

Pacific Journal of Mathematics

**INCREASING PATHS ON THE ONE-SKELETON OF A CONVEX
COMPACT SET IN A NORMED SPACE**

LEONI DALLA

INCREASING PATHS ON THE ONE-SKELETON OF A CONVEX COMPACT SET IN A NORMED SPACE

LEONI DALLA

Let C be a convex compact set in a normed space E and let $\text{skel}_1 C$ be the subset of C that contains those boundary points of C which are not centres of 2-dimensional balls in C . When l is a continuous functional on E , we say that the path $P = g([\alpha, \beta])$ is l -strictly increasing if $l(g(t_1)) < l(g(t_2))$ for every t_1, t_2 such that $\alpha \leq t_1 < t_2 \leq \beta$. D. G. Larman proved the existence of an l -strictly increasing path on the one skeleton of C with $l(g(\alpha)) = \min_{x \in C} l(x)$ and $l(g(\beta)) = \max_{x \in C} l(x)$.

In this paper we prove a theorem concerning the number of l -strictly increasing paths on the one-skeleton of C , that are mutually disjoint and along each of which l assumes values in a range arbitrarily close to its range on C .

1. The results. We quote and prove the following theorem

THEOREM 1. *Let C be a compact convex set of infinite dimension in a normed space E and l be a continuous linear functional on E , which is non constant on C . Let $\varepsilon > 0$ be given, $M = \max_{x \in C} l(x)$ and $m = \min_{x \in C} l(x)$. Then, for every $n = 1, 2, 3, \dots$ there exist n l -strictly increasing paths, $P_k = g_k([\alpha, \beta])$, $k = 1, 2, \dots, n$ on the one-skeleton of C , such that $\text{relint } P_i \cap \text{relint } P_j = \emptyset$ with $i \neq j$, $l(g_k(\alpha)) = m + \varepsilon$ and $l(g_k(\beta)) = M - \varepsilon$ for $k = 1, 2, \dots, n$.*

Proof. Consider the sets $K_0 = \{x \in C: l(x) = M - \varepsilon\}$ and $K_1 = \{x \in C: l(x) = m - \varepsilon\}$. These sets are of infinite dimension and lie on two parallel hyperplanes. We define

$$A = C \cap \{x \in E: l(x) \geq m + \varepsilon\} \cap \{x \in E: l(x) \leq M - \varepsilon\}$$

Then we may select n linearly independent vectors e_1, e_2, \dots, e_n and n linear functionals $l_1 = l, l_2, \dots, l_n$ on E such that the following properties hold:

- (i) $l_1(e_1) = 1$, $l_i(e_i) \neq 0$ for $i = 2, 3, \dots, n$ and $l_i(e_j) = 0$ for $i \neq j$
- (ii) Let $E_n = [e_1, e_2, \dots, e_n]$ be the n -dimensional subspace of E spanned by e_1, e_2, \dots, e_n and π_0 be the projection map on E , defined by $\pi_0(x) = l_1(x)e_1 + \dots + l_n(x)e_n$. Then $\dim \pi_0(K_0) = \dim \pi_0(K_1) = n - 1$.

From the previous, it follows that $C_n = \pi_0(A)$ is a convex body in E_n , $\pi_0(K_0) = \{x \in C_n: l(x) = M - \varepsilon\}$ and $\pi_0(K_1) = \{x \in C_n: l(x) = m + \varepsilon\}$.

Let $u \in E_n$ be a unit vector perpendicular to e_1 . Then according to the results proved in [3] we may choose a unit vector $u' \in E_n$ orthogonal to e_1 , as close as we please to u and such that there are no line segments in the direction u' on the boundary of $C_n - \text{rel int } \pi_0(K_0) - \text{rel int } \pi_0(K_1)$. Then the projection σ_{n-1} of E_n onto the hyperplane E_{n-1} perpendicular to u' has an inverse function from $\text{bd } \sigma_{n-1}(C_n) - \text{rel int } \sigma_{n-1}(\pi_0(K_0)) - \text{rel int } \sigma_{n-1}(\pi_0(K_1))$ back to C_n .

