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Fine homotopy equivalences from s onto complete separable AR's are constructed that are analogs of certain cell-like maps defined on Euclidean space. In particular, (i) there is a fine homotopy equivalence f from s onto a complete separable AR X such that the collection of nondegenerate values N_f of f is a singleton whose pre-image under f is a 1-dimensional AR widely embedded in s, and (ii) there is a fine homotopy equivalence g from s onto a complete separable AR Y such that N_g is a Cantor set and every nondegenerate fiber of g is a tame Z-set in s. Neither X nor Y is homeomorphic to s but both become homeomorphic to s upon multiplication by a certain complete 1-dimensional AR.

1. Introduction. In this paper, we construct examples of 'decompositions' of $s = \prod_{i=1}^{\infty}(-1,1)_i$ that are analogs in the non-locally compact setting of s of certain nonshrinkable cell-like upper-semicontinuous decompositions of E^n ($n \geq 3$). In particular, we focus on two types of examples from the cell-like decomposition theory of E^n. The first consists of examples of cell-like but non-cellular decompositions of E^n with exactly one nondegenerate element and the second consists of the 'dog bone' decompositions of E^n into points and tame arcs such that the associated decomposition spaces are distinct from E^n **[Bi]**, **[Ea]**. Since all cell-like decompositions of s that yield ANR decomposition spaces are shrinkable **[Mo]**, such examples do not exist among the cell-like decompositions of s. Consequently, instead of concentrating upon the decompositions of E^n themselves, we focus upon the properties of the associated decomposition maps, which are cell-like and hence fine homotopy equivalences **[Ha]**.

In §2, we construct a fine homotopy equivalence f from s onto a complete separable AR X such that the collection of nondegenerate values N_f of f is a singleton whose pre-image under f is a 1-dimensional AR wildly embedded in s. X is not homeomorphic to s but there is a complete 1-dimensional AR A such that $X \times A$ is homeomorphic to $s \times A \approx s$. This example corresponds to standard examples of cell-like non-cellular decompositions of E^n ($n \geq 3$) that one obtains by threading an arc through a wild Cantor set in E^n and using the decomposition whose only nondegenerate element is that arc. Since the fine homotopy equivalence f is a UV^∞-map, it fails to be cell-like only in that its nondegenerate fiber fails to be compact.

In §3, we construct a fine homotopy equivalence f from s onto a complete separable AR X such that all the nondegenerate fibers of f are tame Z-set copies of s and N_f is a Cantor set. X is not homeomorphic to s though $X \times A$ is homeomorphic to $s \times A \approx s$. This example corresponds to the dog bone decompositions of E^n. Again the fine homotopy equivalence f is a UV^∞-map and fails to be cell-like only in that its nondegenerate fibers are not compact.

In §4, we prove a Stabilization Theorem that allows us to prove that the examples constructed in §§2 and 3 stabilize (i.e., become homeomorphic to s) upon multiplication by a 1-dimensional complete AR A. The Stabilization Theorem also allows us to improve upon our dog bone example by obtaining such an example where all the nondegenerate fibers are tame Z-set copies of a 1-dimensional AR. This and further examples are discussed in §5.

We include an Appendix at the end of the paper that includes pertinent facts about the Hilbert cube and s-manifolds that we use in the paper. The results in Part 1 of the Appendix are generally known, but the author could find no reference for them in the literature and thus short proofs have been included. We cite references for the standard results stated in Part 2 of the Appendix. Part 3 of the Appendix contains statements of results that are as yet unpublished. These results and their corresponding proofs will appear in [**BBMW**].

The results of this paper appear as part of the author's doctoral dissertation written under the direction of J. J. Walsh at The University of Tennessee [**Bo$_1$**].

Terminology and notation. Generally, all spaces are separable and metric. A *mapping* or a *map* is a continuous function and two maps f and g from a space X to a space Y are *$\mathscr{U}$-close*, where $\mathscr{U}$ is an open cover of Y, provided $\{\{f(x), g(x)\}\}_{x \in X}$ refines $\mathscr{U}$. *Fine homotopy equivalence* and *UV^∞-maps* between ANR's are defined in the Appendix.

2. s modulo α—a 'cell-like non-cellular decomposition' of s. For a map $f\colon X \to Y$ between topologically complete spaces, a point $y \in Y$ is a *nondegenerate value* of f provided $y \in \mathrm{Cl}_Y f(X)$ and either $f^{-1}(y) = \varnothing$, or $f^{-1}(y)$ contains at least two points, or $f^{-1}(y) = \{x\}$ but $f^{-1}(\mathscr{B})$ is not a basis for x where $\mathscr{B}$ is a basis for y. The set of nondegenerate values of f is denoted by N_f and is an F_σ-subset of Y. Furthermore, if $f(X)$ is dense in Y, then the restriction of f is a homeomorphism of $f^{-1}(Y - N_f)$ onto

$Y - N_f$. In case f is proper, in particular if f is cell-like, $N_f = \{y \mid f^{-1}(y) \neq \text{point}\}$. In the non-locally compact setting of s, generally a fine homotopy equivalence is not onto, though it has dense image, and points not in the iamge are necessarily nondegenerate values. See **[BBMW]**. A map $f\colon X \to Y$ is a *near-homeomorphism* provided for every open cover $\mathcal{U}$ of Y, there is a $\mathcal{U}$-approximation h to f such that h is a homeomorphism. A closed subset F of an ANR X is a *Z-set* provided for every open cover $\mathcal{U}$ of X, there is a map $f\colon X \to X - F$ $\mathcal{U}$-close to id_X.

A standard method for obtaining an example of a cell-like non-cellular decomposition of E^n $(n \geq 3)$ is to decompose E^n into points and an arc whose complement in E^n is not simply connected (thread an arc through a wild Cantor set). This method produces a map $f\colon E^n \to X$ from E^n onto a locally compact AR X, where X is not homeomorphic to E^n, with the following properties:

(1) f is cell-like, hence a fine homotopy equivalence;

(2) $N_f = \{x\}$ where $f^{-1}(x)$ is an arc wildly embedded in E^n;

(3) $f \times \mathrm{id}\colon E^n \times E^1 \to X \times E^1$ is a near-homeomorphism and $X \times E^1 \approx E^{n+1}$. The same technique also produces corresponding examples for the Hilbert cube $I^\infty = \prod_{i=1}^{\infty}[-1,1]_i$ and the n-cell $I^n = \prod_{i=1}^{n}[-1,1]_i$ $(n \geq 3)$.

