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**TANGENTS TO A MULTIPLE PLANE CURVE**

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## TANGENTS TO A MULTIPLE PLANE CURVE

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**The limiting behavior of the tangents and the flexes are computed as a reduced plane curve degenerates into a multiple plane curve.**

**0. Introduction.** In this paper, we consider the degeneration of a reduced irreducible plane curve to a multiple plane curve. We study the associated degeneration of tangent lines by viewing a line as a linear imbedding  $\mathbf{P}^1 \hookrightarrow \mathbf{P}^2$  and studying deformations of this imbedding. We compute the limiting behavior of the dual curve and the flexes. A similar computation yields the limiting behavior of the bitangents; this will appear later in a separate paper. The main result is stated as Proposition (2.1).

The author takes this opportunity to thank Professor William Fulton for introducing him to this problem.

**1. The dual of a multiple curve.** Let  $C \subset \mathbf{P}_C^2$  be a smooth curve of degree  $d$ .  $C^* \subset \mathbf{P}^{2*}$  will denote the dual curve of tangents to  $C$ .

Let  $n$  be a positive integer,  $n \geq 2$ . Let

$$(1.1) \quad G^n + tF = 0$$

be a generic pencil of plane curves, with  $\deg G = d$ ,  $\deg F = nd$ . We will freely abuse notation by using the same letter to denote a polynomial or its zero locus. Here, generic means that  $G$ ,  $F$  are smooth, and meet transversely at their  $nd^2$  points of intersection, the base points of the pencil.  $G^*$  is assumed to have only nodes and cusps as singularities. The pencil (1.1) will be denoted by  $C_t$ . Let  $C_0^* = \lim_{t \rightarrow 0} C_t^*$ . The goal of this section is to prove the following.

(1.2) PROPOSITION.  $C_0^*$  is the union of  $G^*$  with multiplicity  $n$ , together with the  $nd^2$  pencils of lines through the base points, each pencil having multiplicity  $(n - 1)$ .

REMARKS. (1) Proposition (1.2) is quite elementary. It is not much more difficult than the case  $n = 2$ ,  $d = 1$  implicitly worked out in [4]. The

value in this method of proof lies purely in its expository value as a prelude to §2.

(2) By a standard formula for plane curves ([2], for example)  $\text{deg } C_t^* = nd(nd - 1)$  for  $t \neq 0$ , while  $\text{deg } C_0^* = nd(d - 1) + (n - 1)nd^2 = nd(nd - 1)$ .

The techniques used are a variant of the techniques of [3], which were inspired by the work of Clemens. Given a line  $L \subset \mathbf{P}^2$ , we look for a family of lines  $L_s$  with  $L_0 = L$  and  $L_s$  tangent to  $C_t$  with  $t = s^r$  for a positive integer  $r$ . Then  $L$  would correspond to a general point of a multiplicity  $r$  component of  $C_0^*$  with cyclic local monodromy.

We choose an isomorphism  $\alpha: \mathbf{P}^1 \rightarrow L$  given by three homogeneous linear forms  $\alpha = (\alpha_0(u, v), \alpha_1(u, v), \alpha_2(u, v))$ , where  $(u, v)$  are homogeneous coordinates on  $\mathbf{P}^1$ . We single out  $(1, 0) \in \mathbf{P}^1$  as the candidate for a point of tangency of  $L$  with  $C_0$ . We look for an extension of  $\alpha$  to  $\alpha(s)$ , holomorphic in  $s$  for  $|s| < \varepsilon$ , with  $\alpha(0) = \alpha$ , and satisfying

$$(1.3) \quad (G^n + s^r F) \circ \alpha(s) \equiv 0 \pmod{v^2} \quad \text{for } |s| < \varepsilon.$$

We attempt to solve (1.3) by power series in  $s$ . We show that this is possible when either  $L$  is tangent to  $G$  or when  $L$  passes through a base point. In the former case, for general  $L$ , we must take  $r = n$ , while in the latter case, we take  $r = n - 1$ . By consideration of degrees, i.e. Remark (2), no other components are present, proving Proposition (1.2).