If $\{e_1, u_2, \dots, u_{n-1}, u\}$ is an orthogonal system in E_n then we can choose, using induction, unit vectors u'_{n-1}, \dots, u'_3 orthogonal to e_1 and as close as we please in direction to the projections of the vectors u_{n-1}, \dots, u_3 onto the subspaces $E_{n-1} \subseteq [u']^\perp$, $E_{n-2} \subseteq [u', u'_{n-1}]^\perp, \dots, E_3 \subseteq [u', u'_{n-1}, \dots, u'_4]^\perp$ and in such a way the projections $\sigma_k: E_k \rightarrow E_{k-1}, k = n-2, \dots, 3$ have unique inverses from

$$\begin{aligned} & \text{bd } \sigma_k \circ \sigma_{k+1} \circ \dots \circ \sigma_{n-1}(C_n) - \text{rel int } \sigma_k \circ \sigma_{k+1} \circ \dots \circ \sigma_{n-1}(\pi_0(K_0)) \\ & - \text{rel int } \sigma_k \circ \sigma_{k+1} \circ \dots \circ \sigma_{n-1}(\pi_0(K_1)) \end{aligned}$$

back to $\sigma_{k+1} \circ \dots \circ \sigma_{n-1}(C_n)$. We complete the orthonormal system $u', u'_{n-1}, \dots, u'_2, u'_1$ by taking $u'_1 = e_1$ and u'_2 to be the unit vector perpendicular to $u', u'_{n-1}, \dots, u'_3, u'_1 = e_1$ and closest to u_2 .

Write now $\omega_{u'} = \sigma_2 \circ \dots \circ \sigma_{n-1}$ for the projection of E_n on the two dimensional subspace E_2 . For each t such that $m + \varepsilon \leq t \leq M - \varepsilon$, we define by $\xi_0(t)$ the point on the line segment $\{x \in \omega_{u'}(C_n): l_1(x) = t\}$ whose second coordinate attains its maximum value. On the other hand we may suppose, by making appropriate transformation of C , that there exists a cylinder B in the convex body C_n of E_n such that $B = \overline{\text{con}(S_0 \cup S_1)}$, where S_0 and S_1 are $(n-1)$ -dimensional balls of diameter δ with the property $S_i \subseteq \text{rel int } \pi_0(K_i), i = 0, 1$ and the axis of B in the direction of e_1 .

Let ε_0 be such that $0 < \varepsilon_0 < \min\{d(\text{bd } \pi_0(K_0), S_0), d(\text{bd } \pi_0(K_1), S_1)\}$ where d is the usual distance between two sets. The convexity of C_n implies $d(\text{bd } C_n - \pi_0(K_0) - \pi_0(K_1), B) > \varepsilon_0$. Then there exist a linear functional $l_{u'}$ on E_n such that $l_{u'}(u') = 0, l_{u'}(u'_2) = 1$ and $l_{u'}(\xi_0(t)) > \varepsilon_0 + \delta/2 > 0$.

Now let $\xi'_0(t)$ be the point on the line segment $\{x \in \omega_{u'}(C_n): l_1(x) = t\}$ whose second coordinate attains its minimum value, then $l_{u'}(\xi'_0(t)) < -(\varepsilon_0 + \delta/2) < 0$. Because of the choice of $u', u'_{n-1}, \dots, u'_3$ the inverse function $\omega_{u'}^{-1}$ is uniquely defined from the curves $\xi_0(t)$ and $\xi'_0(t)$

back to the one-skeleton of C_n . Consider now $x_0(t) = \omega_u^{-1}(\xi_0(t))$ and $x'_0(t) = \omega_u^{-1}(\xi'_0(t))$ where $m + \epsilon \leq t \leq M - \epsilon$. Then $x_0(t)$ and $x'_0(t)$ where $m + \epsilon \leq t \leq M - \epsilon$ are paths on the one-skeleton of C_n . By construction $l_1(x_0(t)) = t, l_1(x'_0(t)) = t$,

$$(1) \quad l_u(x_0(t)) > \epsilon_0 + \frac{\delta}{2}, \quad l_u(x'_0(t)) < -\left(\epsilon_0 + \frac{\delta}{2}\right)$$

for $m + \epsilon \leq t \leq M - \epsilon$.

We say then that $\{x_0(t), m + \epsilon \leq t \leq M - \epsilon\}$ and $\{x'_0(t), m + \epsilon \leq t \leq M - \epsilon\}$ are paths on the one-skeleton of C_n "in the direction near u ". Following the methods developed in Theorem 1 in [2] we construct two l -strictly increasing paths $z_0(t)$ and $z'_0(t)$, $m + \epsilon \leq t \leq M - \epsilon$ on the one-skeleton of A such that

$$(2) \quad l_1(z_0(t)) = t, \quad l_1(z'_0(t)) = t \quad \text{and}$$

$$\|\pi_0(z_0(t)) - x_0(t)\| < \frac{\epsilon_0}{3}, \quad \|\pi_0(z'_0(t)) - x'_0(t)\| < \frac{\epsilon_0}{3}$$

where $m + \epsilon \leq t \leq M - \epsilon$.