Examples similar to those above for s in place of E^n cannot be obtained by "modding out" a wild arc since s contains no wild compact subsets **[Ch]**. However, we do obtain such examples for s by threading a 1-dimensional noncompact AR α through a wild 0-dimensional closed subset of s, and then putting a metric topology on s/α. Let $\bar{A}$ be a dendrite (compact 1-dimensional AR = uniqely arcwise connected Peano continuum) whose endpoints are dense and let $A = \bar{A} - F_0$ for some dense σ-compact collection F_0 of endpoints of $\bar{A}$.

EXAMPLE 1. There is a map $\pi\colon s \to s/\alpha$ of s onto a topologically complete separable AR s/α not homeomorphic to s that satisfies:

(1.1) π is a fine homotopy equivalence;

(1.2) $N_\pi = \{x\}$ where $\pi^{-1}(x)$ is a 1-dimensional AR α wildly embedded in s (i.e., α is not embedded as a Z set);

(1.3) $\pi \times \mathrm{id}\colon s \times A \to s/\alpha \times A$ is a near-homeomorphism and $s/\alpha \times A \approx s \times A \approx s$.

Construction. Let $\bar{C}$ be a wild Cantor set in the Hilbert cube I^∞ whose complement is not simply connected **[Wo]**. Observe that $C = \bar{C} \cap s$ is a closed 0-dimensional subset of s whose complement in s is not simply connected. Since the endpoints of $\bar{A}$, denoted $E(\bar{A})$, is a dense G_δ-subset of $\bar{A}$, $E(\bar{A})$ is homeomorphic to the irrationals **[AU]**;hence, there is an embedding e of $\overline{C}$ into $E(\overline{A})$. Extend e^{-1}: $e(\overline{C}) \to I^\infty$ to an embedding f: $\overline{A} \to I^\infty$ such that $f(\overline{A}) \cap (I^\infty - s) \subset \overline{C}$. This is possible since $I^\infty - s$ is a σ-Z-set (countable union of Z-sets) in I^∞. See Theorem A.1 of the Appendix for details. Identify $f(\overline{A})$ with $\overline{A}$ so that $\overline{C} \subset \overline{A} \subset I^\infty$ and let $\mathscr{a} = \overline{A} \cap s$. Then $\overline{C} \cap s = C \subset \mathscr{a}$ and since $\overline{A} - E(\overline{A}) \subset \mathscr{a} \subset \overline{A}$, $\mathscr{a}$ is a 1-dimensional closed AR subset of s. Let $I^\infty/\overline{A}$ be the quotient space of the decomposition of I^∞ whose only nondegenerate element is $\overline{A}$ and denote the decomposition map $I^\infty \to I^\infty/\overline{A}$ by $\overline{\pi}$. Let $\pi = \overline{\pi}\,|\,s$: $s \to \overline{\pi}(s)$ and denote $\overline{\pi}(s)$ by $s/\mathscr{a}$. Observe that $s/\mathscr{a} - \pi(\mathscr{a})$ is not simply connected and hence $s/\mathscr{a}$ is not homeomorphic to s. Since $B = I^\infty - (s \cup \overline{A})$ is a σ-Z-set in I^∞, $\overline{\pi}(B)$ is a σ-Z-set in the compact AR $I^\infty/\overline{A}$, hence **[To$_2$]** guarantees that $(I^\infty/\overline{A}) - \overline{\pi}(B) = s/\mathscr{a}$ is a topologically complete separable ANR. Since there is a deformation $\{\alpha_t\}$ of $I^\infty/\overline{A}$ with $\alpha_0 = \mathrm{id}$ and $\operatorname{im}\alpha_t \subset s/\mathscr{a}$ for every $t > 0$ (use Theorem A.2 of Appendix), $s/\mathscr{a}$ is an AR.

To prove (1.1), it suffices to prove that π is a UV^∞-map (see Appendix). Let U be an open neighborhood of $\pi(\mathscr{a})$ in $s/\mathscr{a}$ and let $\overline{U}$ be the open neighborhood of $\overline{\pi}\,(\overline{A}) = \pi(\mathscr{a})$ in $I^\infty/\overline{A}$ such that $U = \overline{U} \cap s/\mathscr{a}$. Choose an open neighborhood $\overline{V}$ of $\overline{\pi}(\overline{A})$ in $I^\infty/\overline{A}$ such that $\overline{\pi}^{-1}(\overline{V})$ contracts to a point in $\overline{\pi}^{-1}(\overline{U})$ and let f: $\overline{\pi}^{-1}(\overline{V}) \times [0,1] \to \overline{\pi}^{-1}(\overline{U})$ denote such a contraction. Since $F = (I^\infty - s) \cap \overline{\pi}^{-1}(\overline{U})$ is a σ-Z-set in the I^∞-manifold $\overline{\pi}^{-1}(\overline{U})$ **[Ch]**, there is a deformation g: $\overline{\pi}^{-1}(\overline{U}) \times [0,1] \to \overline{\pi}^{-1}(\overline{U})$ such that $g(x,0) = x$ for all $x \in \overline{\pi}^{-1}(\overline{U})$ and

$$g\big(\overline{\pi}^{-1}(\overline{U}) \times (0,1]\big) \subset \overline{\pi}^{-1}(\overline{U}) - F = \pi^{-1}(U).$$

Let $V = \overline{V} \cap s/\mathscr{a}$ and define h: $\pi^{-1}(V) \times [0,1] \to \pi^{-1}(U)$ by $h(x,t) = g(f(x,t),t)$. Then h is a contraction of $\pi^{-1}(V)$ in $\pi^{-1}(U)$ and π is UV^∞.

(1.2) easily follows from the construction of $s/\mathscr{a}$.

For (1.3), it follows from **[To$_1$]** that $s \times A$ is homeomorphic to s and we prove in §4 that $s/\mathscr{a} \times A$ is homeomorphic to s. Thus $\pi \times \mathrm{id}$: $s \times A \to s/\mathscr{a} \times A$ is a fine homotopy equivalence between s-manifolds and therefore is a near-homeomorphism (see Appendix, **[Fe]**).

3. A dog bone decomposition of s. In **[Bi]**, R. H. Bing constructed the first example of a decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3. This example has become known as Bing's dog bone space. Subsequently, W. T. Eaton **[Ea]** using an idea of R. D. Anderson constructed such dog bone decompositions of E^n for all $n > 3$. In this section we produce a 'decomposition' of s analogous to the dog bone decompositions of Euclidean space and in §5 we use the Stabilization Theorem of §4 to further refine the example of this section.