We now fix some more notation. Let  $P_k$  denote the vector space of homogeneous forms of degree  $k$  on  $\mathbf{P}^1$ . There is a linear map

$$(1.4) \quad \Phi_G: P_1^3 \rightarrow P_d, \quad \Phi_G(\sigma_0, \sigma_1, \sigma_2) = \sum_{j=0}^2 \sigma_j \left( \frac{\partial G}{\partial X_j} \circ \alpha \right)$$

and for each integer  $k \geq 0$ , the related map

$$(1.5) \quad \Phi_G^{(k)}: P_1^3 \xrightarrow{\Phi_G} P_d \rightarrow P_d/(v^{k+1}).$$

(1.6) LEMMA. *For any  $L$ ,  $\Phi_G^{(1)}$  is surjective (hence also  $\Phi_G^{(0)}$ ).*

*Proof.* Since  $G$  is smooth, we may change coordinates so  $\psi = \partial G / \partial X_0 \circ \alpha \not\equiv 0 \pmod{v}$ , so that  $\psi$  is a unit in the graded ring  $R_1 = \bigoplus_j P_j/(v^2)$ . Thus any  $Q \in P_d/(v^2)$  can be divided by  $\psi \pmod{v^2}$  to yield  $\sigma \in P_1$ ; then  $\Phi_G^{(1)}(\sigma, 0, 0) = Q$ . □

We introduce some more notation to facilitate higher order computations. Let

$$\alpha^{(r)} = \left( \frac{d^r \alpha_i}{ds^r} \Big|_{s=0} \right)_{i=0,1,2}, \quad G_{ij} \alpha^{(r)} \alpha^{(s)} = \sum_{i,j} \left( \frac{\partial^2 G}{\partial X_i \partial X_j} \right) \alpha_i^{(r)} \alpha_j^{(s)}.$$

We also note that homogeneous polynomials of degree  $j$  in  $(u, v)$  can be viewed as polynomials of degree  $\leq j$  in  $v$ ; we will hence usually view  $P_j/(v^{k+1}) \subset \mathbf{C}[v]/(v^{k+1})$ , and speak of constant terms, linear terms, etc. We also freely divide truncated polynomials.

We start by specializing to the case  $n = 2$  to fix ideas.

(1.7) PROPOSITION. (1.2) is true for  $n = 2$ .

*Proof.* We set  $n = 2, r = 1$  (so that  $s = t$ ) in (1.3), and let  $t = 0$  to obtain

$$(1.8) \quad G^2 \equiv 0 (v^2)$$

where we have abused notation by viewing  $G$  as a form on  $\mathbf{P}^1$  via  $\alpha$ . This gives

$$(1.9) \quad G \equiv 0 (v).$$

We continue by differentiating (1.3) with respect to  $t$  and setting  $t = 0$ .

$$(1.10) \quad 2G\Phi_G \alpha' + F \equiv 0 (v^2)$$

Using (1.9), (1.10) forces  $F = 0 (v)$ , i.e.

$$(1.11) \quad L \text{ passes through a base point.}$$

To show that the pencil containing  $L$  indeed has multiplicity 1 in  $C_0^*$ , we may take  $L$  general, and so assume  $G$  is not tangent to  $L \simeq \mathbf{P}^1$  at  $(1, 0)$ . We then obtain from (1.10)

$$(1.12) \quad \Phi_G^{(0)} \alpha' = -F/2G.$$

and Lemma 1.6 implies that we can solve (1.12) for  $\alpha'$ . Thus the pencils through the base points deform to first order; these pencils are the only candidates for a multiplicity 1 component of  $C_0^*$ .

For the second order obstruction, we take the second derivative of (1.3) with respect to  $t$  and set  $t = 0$  to obtain

$$(1.13) \quad 2G\Phi_G \alpha'' + 2GG_{ij} \alpha' \alpha' + 2(\Phi_G \alpha')^2 + 2\Phi_F \alpha' \equiv 0 (v^2).$$

In order for (1.13) to have a solution for  $\alpha''$ , we must require that

$$(1.14) \quad 2(\Phi_G \alpha')^2 + 2\Phi_F(\alpha') \equiv 0(v).$$

This can be accomplished by the following lemma.

(1.15) LEMMA.  $\Phi_F^{(0)}|_{\ker \Phi_G^{(1)}}: \ker \Phi_G^{(1)} \rightarrow P_{nd}/(v)$  is surjective.