From relations (1) and (2) it follows that

$$(3) \quad l_u(\pi_0(z_0(t))) > \frac{2}{3}\epsilon_0 + \frac{\delta}{2}, \quad l_u(\pi_0(z'_0(t))) < -\left(\frac{2}{3}\epsilon_0 + \frac{\delta}{2}\right)$$

$$\text{and} \quad \begin{aligned} \pi_0\{z_0(t): m + \epsilon \leq t \leq M - \epsilon\} \cap B &= \emptyset, \\ \pi_0\{z'_0(t): m + \epsilon \leq t \leq M - \epsilon\} \cap B &= \emptyset. \end{aligned}$$

As (2) holds we may say that $z_0(t), z'_0(t)$ are paths on the one-skeleton of A in the direction near u and we write $z_0 = z_u$ and $z'_0 = z'_u$.

Let S be the unit ball in E^n , lying on the hyperplane $l_1(x) = 0$ and let θ be a positive number such that $0 < \theta < (1/2d)(\delta/2 + \epsilon_0/3)$ where $d = \text{diam } C_n$. The compactness of S implies the existence of unit vectors u_1, u_2, \dots, u_m such that for every unit vector u in S , there exists $i_0 \in \{1, 2, \dots, m\}$ with $\|u - u_{i_0}\| < \theta$. Let $Z_{u_i}\{z_{u_i}(t), m + \epsilon \leq t \leq M - \epsilon\}$ and $Z_{u_{m+i}} = \{z'_u(t), m + \epsilon \leq t \leq M - \epsilon\}$ where $i = 1, 2, \dots, m$ be paths on the one-skeleton of A in the direction near u_i . Let $j(Z_{u_1}, Z_{u_2}, \dots, Z_{u_i})$ be the junction set of the paths $Z_{u_1}, Z_{u_2}, \dots, Z_{u_i}$. Suppose now that $\text{card } j(Z_{u_1}, Z_{u_2}, \dots, Z_{u_{\lambda-1}}) < +\infty$ and $\text{card } j(Z_{u_1}, Z_{u_2}, \dots, Z_{u_\lambda}) = +\infty$ for some λ such that $1 \leq \lambda \leq 2m$. Renaming, if necessary, the paths $Z_{u_1}, Z_{u_2}, \dots, Z_{u_\lambda}$ we consider the greatest integer k such that $1 \leq k \leq \lambda - 1$, $\text{card } j(Z_{u_i}, Z_{u_\lambda}) < \infty$ for $i = 1, 2, \dots, k - 1$ and $\text{card } j(Z_{u_i}, Z_{u_\lambda}) = +\infty$ for $i = k, k + 1, \dots, \lambda - 1$.

Let

$$\alpha = \inf \{ t : t \in [m + \varepsilon, M - \varepsilon] \text{ and } z_{u_k}(t) \in j(Z_{u_k}, Z_{u_\lambda}) \}$$

and

$$\beta = \sup \{ t : t \in [m + \varepsilon, M - \varepsilon] \text{ and } z_{u_k}(t) \in j(Z_{u_k}, Z_{u_\lambda}) \}.$$

As z_{u_k} and z_{u_λ} are continuous functions, there is a finite number of closed subintervals $[a_i, b_i]$, $i = 1, 2, \dots, \nu$, of $[m + \varepsilon, M - \varepsilon]$ with the following properties:

- (i) $z_{u_k}(a_i) = z_{u_\lambda}(a_i)$, $z_{u_k}(b_i) = z_{u_\lambda}(b_i)$
- (ii) $z_{u_k}(t) \neq z_{u_\lambda}(t)$, $\alpha_i < t < b_i$
- (iii) $\max_{a_i < t < b_i} \|z_{u_k}(t) - z_{u_\lambda}(t)\| > \varepsilon_0/3$ for $i = 1, 2, \dots, \nu$.

Then

$$\begin{aligned} z_{u_\lambda}(m + \varepsilon, a) \cup z_k(a, a_1) \cup \bigcup_{i=1}^{\nu} z_{u_\lambda}(a_i, b_i) \\ \cup \bigcup_{i=1}^{\nu-1} z_{u_k}(b_i, a_{i+1}) \cup z_{u_k}(b_\nu, b) \cup z_{u_\lambda}(b, M - \varepsilon) \end{aligned}$$

is an l -increasing path, $Z_{u_\lambda}^*$ say, on the one-skeleton of C that is different from Z_{u_λ} on the set

$$\Gamma = z_{u_k}(a, a_1) \cup \bigcup_{i=1}^{\nu-1} z_{u_k}(b_i, a_{i+1}) \cup z_{u_k}(b_\nu, b).$$