A dog bone decomposition of E^n provides a map f: $E^n \to X$ from E^n ($n \geq 3$) onto a locally compact AR X, where X is not homeomorphic to E^n, with the following properties:

(1) f is cell-like, hence a fine homotopy equivalence;

(2) N_f is a Cantor set and the fibers of f are points and tame arcs;

(3) for every $x \in N_f$, there is a near-homeomorphism α: $E^n \to E^n$ such that $N_\alpha = \{y\}$ where $\alpha^{-1}(y) = f^{-1}(x)$ and there is a cell-like map β: $E^n \to X$ such that $N_\beta = N_f - \{x\}$ and $f = \beta \circ \alpha$;

(4) $f \times \mathrm{id}$: $E^n \times E^1 \to X \times E^1$ is a near-homeomorphism and $X \times E^1 \approx E^{n+1}$.

EXAMPLE 2. There is a map π: $s \to s_C$ of s onto a topologically complete separable AR s_C not homeomorphic to s that satisfies:

(2.1) π is a fine homotopy equivalence;

(2.2) N_π is a Cantor set and the fibers of π are points and tame Z-set copies of s;

(2.3) for every $x \in N_\pi$, there is a near-homeomorphism α: $s \to s$ such that $N_\alpha = \{y\}$ where $\alpha^{-1}(y) = \pi^{-1}(x)$ and there is a fine homotopy equivalence β: $s \to s_C$ such that $N_\beta = N_\pi - \{x\}$ and $\pi = \beta \circ \alpha$;

(2.4) $\pi \times \mathrm{id}$: $s \times A \to s_C \times A$ is a near-homeomorphism and $s_C \times A \approx s \times A \approx s$.

Construction. Let C be a wild Cantor set in I^∞ whose complement is not simply connected [**Wo**] and let s_C denote the subspace $s \cup C$ of I^∞. s_C is a topologically complete separable AR [**To$_2$**] and $s_C - C$ is not simply connected. It follows that C is not a Z-set in s_C and s_C is not homeomorphic to s.

Since $s_C - C = s - C$ is an s-manifold, the projection map p: $(s_C - C) \times s \to s_C - C$ is a near-homeomorphism [**Sc**]. For any metric d on s_C, let $\mathcal{U}$ be any open cover of $s_C - C$ such that for each $U \in \mathcal{U}$, $\operatorname{diam}_d U < d(U, C)$. Let h': $(s_C - C) \times s \to s_C - C$ be a homeomorphism that is $\mathcal{U}$-close to p. Define h: $s_C \times s \to s_C$ by $h = h'$ on $(s_C - C) \times s$ and $h(\{c\} \times s) = c$ for all $c \in C$. Our choice of $\mathcal{U}$ ensures that h is continuous on $s_C \times s$ and it is obvious that h is surjective and that $N_h = C$. Since each $c \in C$ is a Z-set in s_C, $\{c\} \times s$ is a Z-set in $s_C \times s$.

By [**To$_1$**], $s_C \times s$ is homeomorphic to s; let $\pi = h \circ g: s \to s_C$ where g is a homeomorphism of s onto $s_C \times s$ and observe that $N_\pi = C$ and the fibers of π consist of points and tame Z-set copies of s in s, hence (2.2) holds. For (2.1), it suffices to show that h is a UV^∞-map. This is a straightforward consequence of our choice of $\mathcal{U}$ and p, for if U is an open neighborhood in s_C of an element c of C, there is a neighborhood W of c such that $W \times s \subset h^{-1}(U)$. This follows since $W \times s = p^{-1}(W)$ and

p is $\mathscr{U}$-close to h. Now choose a neighborhood V of c in s_C such that $st(V,\mathscr{U})$ contracts in W to a point. Observe that

$$h^{-1}(V) \subset p^{-1}(st(V,\mathscr{U})) = st(V,\mathscr{U}) \times s \subset W \times s \subset h^{-1}(U)$$

and since $st(V,\mathscr{U})$ contracts in W to a point and s is contractible, $h^{-1}(V)$ contracts in $h^{-1}(U)$ to a point and h is UV^{∞}.

To prove (2.3), we need some results about "reduced products" and strong-Z-sets (see §4). The Appendix at the end of the paper lists some of the pertinent results and the reader is urged to consult **[BBMW]** for proofs of these results and details of the following argument. Let $x \in N_{\pi}$ and denote the product of s_C and s reduced at x by $(s_C \times s)_{\{x\}}$. It follows from **[BBMW**; Corollary 1.2] and the Strong-Z-set Shrinking Theorem of **[BBMW]** that the projection mapping q: $s_C \times s \to (s_C \times s)_{\{x\}}$ is a near-homeomorphism. Let j be a homeomorphism of $(s_C \times s)_{\{x\}}$ onto s and let $\alpha = j \circ q \circ g : s \to s$ and $\beta = h \circ q^{-1} \circ j^{-1} : s \to s_C$. Obviously α is a near-homeomorphism and $N_{\alpha} = \{y\}$ where $\alpha^{-1}(y) = \pi^{-1}(x)$. β is a fine homotopy equivalence (UV^{∞}-map) and $N_{\beta} = N_{\pi} - \{x\}$. Obviously $\pi = \beta \circ \alpha$ and (2.3) holds.

For (2.4), it follows from **[To$_1$]** that $s \times A$ is homeomorphic to s and we prove in §4 that $s_C \times A$ is homeomorphic to s. Thus $\pi \times \mathrm{id}$: $s \times A \to s_C \times A$ is a fine homotopy equivalence between s-manifolds and therefore is a near-homeomorphism **[Fe]**.

4. Stabilizing the examples. Recall that A denotes the complement of a dense σ-Z-set F_0 in a dendrite $\bar{A}$ whose endpoints are dense. A closed subset F of an ANR X is a *strong-Z-set* provided for every open cover $\mathscr{U}$ of X, there is a map f: $X \to X - N(F)$, where $N(F)$ is an open neighborhood of F in X, such that f is $\mathscr{U}$-close to id_X. The concept of strong-Z-set is introduced in **[BBMW]**. If X is locally compact, then the concepts of Z-set and strong-Z-set coincide; however, if X is not locally compact, the two concepts differ. For examples, see **[BBMW]**.

The following theorem and its corollaries allow us to conclude that the examples s/α and s_C of §§2 and 3, respectively, become homeomorphic to s upon multiplication by the 1-dimensional AR A.