*Proof.* Since  $\dim P_{nd}/(v) = 1$ , the lemma can fail to hold only if  $\ker \Phi_G^{(1)} \subset \ker \Phi_F^{(0)}$ . But since  $F$  and  $G$  intersect transversally, we can change coordinates in  $\mathbf{P}^2$  so that  $X_0 = 0$  is tangent to  $F$ , and  $X_1 = 0$  is tangent to  $G$  at  $\alpha(1, 0)$ . So we may assume that, in the affine coordinate  $v$  near  $(1, 0) \in \mathbf{P}^1$ ,  $(\partial G/\partial X_0)(\alpha(v)) \equiv av(v^2)$ ,  $(\partial G/\partial X_1)(\alpha(v)) \equiv b(v)$ , where  $b \neq 0$ . Then  $(-bu, av, 0) \in \ker \Phi_G^{(1)} - \ker \Phi_F^{(0)}$ .

Now we can replace  $\alpha'$  with  $\alpha' - \tilde{\alpha}$ , where  $\tilde{\alpha} \in \ker \Phi_G^{(1)}$  and  $\Phi_F^{(0)}\tilde{\alpha} \equiv (\Phi_G \alpha')^2(v)$ , by the lemma. Then (1.12) still holds, but now the left-hand side of (1.13) is divisible by  $G$ , since (1.14) now holds. After dividing (1.13) by  $G$ , we can now solve for  $\alpha''$  by using lemma (1.6) again.

For simplicity, we introduce the symbol  $Q_j$  to stand for any expression involving  $\alpha$  only through  $\alpha', \alpha'', \dots, \alpha^{(j)}$ . The higher order obstructions are now handled by the following easily established lemma.

(1.16) LEMMA. For  $n \geq 2$ , the  $n$ th obstruction to (1.3) is

$$\begin{aligned} \frac{d^n}{dt^n}(G^2 + tF) \Big|_{t=0} &\equiv 2G\Phi_G \alpha^{(n)} + n\Phi_F \alpha^{(n-1)} \\ &+ 2n\Phi_G \alpha' \Phi_G \alpha^{(n-1)} + GQ_{n-1} + Q_{n-2} \equiv 0(v^2). \quad \square \end{aligned}$$

We inductively complete the power series solution of (1.3). We suppose that we have solved for  $\alpha', \dots, \alpha^{(n-1)}$ . Then using Lemma 1.15, we modify  $\alpha^{(n-1)}$  so that (1.16) becomes divisible by  $G$ . After dividing by  $G$ , we use Lemma (1.6) once more to solve for  $\alpha^{(n)}$ .

This procedure gives a formal power series solution of (1.3). By Artin's theorem [1] there is a holomorphic solution of (1.3) for  $|t| < \epsilon$ . Thus, the pencils through the base points are each multiplicity 1 components of  $C_0^*$ .

REMARK. The solution for  $\alpha^{(n)}$  is far from unique; in fact, the computation above shows that the ambiguity lies in  $\ker \Phi_G^{(0)} \cap \ker \Phi_F^{(0)}$ , a 4-dimensional vector space. Let  $B \subset GL(2)$  denote the isotropy group of

(1, 0), so that  $\dim B = 3$ . This is the ambiguity arising by representing  $L$  as  $(\mathbf{P}^1, (1, 0))$ . The difference between 4 and 3 reflects that a curve (the pencil) is deforming.

The other component  $2G^*$  is found by letting  $n = 2, t = s^2$  in (1.3). The order zero obstruction again leads to (1.9), which holds for a tangent to  $G$  (in fact,  $G \equiv 0 (v^2)$ ). The first order obstruction is

$$(1.17) \quad 2G\Phi_G\alpha' \equiv 0 (v^2)$$

which is again automatic, and puts no restrictions on  $\alpha'$ .

The second order obstruction is

$$(1.18) \quad 2G\Phi_G\alpha'' + 2(\Phi_G\alpha')^2 + 2GG_{ij}\alpha'\alpha' + 2F \equiv 0 (v^2).$$

This equation can be solved for  $\alpha''$  provided that

$$(1.19) \quad (\Phi_G\alpha')^2 \equiv -F (v^2).$$

We can assume that  $L$  does not pass through a base point (i.e.  $F \not\equiv 0 (v)$ ). After taking a square root, Lemma (1.6) ensures that we can find such an  $\alpha'$ , and (1.18) imposes no conditions on  $\alpha''$ . For the higher order obstructions we need an easy lemma.

