 1 \). Therefore we may write, for \( z \in \partial \Gamma_s \):

\[
\frac{1}{(\varphi(z) - \varphi(\xi))(\varphi'(z) - \varphi'(\xi))} = \frac{1}{\varphi(\xi)\varphi'(\xi)} \cdot \frac{1}{(1 - \varphi(z)/\varphi(\xi))(1 - \varphi'(z)/\varphi'(\xi))}
\]

\[
= \frac{1}{\varphi(\xi)\varphi'(\xi)} \sum_{\alpha,\beta}^{0,\infty} \left( \frac{\varphi(z)}{\varphi(\xi)} \right)^\alpha \left( \frac{\varphi'(z)}{\varphi'(\xi)} \right)^\beta,
\]

with the double series absolutely uniformly convergent on \( \partial \Gamma_s \). It follows that

\[
\int_{\partial \Gamma_s} f(z) \frac{(h_1 h'_2 - h_2 h'_1) dz_1 \wedge dz_2}{(\varphi(z) - \varphi(\xi))(\varphi'(z) - \varphi'(\xi))}
\]

\[
= \sum_{\alpha,\beta}^{0,\infty} \frac{1}{(\varphi(\xi))^{\alpha+1}} \left( \frac{\varphi'(\xi)}{\varphi'(\xi)} \right)^{\beta+1} \int_{\partial \Gamma_s} f \mu_{\alpha,\beta},
\]

where

\[
\mu_{\alpha,\beta} = (h_1 h'_2 - h_2 h'_1)(\varphi(z))^{\alpha} (\varphi'(z))^{\beta} dz_1 \wedge dz_2
\]

\((\alpha, \beta = 0, 1, 2, \ldots)\).

Now, since every \( \mu_{\alpha,\beta} \) is a holomorphic 2-form on \( U \cap U' \), so that \( \overline{\partial} \mu_{\alpha,\beta} = 0 \), Proposition 1.9 implies:

\[
\int_{\partial \Gamma_s} f \mu_{\alpha,\beta} = 0 \quad (\alpha, \beta = 0, 1, 2, \ldots).
\]

Therefore also for \( n = 2 \) we have: \( F^s_h(\xi) = F^s_{h'}(\xi) \).

It remains to show that every \( F^s_h \) is holomorphic, i.e. that, for each \( \xi \in U_s(\varphi) \setminus \partial D \),

\[
\frac{\partial F^s_h}{\partial \xi_\alpha}(\xi) = 0 \quad (\alpha = 1, \ldots, n).
\]

Clearly, we have:

\[
\frac{\partial F^s_h}{\partial \xi_\alpha}(\xi) = \int_{\Gamma_s} f \frac{\partial \omega}{\partial \xi_\alpha}(\xi) - \int_{\partial \Gamma_s} f \frac{\partial \Phi^s_h}{\partial \xi_\alpha}(\xi);
\]

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further, on account of (1.5), (1.6), (1.7), we may rewrite the right side of this equality as:

\[
\int_{\Gamma_x} f \frac{\partial \Omega_\alpha}{\partial \xi_\alpha}(\xi) - \int_{\partial \Gamma_x} f \frac{\partial \Omega_\alpha}{\partial \xi_\alpha}(\xi) + I,
\]

where

\[
I = \begin{cases} 
\int_{\partial \Gamma_x} f \tilde{\Psi}_h^\alpha(\xi) & \text{if } n \geq 3, \\
0 & \text{if } n = 2.
\end{cases}
\]

Since \([\partial \Omega_\alpha/\partial \xi_\alpha]\)(\xi) is defined on all of \(C^n \setminus \xi\), Proposition 1.9 implies that the difference of integrals in (*) is zero. Moreover, by Proposition 1.9 again, I is zero also in case \(n \geq 3\), since \(\Psi_h^\alpha(\xi)\) may be replaced by any \((n, n - 3)\)-form \(\tilde{\Psi}^\alpha\) of class \(C^\infty\) on all of \(C^n\) and equal to \(\Psi_h^\alpha(\xi)\) on an open neighbourhood of \(\partial \Gamma_x\). Hence \([\partial F_h^\alpha/\partial \xi_\alpha]\)(\xi) = 0.