STABILIZATION THEOREM. *Let K be a compact subset of a topologically complete separable ANR X such that $X - K$ is an s-manifold. If $K \times B$ is a strong-Z-set in $X \times A$ for each compact subset B of A, then $X \times A$ is an s-manifold.*

Before we prove the Stabilization Theorem, we examine some of its consequences. A closed subset D of an ANR X has infinite codimension in X provided $H_q(U, U - D; \mathbf{Z}) = 0$ for all integers $q \geq 0$ and all open sets U of X (H_q denotes singular homology).

COROLLARY 1. *Let $X = Y - F$ where F is a (dense) σ-Z-set in the locally compact separable ANR Y. If K is a compact subset of X such that $X - K$ is an s-manifold and K has infinite codimension in X, then $X \times A$ is an s-manifold.*

Proof. By [$\mathbf{To_2}$], X is an ANR. It suffices to show that $K \times B$ is a strong-Z-set in $X \times A$ for every compact subset B of A. First observe that $K \times A$ has infinite codimension in $X \times A$, the proof of which is exactly the proof of [**DW**; Lemma 2.2] with the exception that $\mathscr{B}$ denotes the basis of $X \times A$ consisting of all sets $U \times J$ where U is open in X and J is connected and open in A. The pertinent property of this basis is that if J and J' are connected and open in A, then so is $J \cap J'$, and that each such J is contractible in itself [**Wh**]. It follows now that $K \times B$ has infinite codimension in $X \times A$ for every compact subset B of A [**DW**; Lemma 2.1]. If $x \in K \times B$ and U is a neighborhood of x in $X \times A$, then there exists a neighborhood V of x in $X \times A$ such that loops in $V - (K \times B)$ are contractible in $U - (K \times B)$. This follows because one can push a loop near x that projects to X missing K along an arc in A to a nearby loop that projects to A to an endpoint of A not in B. This loop then contracts missing $K \times B$. The precise details involved in this argument appear in [$\mathbf{Bo_2}$]. We have argued that $K \times B$ is a 1-LCC subset of $X \times A$ that has infinite codimension in $X \times A$ and this implies that $K \times B$ is a Z-set in $X \times A$ [**DW**; Proposition 4.2].

Observe that $X \times A$ is the complement in $Y \times \bar{A}$ of the σ-Z-set $(F \times \bar{A}) \cup (Y \times F_0)$ in $Y \times \bar{A}$. The next lemma ensures that the Z-set $K \times B$ in $X \times A$ is a strong-Z-set in $X \times A$.

LEMMA 1. *Let $X = Y - F$ where F is a σ-Z-set in a locally compact separable ANR Y. If $D \subset X$ is a Z-set in X, then D is a strong-Z-set in X.*

Proof. Let $\mathscr{U}$ be an open cover of X and let $\tilde{\mathscr{U}}$ be a collection of open subsets of Y such that $\mathscr{U} = \tilde{\mathscr{U}} \cap X = \{U \cap X \mid U \in \tilde{\mathscr{U}}\}$. Let $\tilde{Y} = \bigcup\{U \in \tilde{\mathscr{U}}\}$. Then $(F \cap \tilde{Y}) \cup D$ is a σ-Z-set in the locally compact ANR $\tilde{Y}$ [**Ch**] and there is a proper map $\tilde{f}$: $\tilde{Y} \to \tilde{Y}$ whose image misses $F \cup D$ that is $\tilde{\mathscr{U}}$-close to $\mathrm{id}_{\tilde{Y}}$. Since $\tilde{f}$ is proper, $\operatorname{im} \tilde{f}$ is closed in $\tilde{Y}$ and hence there is a neighborhood $\tilde{N}(D)$ of D in $\tilde{Y}$ such that $\tilde{f}(\tilde{Y}) \cap \tilde{N}(D) = \varnothing$. Let $f = \tilde{f} \mid X$ and $N(D) = \tilde{N}(D) \cap X$ and observe that f: $X \to X$ is $\mathscr{U}$-close to id_X and $f(X) \cap N(D) = \varnothing$. This completes the proof of Lemma 1 and Corollary 1.

Both Corollary 1 and Lemma 1 are false without the assumption that X is the complement of a σ-Z-set in a locally compact separable ANR. Also, the Stabilization Theorem is false if $K \times B$ is assumed only to be a Z-set rather than a strong-Z-set in $X \times A$. [**BBMW**] presents an example of a topologically complete separable AR X containing a point x such that $\{x\}$ is a Z-set in X, $X - \{x\}$ is homeomorphic to s, but $X \times A$ is not homeomorphic to s. Though $\{x\}$ is a Z-set and has infinite codimension in X, $\{x\}$ is not a strong Z-set in X.

PROPOSITION 1. *The point $\pi(a)$ has infinite codimension in s/a and the compact set C has infinite codimension in s_C.*

Proof. First, since points in s_C are Z-sets, points in s_C have infinite codimension in s_C and [**DW**; Corollary 2.5] then implies that each finite dimensional closed subset of s_C, and in particular C, has infinite codimension in s_C. Another application of [**DW**; Corollary 2.5] shows that a has infinite codimension in s and since $\pi\colon s \to s/a$ is a fine homotopy equivalence, $\pi(a)$ has infinite codimension in s/a (see Theorem A.7 in the Appendix).

COROLLARY 2. *$s/a \times A$ and $s_C \times A$ are homeomorphic to s.*

Proof. Obviously both s/a and s_C are complements of σ-Z-sets in compact AR's. Since $s/a - \{\pi(a)\}$ and $s_C - C$ are s-manifolds, Proposition 1 and Corollary 1 apply to show that $s/a \times A$ and $s_C \times A$ are s-manifolds and since s/a and s_C are both AR's, both $s/a \times A$ and $s_C \times A$ must be homeomorphic to s [**He**].

Finally we are ready to begin the proof of the Stabilization Theorem. The fundamental tool for recognizing s-manifolds is the characterization theorem of H. Torunczyk [**To$_3$**]. See also [**BBMW**]. Recall that a collection $\mathscr{D}$ of subsets of a space X is *discrete* in X provided each $x \in X$ has a neighborhood that meets at most one member of $\mathscr{D}$.

s-MANIFOLD CHARACTERIZATION THEOREM. *A topologically complete separable ANR X is an s-manifold if and only if for each open cover $\mathscr{U}$ of X and map $f\colon \oplus_{n=1}^{\infty} I^n \to X$ of the countable free union of cells of unbounded dimension into X, there exists a $\mathscr{U}$-close approximation $g\colon \oplus_{n=1}^{\infty} I^n \to X$ to f such that $\{g(I^n)\}_{n=1}^{\infty}$ forms a discrete family in X.*

The above approximation property characterizing s-manifolds is referred to as the *discrete approximation property*. Our proof of the Stabilization Theorem consists of verifying that $X \times A$ satisfies the discrete approximation property. The following Lemma is critical to the proof.