(1.20) LEMMA. For  $n \geq 2$ , the  $n$ th obstruction is

$$\begin{aligned} \frac{d^n}{ds^n}(G^2 + s^2F) \Big|_{s=0} &\equiv 2G\Phi_G\alpha^{(n)} + 2n\Phi_G\alpha'\Phi_G\alpha^{(n-1)} \\ &+ GQ_{n-1} + Q_{n-2} \equiv 0 (v^2). \quad \square \end{aligned}$$

Since  $\Phi_G\alpha'$  is a unit in  $P_d/(v^2)$ , we can choose  $\alpha^{(n-1)}$  to ensure that there is no  $n$ th obstruction, using Lemma (1.6). Thus, there is a formal power series solution of (1.3) with  $t = s^2$ , and Artin's Theorem finishes the proof of Proposition (1.7).  $\square$

REMARK. In the case of tangents, the ambiguity lies in  $\ker \Phi_G^{(1)}$ , which is as before a 4-dimensional vector space.

*Proof of Proposition (1.2).* We start by letting  $t = s^{n-1}$  in (1.3), and attempt to deform a pencil through a base point. There are clearly no obstructions through order  $n - 2$ . The  $(n - 1)$ st obstruction is (since

$G^j \equiv 0 (v^2)$  for  $j \geq 2$ )

$$(1.21) \quad n!G(\Phi_G\alpha')^{n-1} + (n-1)!F \equiv 0 (v^2).$$

We may assume  $L$  is not tangent to  $G$  or  $F$ ; then we can solve (1.21) for  $\alpha'$ .

The  $n$ th order obstruction is seen to be

$$(1.22) \quad n! \binom{n}{2} G(\Phi_G\alpha')^{n-2} \Phi_G\alpha'' + n!(\Phi_G\alpha')^n + n!\Phi_F\alpha' \\ + GQ_n \equiv 0 (v^2).$$

We now can use Lemma (1.15) to modify  $\alpha'$  so that (1.22) is consistent. After dividing (1.22) by  $G$ , and noting that  $\Phi_G^{(1)}\alpha'$  is a unit, we can then solve for  $\alpha''$ .

For the higher order obstructions, we note that for  $r \geq n+1$

$$(1.23) \quad \left. \frac{d^r}{ds^r} (G^n + s^{n-1}F) \right|_{s=0} \\ \equiv n(n-1) \frac{r!}{(r-n+2)!} G(\Phi_G\alpha')^{n-2} \Phi_G\alpha^{(r-n+2)} \\ + n \frac{r!}{(r-n+2)!} (\Phi_G\alpha')^{n-1} \Phi_G\alpha^{(r-n+1)} \\ + \frac{r!}{(r-n+1)!} \Phi_F\alpha^{(r-n+1)} + GQ_{r-n+1} + Q_{r-n} \equiv 0 (v^2).$$

As before, we can use Lemma (1.15) inductively to modify  $\alpha^{(r-n+1)}$  to ensure the consistency of (1.23), then solve for  $\alpha^{(r-n+2)}$  using Lemma (1.6). Finally, Artin's Theorem shows that a pencil through a base point is a multiplicity  $(n-1)$  component of  $C_0^*$ .

Turning next to the tangents to  $G$  (so that  $G \equiv 0 (v^2)$ ), we let  $t = s^n$  in (1.3). There are clearly no obstructions through order  $(n-1)$ .

The  $n$ th order obstruction yields

$$(1.24) \quad n!(\Phi_G\alpha')^n + n!F \equiv 0 (v^2).$$

Assuming that  $L$  does not pass through a base point, we can solve (1.24) for  $\alpha'$ .

For the higher order obstructions, we note that for  $r \geq n+1$

$$(1.25) \quad \left. \frac{d^r}{ds^r} (G^n + s^n F) \right|_{s=0} \equiv n \frac{r!}{(r-n+1)!} (\Phi_G\alpha')^{n-1} \Phi_G\alpha^{(r-n+1)} \\ + Q_{r-n} \equiv 0 (v^2).$$

As  $\Phi_G \alpha'$  is a unit, we can solve for  $\alpha^{(r-n+1)}$ . Artin's Theorem completes the proof.  $\square$

**2. Flexes on a multiple curve.** In the situation of §1, we look at the limiting behavior of the flexes of  $C_t$ .

(2.1) **PROPOSITION.** *The flexes of  $C_t$  degenerate to the flexes of  $G$ , the tangents to  $F$  at a base point, and the tangents to  $G$  at a base point, with multiplicities  $n, n - 2, 2n - 1$  respectively.*

*Proof.* By a standard formula for plane curves [2],  $C_t$  has  $3nd(nd - 2)$  flexes;  $g$  has  $3d(d - 2)$  flexes and  $nd^2$  base points. Also  $3nd(nd - 2) = n(3d(d - 2)) + (n - 2)nd^2 + (2n - 1)nd^2$ . So as in §1, it suffices to construct deformations of the claimed limits with the indicated multiplicities.