By construction the set Γ is within distance $\varepsilon_0/3$ from Z_{u_λ} , hence we have

$$(4) \quad \|z_{u_\lambda}(t) - z_{u_\lambda}^*(t)\| < \varepsilon_0/3 \quad \text{for every } t \in [m + \varepsilon, M - \varepsilon]$$

As $\text{card } j(Z_{u_i}, Z_{u_\lambda}^*) < +\infty$ for $i = 1, 2, \dots, k$, we can replace Z_{u_λ} by $Z_{u_\lambda}^*$ for every $\lambda = 1, 2, \dots, 2m$ with $\text{card } j(Z_{u_1}, \dots, Z_{u_{\lambda-1}}) < +\infty$ and $\text{card } j(Z_{u_1}, \dots, Z_{u_\lambda}) = +\infty$. Then $\text{card } j(Z_{u_1}^*, \dots, Z_{u_{2m}}^*) < +\infty$ and using (3) and (4) we get $|l_{u'}(\pi_0(z_{u_\lambda}^*(t)))| > \delta/2 + \varepsilon_0/3$ where $u' \in S$, $\|u' - u_\lambda\| < \theta$.

Now we can define the graph G with vertex set $V = \{K_0\} \cup \{K_1\} \cup j(Z_{u_1}^*, \dots, Z_{u_{2m}}^*)$, where an ordered pair of these nodes is said to form a directed subgraph of G if they are joined by an l -increasing arc from $\bigcup_{i=1}^{2m} Z_{u_i}^*$, which contains no other node of G . The required result now follows from Menger-Whitney theorem for the finite graph G , if we are able to show that the removal of $(n - 1)$ vertices from $j(Z_{u_1}^*, \dots, Z_{u_{2m}}^*)$ still allows an l -increasing path running from K_0 to K_1 .

Let y_1, y_2, \dots, y_{n-1} be $(n - 1)$ vertices from $j(Z_{u_1}^*, \dots, Z_{u_{2m}}^*)$. For the points $\pi_0(y_1), \pi_0(y_2), \dots, \pi_0(y_{n-1})$ of E_n , there exists a linear functional l' on E_n such that $l'(\pi_0(y_i)) \geq 0, i = 1, 2, \dots, n - 1, l'(e_1) = 0$ and $l'(v) = 1$ for some $v \in S$. Let now $u \in S$ be an arbitrary vector such that $l'(u) = 0$ and $l_1(u) = 0$. For the vector u there exists a vector $u_k \in S$ such that $\|u - u_k\| \leq \theta$. Let $Z_{u_{m+k}}^*$ be the path on the one-skeleton of C in the direction near u_k , with

$$(5) \quad l_{u_k}(\pi_0(z_{u_{m+k}}^*(t))) < -\left(\frac{\delta}{2} + \frac{\epsilon_0}{3}\right), \quad m + \epsilon \leq t \leq M - \epsilon$$

We can also select u in such a way that $l'(u) = 0$ and $l_1(u) = 0$ for which the corresponding l_{u_k} has the property $l_{u_k}(v') = 1$ for some $v' \in S$ with $\|v - v'\| < \theta$.

Now, we may suppose that

$$(6) \quad \begin{aligned} l_{u_k}(\pi_0(y_i)) &\geq 0 \quad \text{for } i = 1, 2, \dots, \mu \quad \text{and} \\ l_{u_k}(\pi_0(y_i)) &< 0 \quad \text{for } i = \mu + 1, \dots, n - 1 \end{aligned}$$

Relations (5) and (6) imply that

$$(7) \quad \pi_0(y_i) \notin \pi_0(Z_{u_{m+k}}^*) \quad \text{for } i = 1, 2, \dots, \mu$$

We have that $l'(v) = 1, l_{u_k}(v') = 1$ with $\|v - v'\| < \theta$ and $l'(\pi_0(y_i)) \geq 0, l_{u_k}(\pi_0(y_i)) < 0$ for $i = \mu + 1, \dots, n - 1$. Hence

$$(8) \quad l_{u_k}(\pi_0(y_i)) \geq -d\theta - \left(\frac{\delta}{2} + \frac{\epsilon_0}{3}\right), \quad i = \mu + 1, \dots, n - 1.$$

From (5) and (8) we have that $\pi_0(y_i) \notin \pi_0(Z_{u_{m+k}}^*)$ for $i = \mu + 1, \dots, n - 1$. Hence, from (7) and (8) follows that $y_i \notin Z_{u_{m+k}}^*, i = 1, 2, \dots, n - 1$ which completes the proof of the theorem.

From the above theorem one can deduce the following corollaries whose proofs are omitted as obvious.