The proof of Proposition 2.5 is then completed.

Next, we have:

**Proposition 2.6.** Let \(V\) be an open neighbourhood of \(\partial D \setminus K\), contained in \(\bigcup_{U \supset D} \bigcup_{\varphi \in \mathcal{O}(U)} U(\varphi)\), such that \(V \setminus (\partial D \setminus K) = V_+ \cup V_-\), where \(V_+, V_-\) are connected separated open sets and \(V_- \subset C^n \setminus \overline{D}\). Then \(F = 0\) on \(V_-\).

**Proof.** We first point out that, given an open neighbourhood \(U\) of \(\overline{D}\) and a function \(\varphi \in \mathcal{O}(U)\), if \(\xi\) is a point in \(U\) such that \(|\varphi(\xi)| > \max_{\overline{D}}|\varphi|\) (which obviously implies that \(\xi \in U_1(\varphi) \setminus \overline{D}\)), then \(F(\xi) = 0\). As a matter of fact, if \(h \in \mathcal{O}_{\varphi}^r(U \times U)\), we have:

\[
F(\xi) = F_h^1(\xi) = \int_{\Gamma_1} f \omega(\xi) - \int_{\partial \Gamma_1} f \Phi_h(\xi),
\]

and, since \(\overline{D} \subset U \setminus L_1(\varphi)\), on an open neighbourhood of \(\overline{D}\) \(\omega(\xi)\), \(\Phi_h(\xi)\) are both defined and satisfy \(\omega(\xi) = \overline{\Phi}_h(\xi)\). Hence Proposition 1.9 implies that \(F(\xi) = 0\).

Now, take \(U\) and \(\varphi\) such that \(U(\varphi) \cap D \neq \emptyset\); then \(\max_{\overline{D}}|\varphi| > \max_{K}|\varphi|\), so that \(\varphi\) is not constant on the connected component of \(U\) containing \(\overline{D}\) and, further, any point \(\xi^0 \in \partial D\) where \(|\varphi|\) attains the value

\[\text{Such a } V \text{ does exist, because } \partial D \setminus K \text{ is connected. For example, we may take as } V \text{ a small tubular neighbourhood of } \partial D \setminus K \text{ in } C^n \setminus K.\]
max\_D |\varphi| must belong to \partial D \setminus K. One can actually find such a point \xi^0 by the well known “maximum principle”. Then \xi^0 is a limit point of the open set \( W = \{ \xi \in U; |\varphi(\xi)| > \max\_D |\varphi| \} \) (by the maximum principle again), and, since \( \xi^0 \in \partial D \setminus K \), this obviously implies that \( W \cap V_- \neq \emptyset \). But we already know that \( F \) is zero on \( W \cap V_- \); it follows that \( F \) is zero on all of \( V_- \), because \( V_- \) is connected.

Finally, we are in a position to prove that \( F \) is a continuous extension of \( f \) to \( \overline{D} \setminus K \), i.e., the following holds:

**Proposition 2.7.** For every point \( z^0 \in \partial D \setminus K \) we have:

\[
\lim_{\xi \to z^0} F(\xi) = f(z^0),
\]

the limit being evaluated for \( \xi \in D \).