We now need to solve

$$(2.2) \quad (G^n + s^r F) \circ \alpha(s) \equiv 0 \pmod{v^3} \quad \text{for } |s| < \epsilon$$

for  $r = n$  in the case of a flex of  $G$ , for  $r = n - 2$  in the case of a tangent to  $F$  at a base point, and for  $r = 2n - 1$  in the case of a tangent to  $G$  at a base point.

We first check the flexes of  $G$ , starting with a lemma.

(2.3) **LEMMA.** *If  $L$  is an ordinary inflectional tangent to  $G$ , then  $\Phi_G^{(2)}$  is surjective.*

*Proof.* We can change coordinates so that  $L$  has equation  $X_1 = 0$ , and  $G$  has an equation of the form  $X_1 f + X_0^3 g$ , where  $f(0, 0, 1), g(0, 0, 1) \neq 0$ . We may as well let  $\alpha: \mathbf{P}^1 \rightarrow L$  be  $\alpha(u, v) = (v, 0, u)$ . Then, using subscript notation for partial derivatives, we find that

$$G_0 \circ \alpha = 3v^2 g + v^3 g_0 \quad G_1 \circ \alpha = f + v^3 g_1$$

and so  $\Phi_G^{(2)}$  is surjective by inspection.  $\square$

The proof of the case of flexes is now completed by mimicking the computation of the component  $nG^*$  of §1, using Lemma (2.3) in place of Lemma (1.6).

We turn next to the case of a tangent to  $F$  at a base point, i.e.  $G \equiv 0(v), F \equiv 0(v^2), t = s^{n-2}$ .

There are clearly no obstructions through order  $n - 3$ . For the order  $n - 2$  obstruction, we note that

$$(2.4) \quad \left. \frac{d^{n-2}}{ds^{n-2}}(G^n + s^{n-2}F) \right|_{s=0} \equiv \frac{n!}{2} G^2(\Phi_G \alpha')^{n-2} + (n-2)!F \equiv 0 \pmod{v^3}$$

and since  $F, G$  have order exactly 2, 1 respectively as polynomials in  $v$ ,  $F/G^2$  is a unit, so we can extract an  $(n - 2)$  root and solve for  $\Phi_G^{(0)}\alpha'$  in (2.4).

The higher order obstructions are given by

$$(2.5) \quad \begin{aligned} & \left. \frac{d^k}{ds^k} (G^n + s^{n-2}F) \right|_{s=0} \\ &= \frac{n(n-1)(n-2)}{2} \frac{k!}{(k+3-n)!} G^2(\Phi_G \alpha')^{n-3} \Phi_G \alpha^{(k+3-n)} \\ & \quad + G^2 Q_{k+2-n} + n(n-1) \frac{k!}{(k+2-n)!} G(\Phi_G \alpha')^{n-2} \Phi_G \alpha^{(k+2-n)} \\ & \quad + \frac{k!}{(k+2-n)!} \Phi_F \alpha^{(k+2-n)} + Q_{k+1-n} \pmod{v^3}. \end{aligned}$$

This case is finished by a couple of lemmas.

(2.6) LEMMA.  $\Phi_F^{(1)}|_{\ker \Phi_G^{(0)}}: \ker \Phi_G^{(0)} \rightarrow P_{nd}/(v^2)$  is surjective.

*Proof.* Lemma 1.15 says that  $\dim \ker \Phi_G^{(1)} \cap \ker \Phi_F^{(0)} = 3$ . Reversing the roles of  $F$  and  $G$  yields the lemma. □

(2.7) LEMMA. *After solving for the  $k$ th obstruction, we have  $\infty^3$  solutions for  $\alpha^1, \dots, \alpha^{(k+2-n)}$ , and  $\Phi_G^{(0)}\alpha^{(k+3-n)}$  is determined.*

*Proof.* Inductively, we equate the linear plus constant term of (2.5) to 0  $\pmod{v^2}$ , using Lemma (2.6) to modify  $\alpha^{(k+2-n)}$ .  $\Phi_G^{(0)}\alpha^{(k+3-n)}$  is now found by Lemma (1.6). □

An application of Artin’s Theorem completes the proof of the case of a tangent to  $F$  at a base point.