$E(A)$ denotes the endpoints of A. The reader should observe that since $E(\bar{A})$ is a dense G_δ in $\bar{A}$ and $A = \bar{A} - F_0$ where F_0 is a σ-compact subset of $E(\bar{A})$, the Baire property for $\bar{A}$ guarantees that $E(A)$ is not only non-empty, but also dense in A.

LEMMA 2. *Let L be a closed subset of A contained in $E(A)$ and let $\mathcal{U}$ be a collection of open subsets of A that covers L. Then there is a countable collection $\mathcal{V}$ of (pairwise disjoint) open subsets of A that refines $\mathcal{U}$ and covers L such that each element of $\mathcal{V}$ is a component of the complement of some point in A and for which $\{\mathrm{Cl}_A V \mid V \in \mathcal{V}\}$ is discrete in A.*

Proof. Use the fact that $E(A)$ is 0-dimensional to obtain a countable pairwise disjoint collection $\mathcal{W}$ of connected open subsets of A that refines $\mathcal{U}$ and covers L, and assume that each element of $\mathcal{W}$ meets L. For each W in $\mathcal{W}$, choose an open subset $\bar{W}$ in $\bar{A}$ such that $\bar{W} \cap A = W$ and let w be a cutpoint of A in W **[Wh]**. Let $[w]$ denote the union of all arcs from w to points in the frontier of $\bar{W}$ in $\bar{A}$. $[w]$ is a closed subset of $\bar{A}$, hence $W - [w]$ is open in A and therefore each component of $W - [w]$ is open in A. For each component D of $W - [w]$, observe that $\mathrm{Fr}_A D \subset [w]$ ($\mathrm{Fr}_A D$ denotes the topological frontier of D in A). Suppose $x, y \in \mathrm{Fr}_A D$ and $x \neq y$. Since $D \cup \{x, y\}$ is arc connected **[Wh]**, the arc $[x, y]$ from x to y is contained in $D \cup \{x, y\}$. But $[x, y] \subset [x, w] \cup [w, y] \subset [w]$, a contradiction. Therefore, $\mathrm{Fr}_A D$ consists of exactly one point. Use the fact that L is closed in A to obtain a cutpoint a_D in D such that $L \cap D$ is contained in a component of $A - \{a_D\}$ contained in D. In this way, we obtain a countable pairwise disjoint collection $\mathcal{V}$ of connected open subsets of A that refines $\mathcal{W}$ and covers L such that each V in $\mathcal{V}$ meets L and is a component of the complement of a cutpoint of A. Moreover, by our choice of cutpoint a_D corresponding to D, $\{\mathrm{Cl}_A V \mid V \in \mathcal{V}\}$ is pairwise disjoint. Now observe that $\{\mathrm{Cl}_A V \mid V \in \mathcal{V}\}$ is discrete in A since L is closed in A, each V in $\mathcal{V}$ meets L, and the fact that $\bar{A}$ is a locally connected, uniquely arcwise connected compactum implies that only finitely many of the $\mathrm{Cl}_A V$'s can have diameter greater than any given positive number.

Proof of the Stabilization Theorem. Let $\mathcal{U}$ be an open cover of $X \times A$ that consists of product open sets $V \times W$ where V and W are open in X and A, respectively, and let $f\colon \bigoplus_{n=1}^{\infty} I^n \to X \times A$ be a map. Since $A - E(A)$ is σ-compact, our hypothesis implies that $K \times (A - E(A))$ is a countable union of strong-Z-sets in $X \times A$ (strong-σ-Z-set) and thus there exists a map $\alpha\colon X \times A \to X \times A$ that is $\mathcal{U}$-close to $\mathrm{id}_{X \times A}$ that satisfies

$$(*) \qquad [\mathrm{Cl}_{X\times A}\,\alpha(X\times A)] \cap [K\times(A - E(A)] = \varnothing.$$

Let $L(\alpha\circ f)$ denote the *limit points of the map* $\alpha\circ f$, that is, $L(\alpha\circ f)$ consists of all points x in $X\times A$ such that every neighborhood of x meets infinitely many sets from $\{\alpha(f(I^n))\}_{n=1}^{\infty}$. Easily, $L(\alpha\circ f)$ is a closed subset of $X\times A$ and since K is compact, this implies that $L = p_A((K\times A)\cap L(\alpha\circ f))$ is a closed subset of A where p_A denotes the obvious projection map. By $(*)$, L consists only of endpoints of A, that is, $L\subset E(A)$.

For the moment, fix $a\in L$. Choose finitely many sets $V_1\times W_1,\ldots,V_k\times W_k$ in $\mathscr{U}$ so that $(K\times\{a\})\cap(V_i\times W_i)\neq\varnothing$ for each i and $K\times\{a\}\subset\bigcup_{i=1}^{k}(V_i\times W_i)$. Let $V_a = V_1\cup\cdots\cup V_k$ and $W_a = W_1\cap\cdots\cap W_k$ and observe that $K\times\{a\}\subset V_a\times W_a$ and if $v\in V_a$ and $W\subset W_a$, then $\{v\}\times W\subset U$ for some $U\in\mathscr{U}$. Obtain such sets V_a and W_a for each element a of L.

Since $L\subset E(A)$ is closed in A and $\{W_a \mid a\in L\}$ is a cover of L by sets open in A, Lemma 2 applies to produce a refinement $\{W_\gamma\}_{\gamma=1}^{\infty}$ of $\{W_a \mid a\in L\}$ by pairwise disjoint open sets in A that cover L and such that each W_γ is a component of $A - \{a_\gamma\}$ for some cutpoint a_γ in A. Also, if $\overline{W}_\gamma = W_\gamma\cup\{a_\gamma\} = \mathrm{Cl}_A W_\gamma$, then $\{\overline{W}_\gamma\}_{\gamma=1}^{\infty}$ is discrete in A. For each γ, choose some W_a so that $W_\gamma\subset W_a$ and define V_γ to be V_a. The collection $\{\{v\}\times W_\gamma \mid v\in V_\gamma,\ \gamma = 1,2,\ldots\}$ refines $\mathscr{U}$ and the collection $\{V_\gamma\times W_\gamma\}_{\gamma=1}^{\infty}$ is an open cover of $(K\times A)\cap L(\alpha\circ f)$ by pairwise disjoint open subsets of $X\times A$.