**Proof.** For every \( w \in \partial D \setminus K \), denote by \( \bar{v}(w) \) the unit vector perpendicular to \( \partial D \setminus K \) at \( w \), inward pointing with respect to \( D \). We first prove that

\[
(*) \quad \lim_{t \to 0^+} F(w + tv(w)) = f(w),
\]

with the limit uniform on compact subsets of \( \partial D \setminus K \). Given \( w \in \partial D \setminus K \), we can find an open neighbourhood \( U \) of \( w \), a function \( \varphi \in \mathcal{O}(U) \) and a positive integer \( s \) such that \( w \in U_s(\varphi) \cap (\Gamma_s \setminus \partial \Gamma_s) \). Then, for \( t > 0 \) small enough, we have:

\[
w + tv(w) \in U_s(\varphi) \cap D, \quad w - tv(w) \in U_s(\varphi) \cap V_-,
\]

with \( V_- \) as in Proposition 2.6, and hence, if \( h \in \mathcal{O}^n(U \times U) \), it follows that

\[
F(w + tv(w)) = F^s_h(w + tv(w)),
\]

\[
F(w - tv(w)) = F^s_h(w - tv(w)) = 0.
\]

Therefore we may write:

\[
F(w + tv(w)) = F^s_h(w + tv(w)) - F^s_h(w - tv(w)) = I_1(w, t) - I_2(w, t),
\]

where

\[
I_1(w, t) = \int_{\Gamma_s} f[\omega(w + tv(w)) - \omega(w - tv(w))],
\]

\[
I_2(w, t) = \int_{\partial \Gamma_s} f[\Phi_h(w + tv(w)) - \Phi_h(w - tv(w))].
\]
Now, it can be shown that, for any \( f \in C^0(\Gamma_s) \) (not necessarily a CR-function) and \( w \in \Gamma_s \setminus \partial \Gamma_s \),
\[
\lim_{t \to 0^+} I_1(w, t) = f(w),
\]
with the limit uniform on compact subsets of \( \Gamma_s \setminus \partial \Gamma_s \). A similar result can be found in Harvey-Lawson [4], pp. 251–252, and the proof given there (based on a suitable estimate for \( ||\omega(w + tv(w)) - \omega(w - tv(w))|| \)) works essentially for the present case as well.\(^9\) Next, since the function \( \xi \mapsto \int_{\partial \Gamma_s} f\Phi_h(\xi) \) is defined and real analytic on all of \( U_s(\varphi) \), it is plain that, for \( w \in U_s(\varphi) \cap (\Gamma_s \setminus \partial \Gamma_s) \),
\[
\lim_{t \to 0^+} I_2(w, t) = 0,
\]
with the limit uniform on compact subsets of \( U_s(\varphi) \cap (\Gamma_s \setminus \partial \Gamma_s) \). Hence (*) follows.

After that, it is easy to prove Proposition 2.7. Given \( \varepsilon > 0 \), let \( N_{z^0} \) be an open neighbourhood of \( z^0 \) in \( \partial D \setminus K \) such that \( |f(w) - f(z^0)| < \varepsilon/2 \), for every \( w \in N_{z^0} \), and \( N_{z^0} \subset \partial D \setminus K \). Further, let \( t_0 > 0 \) be such that \( |F(w + tv(w)) - f(w)| < \varepsilon/2 \), for every \( t \leq t_0 \) and \( w \in N_{z^0} \). Clearly, if \( \xi \) is a point of \( D \) close enough to \( z^0 \), there exist exactly a point \( w \in N_{z^0} \) and a positive number \( t \leq t_0 \) such that \( \xi = w + tv(w) \). It follows that
\[
|F(\xi) - f(z^0)| \leq |F(w + tv(w)) - f(w)| + |f(w) - f(z^0)| < \varepsilon,
\]
which proves Proposition 2.7.

Now the proof of Theorem 1 is completed.

REFERENCES


\(^9\) The parallel result for \( n = 1 \) and \( \omega(\xi) = (1/2\pi i) \cdot dz/(z - \xi) \) (the Cauchy kernel) goes back to Plemelj (cf. Muskhelishvili [9], pp. 43–45).


Received October 22, 1984.

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\(^\text{10}\)Added in proof.
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