Finally, we turn to a tangent to  $G$  at a base point, i.e.  $G \equiv 0 \pmod{v^2}$ ,  $F \equiv 0 \pmod{v}$ ,  $t = s^{2n-1}$ .

There are clearly no obstructions through order  $n - 2$ . The order  $n - 1$  obstruction is

$$(2.8) \quad \left. \frac{d^{n-1}}{ds^{n-1}}(G^n + s^{2n-1}F) \right|_{s=0} \equiv n!G(\Phi_G \alpha')^{n-1} \equiv 0 \pmod{v^3}$$

which forces

$$(2.9) \quad \Phi_G^{(0)}\alpha' = 0.$$

We change notation slightly, putting  $G^{(j)} = d^j(G \circ \alpha(s))/ds^j|_{s=0}$ , noting that  $G^{(j)} = \Phi_G \alpha^{(j)} + Q_{j-1}$ . With the additional information (2.9), we now see that there are no obstructions through order  $2n - 3$ . The order  $2n - 2$  obstruction is given by

$$(2.10) \quad \begin{aligned} & \frac{d^{2n-2}}{ds^{2n-2}}(G^n + s^{2n-1}F) \Big|_{s=0} \\ &= \frac{n(2n-2)!}{2^{n-1}} G(G'')^{n-1} + \frac{n(n-1)(2n-2)!}{2^{n-1}} (G')^2 (G'')^{n-2} \\ &\equiv 0 \pmod{v^3}. \end{aligned}$$

This leads to

$$(2.11) \quad G'' \equiv -(n-1)(G')^2/G \pmod{v}.$$

The order  $2n - 1$  obstruction is

$$(2.12) \quad \begin{aligned} & \frac{d^{2n-1}}{ds^{2n-1}}(G^n + s^{2n-1}F) \Big|_{s=0} \\ &= \frac{n(n-1)(2n-1)!}{6 \cdot 2^{n-2}} G(G'')^{n-2} G''' \\ & \quad + \frac{n(n-1)(n-2)(2n-1)!}{6 \cdot 2^{n-2}} (G')^2 (G'')^{n-3} G''' \\ & \quad + \frac{n(2n-1)!}{2^{n-1}} G'(G'')^{n-1} + (2n-1)!F \\ &\equiv 0 \pmod{v^3} \end{aligned}$$

looking at the linear term, and using (2.11), we find

$$(2.13) \quad (G')^{2n-1}/G^{n-1} \equiv (-1)^n \frac{2^{n-1}}{n(n-1)^{n-1}} F \pmod{v^2}.$$

(2.12) implies that we can solve for  $\Phi_G^{(1)}\alpha'$ , and that  $G''$  is a unit, using (2.11) again.

Turning to the quadratic term of (2.12), we see that we must solve for  $G'''(v)$ , or equivalently, for  $\Phi_G^{(0)}\alpha'''$ . This is possible exactly when the expression multiplying  $G'''$  in (2.12) is divisible by  $v^2$ , but not by  $v^3$ . But this expression is a multiple of

$$(2.14) \quad (G'')^{n-3} [GG'' + (n-2)(G')^2]$$

which satisfies the indicated requirement, by (2.11) and the fact that  $G''$  is a unit.

Notice that  $\Phi_G^{(0)}\alpha''$  depends only on  $\alpha'$ , while  $\Phi_G^{(0)}\alpha'''$  depends on  $\Phi_G^{(1)}\alpha''$  and  $\alpha'$ ; however, it is a non-trivial linear expression in the linear term of  $\Phi_G^{(1)}\alpha''$ , as revealed by an examination of our solution of (2.12).

The higher order obstructions are given by

$$\begin{aligned}
 (2.15) \quad & \left. \frac{d^k}{ds^k} (G^n + s^{2n-1}F) \right|_{s=0} \\
 &= \frac{n(n-1)k!}{2^{n-2}(k+4-2n)!} G(G'')^{n-2} G^{(k+4-2n)} \\
 &+ \frac{n(n-1)(n-2)k!}{2^{n-2}(k+4-2n)!} (G')^2 (G'')^{n-3} G^{(k+4-2n)} + GQ_{k+3-2n} \\
 &+ (G')^2 \tilde{Q}_{k+3-2n} + \frac{n(n-1)k!}{2^{n-2}(k+3-2n)!} G'(G'')^{n-2} G^{(k+3-2n)} \\
 &+ G'Q_{k+2-2n} + \frac{nk!}{2^{n-1}(k+2-2n)!} (G'')^{n-1} G^{(k+2-2n)} \\
 &+ Q_{k+1-2n} + \frac{k!}{(k+1-2n)!} \Phi_F(\alpha^{(k+1-2n)}) \\
 &\equiv 0 \pmod{v^3}.
 \end{aligned}$$

Equation (2.15) can be solved inductively.