For each positive integer γ, choose a sequence $\{b_\gamma(i)\}_{i=1}^{\infty}$ of points in W_γ such that $\{b_\gamma(i)\}_{i=1}^{\infty}$ is discrete in A. Since $\{\overline{W}_\gamma\}_{\gamma=1}^{\infty}$ is discrete, $\{b_\gamma(i) \mid \gamma, i = 1,2,\ldots\}$ forms a discrete collection in A. Fix γ and for each i, let $H_i\colon \overline{W}_\gamma\times[0,1]\to\overline{W}_\gamma$ be a contraction of $\overline{W}_\gamma$ to the point $b_\gamma(i)$ that keeps $b_\gamma(i)$ fixed. Choose open subsets $S_0(\gamma)$ and $S_1(\gamma)$ in X such that

(i) $K\subset S_0(\gamma)\subset\overline{S}_0(\gamma) = \mathrm{Cl}_X S_0(\gamma)\subset S_1(\gamma)\subset\overline{S}_1(\gamma) = \mathrm{Cl}_X S_1(\gamma)\subset V_\gamma$,

(ii) $[\overline{S}_1(\gamma)\times\{a_\gamma\}]\cap[\mathrm{Cl}_{X\times A}\,\alpha(X\times A)] = \varnothing$.

Let $C = (X\times A) - (S_1(\gamma)\times W_\gamma)$ and $D = \overline{S}_0(\gamma)\times\overline{W}_\gamma$ and let $C' = (\alpha\circ f)^{-1}(C)$ and $D' = (\alpha\circ f)^{-1}(D)$. For each positive integer i, the sets $C_i' = C'\cap I^i$ and $D_i' = D'\cap I^i$ are closed subsets of I^i and since $C\cap D = \overline{S}_0(\gamma)\times\{a_\gamma\}$ misses $(\alpha\circ f)(I^i)$, $C_i'\cap D_i' = \varnothing$. Let $\theta_i\colon I^i\to[0,1]$ be a Urysohn function such that $\theta_i(C_i') = \{0\}$ and $\theta_i(D_i') = \{1\}$ and define f_γ

on I^i by the formula

$$f_\gamma(x) = \begin{cases} ((p_X \circ \alpha \circ f)(x), H_i((p_A \circ \alpha \circ f)(x), \theta_i(x))) & \\ & \text{if } (\alpha \circ f)(x) \in \bar{S}_1(\gamma) \times \overline{W}_\gamma \\ (\alpha \circ f)(x) & \text{if } (\alpha \circ f)(x) \in C \end{cases}$$

where p_X and p_A are the obvious projections. In this way, we obtain a continuous map $f_\gamma \colon \oplus_{n=1}^\infty I^n \to X \times A$. Observe that f_γ is $\mathscr{U}$-close to $\alpha \circ f$ since f_γ differs from $\alpha \circ f$ only in a movement of the second coordinate that takes place in W_γ. We obtain such a map f_γ for each positive integer γ.

Define a function $g \colon \oplus_{n=1}^\infty I^n \to X \times A$ via $g(x) = f_\gamma(x)$ if $(\alpha \circ f)(x) \in \bar{S}_1(\gamma) \times \overline{W}_\gamma$ and $g(x) = (\alpha \circ f)(x)$ if $(\alpha \circ f)(x)$ is not in any $S_1(\gamma) \times W_\gamma$. We make three claims about g.

(1) *g is well-defined and $\mathscr{U}$-close to $\alpha \circ f$.* Since $\overline{W}_\gamma \cap \overline{W}_{\gamma'} = \varnothing$ if $\gamma \neq \gamma'$ and $f_\gamma = \alpha \circ f$ off $S_1(\gamma) \times W_\gamma$, there is at most one value $g(x)$ assigned to any x. g is $\mathscr{U}$-close to $\alpha \circ f$ since each f_γ is $\mathscr{U}$-close to $\alpha \circ f$.

(2) *g is continuous.* Since $\{\overline{W}_\gamma\}_{\gamma=1}^\infty$ is discrete in A,

$$T_1 = \bigcup_{\gamma=1}^\infty \{\bar{S}_1(\gamma) \times \overline{W}_\gamma\}$$

is a closed subset of $X \times A$, and obviously

$$T_2 = (X \times A) - \bigcup_{\gamma=1}^\infty \{S_1(\gamma) \times W_\gamma\}$$

is a closed subset of $X \times A$. It is clear that $g|(\alpha \circ f)^{-1}(T_1)$ and $g|(\alpha \circ f)^{-1}(T_2)$ are both continuous and agree with each other on $(\alpha \circ f)^{-1}(T_1 \cap T_2)$. Hence g is continuous.

(3) *$L(g) \cap (K \times A) = \varnothing$ where $L(g)$ denotes the limit points of the map g.* Recall that $L(g)$ consists of all points in $X \times A$ each of whose neighborhoods meets infinitely many members of $\{g(I^i)\}_{i=1}^\infty$. Suppose that $g(y_i) \to (x, a) \in K \times A$ where $y_i \in I^{k(i)}$ and $k(i) \to \infty$ as $i \to \infty$. If $a \in W_\gamma$ for some γ, then eventually $g(y_i) \in S_0(\gamma) \times W_\gamma$ (a neighborhood of (x, a)) and $g(y_i) = f_\gamma(y_i)$. In this case, $p_A(f_\gamma(y_i)) = b_\gamma(k(i))$ and thus $b_\gamma(k(i)) \to a$ as $i \to \infty$. This contradicts the fact that $\{b_\gamma(i)\}_{i=1}^\infty$ is discrete in A. Suppose then that $a \notin \overline{W}_\gamma$ for all γ. Then eventually, since $\{\overline{W}_\gamma\}_{\gamma=1}^\infty$ is discrete in A, $p_A(g(y_i)) \notin \overline{W}_\gamma$ for all γ. Then $g(y_i) = (\alpha \circ f)(y_i)$ and $(x, a) \in L(\alpha \circ f)$. But then $a \in L$ and we reach a contradiction since L is covered by $\{W_\gamma\}_{\gamma=1}^\infty$. Thus, we must have that $a = a_{\gamma_0}$ for some γ_0. Since $p_A(g(y_i)) \to a_{\gamma_0} = \mathrm{Fr}_A(W_{\gamma_0})$ and $\{\overline{W}_\gamma\}_{\gamma=1}^\infty$ is

discrete, either $p_A(g(y_i)) \in W_{\gamma_0}$ for infinitely many i or $p_A(g(y_i)) \notin \overline{W}_\gamma$ for all γ for infinitely many i. In the former case we have $g(y_i) = f_{\gamma_0}(y_i)$ for infinitely many i and in the latter we have $g(y_i) = (\alpha \circ f)(y_i)$ for infinitely many i. These give rise to the same contradictions exhibited above and we finally conclude that $L(g) \cap (K \times A) = \varnothing$.