COROLLARY 1. *Suppose that C and l are defined as in Theorem 1, the faces*

$$F_0 = \left\{ x \in C : l(x) = \min_{y \in C} l(y) \right\} \quad \text{and} \quad F_1 = \left\{ x \in C : l(x) = \max_{y \in C} l(y) \right\}$$

are such that the dimension of $F'_0 \cap F'_1$ is infinite, where F'_0 and F'_1 are the corresponding subspaces translates of F_0 and F_1 correspondingly. Then for every $n = 1, 2, \dots$ there are n l -strictly increasing paths on the one-skeleton of C mutually disjoint that join F_0 to F_1 .

COROLLARY 2. *Suppose that C a compact convex set of infinite dimension in a normed space E . Then the one-dimensional Hausdorff measure of the one-skeleton is infinite.*

We may remark that the n -dimensional Hausdorff measure of the n -skeleton of a set C as in Corollary 2 is infinite for every $n = 1, 2, \dots$. For a direct proof of this result see [1].

REFERENCES

- [1] Leoni Dalla, *The n -dimensional Hausdorff measure of the n -skeleton of a convex weakly compact set*, *Mathematische Nachrichten*, (to appear).
- [2] D. G. Larman, *On the one-skeleton of a compact convex set in a Banach space*, *Proc. London Math. Soc.*, (3) **34** (1977), 117–144.
- [3] D. G. Larman and C. A. Rogers, *Increasing paths on the one-skeleton of a convex body and the directions of line segments on the boundary of a convex body*, *Proc. London Math. Soc.*, **23** (1971), 683–698.

Received March 19, 1985.

DEPARTMENT OF MATHEMATICS
SECTION OF MATHEMATICAL ANALYSIS
AND ITS APPLICATIONS
PANEPISTEMIOPOLIS, 157 81 ATHENS
GREECE

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

V. S. VARADARAJAN
(Managing Editor)
University of California
Los Angeles, CA 90024

HERBERT CLEMENS
University of Utah
Salt Lake City, UT 84112

R. FINN
Stanford University
Stanford, CA 94305

HERMANN FLASCHKA
University of Arizona
Tucson, AZ 85721

RAMESH A. GANGOLLI
University of Washington
Seattle, WA 98195

VAUGHAN F. R. JONES
University of California
Berkeley, CA 94720

ROBION KIRBY
University of California
Berkeley, CA 94720

C. C. MOORE
University of California
Berkeley, CA 94720

H. SAMELSON
Stanford University
Stanford, CA 94305

HAROLD STARK
University of California, San Diego
La Jolla, CA 92093

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH
(1906–1982)

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF HAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph must be capable of being used separately as a synopsis of the entire paper. In particular it should contain no bibliographic references. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California 90024.

There are page-charges associated with articles appearing in the Pacific Journal of Mathematics. These charges are expected to be paid by the author's University, Government Agency or Company. If the author or authors do not have access to such Institutional support these charges are waived. Single authors will receive 50 free reprints; joint authors will receive a total of 100 free reprints. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$190.00 a year (5 Vols., 10 issues). Special rate: \$95.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

The Pacific Journal of Mathematics at P.O. Box 969, Carmel Valley, CA 93924 (ISSN 0030-8730) publishes 5 volumes per year. Application to mail at Second-class postage rates is pending at Carmel Valley, California, and additional mailing offices. Postmaster: send address changes to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Copyright © 1986 by Pacific Journal of Mathematics

Pacific Journal of Mathematics

Vol. 124, No. 2

June, 1986

Philip Lee Bowers , Nonshrinkable “cell-like” decompositions of s	257
Aurelio Carboni and Ross Street , Order ideals in categories	275
Leoni Dalla , Increasing paths on the one-skeleton of a convex compact set in a normed space	289
Jim Hoste , A polynomial invariant of knots and links	295
Sheldon Katz , Tangents to a multiple plane curve	321
Thomas George Lucas , Some results on Prüfer rings	333
Pham Anh Minh , Modular invariant theory and cohomology algebras of extra-special p -groups	345
Ikuko Miyamoto , On inclusion relations for absolute Nörlund summability	365
A. Papadopoulos , Geometric intersection functions and Hamiltonian flows on the space of measured foliations on a surface	375
Richard Dean Resco, J. Toby Stafford and Robert Breckenridge Warfield, Jr. , Fully bounded G -rings	403
Haskell Paul Rosenthal , Functional Hilbertian sums	417
Luen-Fai Tam , Regularity of capillary surfaces over domains with corners: borderline case	469
Hugh C. Williams , The spacing of the minima in certain cubic lattices	483