Let d be any metric for $X \times A$ and let $\mathscr{V}$ be an open cover of $(X - K) \times A$ such that each V in $\mathscr{V}$ is contained in some U in $\mathscr{U}$ and such that for all $V \in \mathscr{V}$, $\operatorname{diam}_d V < d(V, K \times A)$. Let

$$D_n = g^{-1}((X - K) \times A) \cap I^n$$

and denote $g \mid \oplus_{n=1}^\infty D_n$ by $\tilde{g}$. Since $X - K$ is an s-manifold, [**To**$_1$] applies to show that $(X - K) \times A$ is an s-manifold and since $\oplus_{n=1}^\infty D_n$ is a complete separable metric space, [**To**$_3$; 2.1] provides a map $\tilde{h}\colon \oplus_{n=1}^\infty D_n \to (X - K) \times A$ such that $\tilde{h}$ is $\mathscr{V}$-close to $\tilde{g}$ and $\{\tilde{h}(D_n)\}_{n=1}^\infty$ forms a discrete family in $(X - K) \times A$. Define $h\colon \oplus_{n=1}^\infty I^n \to X \times A$ by $h(x) = \tilde{h}(x)$ if $x \in \oplus_{n=1}^\infty D_n$ and $h(x) = g(x)$ otherwise. It is clear that h is $\mathscr{U}$-close to g and our choice of $\mathscr{V}$ guarantees that h is continuous. It is straightforward to prove that $L(h) = \varnothing$, and this implies that each point in $X \times A$ has a neighborhood that meets at most finitely many members of $\{h(I^n)\}_{n=1}^\infty$, hence $\{h(I^n)\}_{n=1}^\infty$ is locally finite in $X \times A$.

Given f, we have found a map h that is $\operatorname{st}^2 \mathscr{U}$-close to f for which $\{h(I^n)\}_{n=1}^\infty$ is locally finite. Since this implies that compact subsets of $X \times A$ are Z-sets, we may assume by a further small adjustment that $\{h(I^n)\}_{n=1}^\infty$ is both pairwise disjoint and locally finite, hence discrete. The s-manifold Characterization Theorem then implies that $X \times A$ is an s-manifold.

5. Further examples. The Stabilization Theorem provides a refinement of our dog bone example by exhibiting such an example where all the fibers of π are points and tame Z-set copies of the 1-dimensional AR A.

EXAMPLE 3. There is a map $\pi\colon s \to s_C$ that satisfies (2.1) through (2.4) of Example 2 with the exception that the nondegenerate fibers of π are tame Z-set copies of the 1-dimensional AR A.

Construction. Since $s_C - C = s - C$ is an s-manifold, the projection map $p\colon (s_C - C) \times A \to s_C - C$ is a near-homeomorphism (see Appendix). Construct h from p using exactly the method used in Example 2. This produces a surjective map $h\colon s_C \times A \to s_C$ such that $N_h = C$ and such that $\{c\} \times A$ is a Z-set in $s_C \times A$ for all $c \in C$. By Corollary 2 of §4, $s_C \times A$ is homeomorphic to s; let $\pi = h \circ g\colon s \to s_C$ where g is a homeomorphism of s onto $s_C \times A$. Then $\pi\colon s \to s_C$ satisfies (2.1) through (2.4) except that $\pi^{-1}(c)$ is a Z-set copy of A in s for each $c \in C$. The verification of (2.1) through (2.4) is exactly as in Example 2 with A in place of s where pertinent.

In **[BBMW]**, various examples of complete separable AR's not homeomorphic to s are constructed that possess the following properties: (i) each example contains a point that is a Z-set but not a strong-Z-set and the complement of this point is homeomorphic to s, (ii) each example satisfies various weaker versions of the discrete approximation property, (iii) none of the examples has a "nice" ANR local compactification in the sense that the examples do not arise as complements of σ-Z-sets in locally compact ANR's (this follows either from **[Bo$_3$]** or from (i) and Lemma 1 of §4), (iv) none of the examples stabilize upon multiplication by any finite product of A with itself (this is proved in **[Be]**). The question arises as to whether or not (iv) is independent of (i), (ii), and (iii); in particular, is (iv) an intrinsic property of complete separable AR's that have no nice ANR local compactification? The example s/α can be used to construct an example that satisfies (i) through (iii) and which becomes homeomorphic to s upon multiplication by A, thus answering the above question negatively.

EXAMPLE 4. There is a fine homotopy equivalence $f: s \to X$ where X is a complete separable AR not homeomorphic to s that satisfies:

(4.1) $N_f = \{x\}$ is a Z-set in X but not a strong-Z-set in X and $X - \{x\}$ is homeomorphic to s;

(4.2) X satisfies the discrete 1-cells property and the discrete carriers property, but not the discrete 2-cells property (see **[BBMW]** for definitions);

(4.3) X necessarily does not have a nice ANR local compactification;

(4.4) $f \times \mathrm{id}: s \times A \to X \times A$ is a near-homeomorphism and $X \times A \approx s \times A \approx s$.

We delete the construction of Example 4 since Bestvina **[Be]**, using different techniques, produces examples having the properties of Example 4.

Examples 1 through 4 are examples of complete separable AR's not homeomorphic to s but which become homeomorphic to s upon multiplication by A. On the other hand, the examples of **[BBMW]** are complete separable AR's not homeomorphic to s and which do not become homeomorphic to s upon multiplication by any finite product of A with itself. The following question arises.

Question. If X is a complete separable AR and $X \times A^n$ is homeomorphic to s for some positive integer n, where A^n denotes the n-fold product of A with itself, is $X \times A$ homeomorphic to s?

If X has a nice ANR local compactification, then the answer is yes and appears in [**Bo**$_4$]. The question remains open in the general case, though Bestvina [**Be**] constructs, modulo the construction of certain examples in homotopy theory, an example for each n of a space X_n such that $X_n \times A^n$ is not homeomorphic to s while $X_n \times A^{n+1}$ is homeomorphic to s.

APPENDIX.

1. Adjusting maps into I^∞. A closed subset F of an ANR X is a *Z_n-set* for some non-negative integer n provided for every $\varepsilon > 0$ and map $f: I^n \to X$ of the n-cell into X, there is a map $g: I^n \to X - F$ that is ε-close to f. An embedding into X is a *Z_n-embedding* provided its image is a Z_n-set.

THEOREM A.1. *Let (D, D_0) be a compact metric pair of dimension at most n and let $f: D \to I^\infty$ be a map such that $f \mid D_0$ is a Z_n-embedding where n is some non-negative integer. Then for every $\varepsilon > 0$, there exists an embedding $g: D \to I^\infty$ such that $g \mid D_0 = f \mid D_0$ and g is ε-close to f. Furthermore, if B is a σ-Z-set in I^∞, g may be chosen so that $g(D - D_0) \cap B = \varnothing$.*

Proof. The space $C(D, I^\infty)$ of maps $D \to I^\infty$ with the sup-norm metric is complete [**Du**]. Let d be a metric for I^∞ and for $\delta > 0$ define

$$C_\delta(D, I^\infty) = \{g \in C(D, I^\infty) \mid d(g(a), f(a)) < \delta \text{ for all } a \in D_0\},$$
$$C_0(D, I^\infty) = \{g \in C(D, I^\infty) \mid g|D_0 = f \mid D_0\}.$$

Then for each $\delta > 0$, $C_\delta(D, I^\infty)$ is open in $C(D, I^\infty)$ and since $C_0(D, I^\infty) = \bigcap_{n=1}^\infty C_{1/n}(D, I^\infty)$, $C_0(D, I^\infty)$ is a G_δ-subset of a complete space, hence $C_0(D, I^\infty)$ is a Baire space.

Let $\{(D_i, D_i')\}_{i=1}^\infty$ be a countable collection of pairs of compact subsets of $D - D_0$ such that (i) $\operatorname{Int}_D D_i \neq \varnothing \neq \operatorname{Int}_D D_i'$, (ii) $D_i \cap D_i' = \varnothing$ for all i, and (iii) for each $x, y \in D - D_0$ with $x \neq y$, there exists an i such that $x \in D_i$ and $y \in D_i'$. For each i, let

$$U_i = \{g \in C_0(D, I^\infty) \mid g(D_i) \cap g(D_i') = \varnothing \text{ and } g(D_i \cup D_i') \cap g(D_0) = \varnothing\}.$$

Easily U_i is open in $C_0(D, I^\infty)$ and since $f \mid D_0$ is an Z_n-embedding and D is at most n-dimensional, U_i is dense in $C_0(D, I^\infty)$.

Let $B = \bigcup_{i=1}^\infty B_i$ where B_i is a Z-set in I^∞ and let $D - D_0 = \bigcup_{i=1}^\infty C_i$ where each C_i is compact. Define

$$V_{i,j} = \{g \in C_0(D, I^\infty) \mid g(C_i) \cap B_j = \varnothing\}, \qquad i, j = 1, 2 \ldots.$$

Each $V_{i,j}$ is open and dense in $C_0(D, I^\infty)$, hence $(\bigcap_i U_i) \cap (\bigcap_{i,j} V_{i,j})$ is dense in $C_0(D, I^\infty)$. Any g in $(\bigcap_i U_i) \cap (\bigcap_{i,j} V_{i,j})$ is an embedding with $g \mid D_0 = f \mid D_0$ and $g(D - D_0) \cap B = \varnothing$.

THEOREM A.2. *Let (D, D_0) be a compact pair and let B be a σ-Z-set in I^∞. Then for every map $f: D \to I^\infty$ and for every $\varepsilon > 0$, there exists a map g: $D \to I^\infty$ ε-close to f such that $g \mid D_0 = f \mid D_0$ and $g(D - D_0) \cap B = \varnothing$.*

The proof of A.2 is contained in the proof of A.1.

2. Fine homotopy equivalence, UV^∞-maps, and near-homeomorphisms. Let $f: X \to Y$ be a map between complete separable ANR's. f is a *fine homotopy equivalence* provided for every open cover $\mathscr{U}$ of Y, there is a map $g: Y \to X$ such that $f \circ g$ is $\mathscr{U}$-homotopic to id_Y and $g \circ f$ is $f^{-1}(\mathscr{U})$-homotopic to id_X. f is a *UV^∞-map* if for every $y \in Y$ and neighborhood U of y, there exists a neighborhood V of y such that $f^{-1}(V)$ contracts to a point in $f^{-1}(U)$. f is a *near-homeomorphism* provided for every open cover $\mathscr{U}$ of Y, there exists a homeomorphism h: $X \to Y$ that is $\mathscr{U}$-close to f. The following sresults are used in this paper.

THEOREM A.3. **[Ko]**. *f is a fine homotopy equivalence if and only if f is a UV^∞-map.*

COROLLARY A.4. *If f_1: $X_1 \to Y_1$ and f_2: $X_2 \to Y_2$ are fine homotopy equivalences, then $f_1 \times f_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ is a fine homotopy equivalence.*

THEOREM A.5. **[Fe]**. *A fine homotopy equivalence between s-manifolds is a near-homeomorphism.*

THEOREM A.6. **[Sc]**, **[To$_1$]**. *The projection mapping p: $M \times X \to M$ where M is an s-manifold and X is a complete separable AR is a near-homeomorphism.*

Proof. By **[To$_1$]**, $M \times X$ is an s-manifold and easily p is a UV^∞-map. Theorems A.3 and A.5 imply that p is a near-homeomorphism.

THEOREM A.7. *If $f: X \to Y$ is a fine homotopy equivalence and K is a compact subset of Y such that $f^{-1}(K)$ has infinite codimension in X, then K has infinite codimension in Y.*

3. Strong-Z-sets and reduced products. Below, we list two results that will appear in **[BBMW]** and that are used in this paper. *Strong-Z-sets* are defined in §4. Given spaces Z and F and a closed subset $L \subset Z$, the *product of Z and F reduced about L*, or the *reduced product of Z and F*, is the space $[(Z - L) \times F] \cup L$ equipped with the topology generated by open subsets of $(Z - L) \times F$ and sets of the form $(U \cap L) \cup (U - L) \times F$, where $U \subset Z$ is open. The reduced product is denoted by $(Z \times F)_L$.

THEOREM A.8. [**BBMW**; Corollary 1.2]. *A closed subset L of an ANR Z is a strong-Z-set in Z if and only if L is a strong-Z-set in $(Z \times s)_L$.*

THEOREM A.9. [**BBMW**; Strong-Z-set Shrinking]. *If $f: M^s \to X$ is a fine homotopy equivalence from an s-manifold to a topologically complete separable ANR and $\mathrm{Cl}_X N_f$ is a strong-Z-set in X, then f is a near-homeomorphism.*

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