(2.16) LEMMA. *After solving for the  $k$ th obstruction, we have  $\infty^3$  solutions for  $\alpha', \dots, \alpha^{(k+1-2n)}$ , we have found  $\Phi_G^{(1)}\alpha^{(k+2-2n)}$ , and we have found  $\Phi_G^{(0)}\alpha^{(k+4-2n)}$ . This last depends non-trivially and linearly on the linear term of  $\Phi_G\alpha^{(k+3-2n)}$ , and on terms of lower order.*

*Proof.* By induction. We start by examining the constant term of (2.15). We observe that the constant term of  $G^{(k+2-2n)}$  depends on  $\Phi_G^{(1)}\alpha^{(k+1-2n)}$  and lower derivatives of  $\alpha$ . Also we note that the expression  $Q_{k+1-2n}$  in (2.15) depends on  $\Phi_G^{(0)}\alpha^{(k+1-2n)}$  and lower derivatives of  $\alpha$ . So Lemma (1.15) applies to allow for the modification of  $\alpha^{(k+1-2n)}$  as before.

Next, we consider the linear term of (2.15). We observe that the constant term of  $Q_{k+2-2n}$  depends on  $\Phi_G^{(0)}\alpha^{(k+2-2n)}$  and lower derivatives of  $\alpha$ , while inductively the constant term of  $G^{(k+3-2n)}$  depends non-trivially and linearly on the linear term of  $\Phi_G\alpha^{(k+2-2n)}$  and on lower order

terms, so that after equating the linear term of (2.15) to 0, we can first solve for the linear term of  $\Phi_G \alpha^{(k+2-2n)}$  (hence for  $\Phi_G^{(1)} \alpha^{(k+2-2n)}$ , as we inductively know the constant term). Lemma (1.16) allows us to solve for  $\alpha^{(k+2-2n)}$ .

Finally, we turn to the quadratic term. Exactly as in the order  $2n - 1$  obstruction, we see that  $\Phi_G^{(0)} \alpha^{(k+4-2n)}$  is multiplied by a constant multiple of (2.14), which we have seen is divisible by  $v^2$ , but not by  $v^3$ . So we can solve for  $\Phi_G^{(0)} \alpha^{(k+4-2n)}$ , and apply Lemma (1.6). Note that the quadratic term of (2.15) involves  $\alpha^{(k+3-2n)}$  only non-trivially and linearly through the linear term of  $\Phi_G \alpha^{(k+3-2n)}$ , completing the induction.  $\square$

An application of Artin's Theorem now finishes the case of tangents to  $G$  through a base point, as well as the proof of Proposition (2.1).  $\square$

#### REFERENCES

- [1] M. Artin, *On the solution of analytic equations*, Invent. Math., **5** (1968), 277–291.
- [2] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, N.Y. 1978.
- [3] S. Katz, *Degeneration of quintic threefolds and their lines*, Duke Math. J., **50** (1983), 1127–1135.
- [4] H. Schubert, *Kalkül der Abzählenden Geometrie*, Springer-Verlag, Berlin, 1979.

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# Pacific Journal of Mathematics

Vol. 124, No. 2

June, 1986

<b>Philip Lee Bowers</b> , Nonshrinkable “cell-like” decompositions of $s$ .....	257
<b>Aurelio Carboni and Ross Street</b> , Order ideals in categories .....	275
<b>Leoni Dalla</b> , Increasing paths on the one-skeleton of a convex compact set in a normed space .....	289
<b>Jim Hoste</b> , A polynomial invariant of knots and links .....	295
<b>Sheldon Katz</b> , Tangents to a multiple plane curve .....	321
<b>Thomas George Lucas</b> , Some results on Prüfer rings .....	333
<b>Pham Anh Minh</b> , Modular invariant theory and cohomology algebras of extra-special $p$ -groups .....	345
<b>Ikuko Miyamoto</b> , On inclusion relations for absolute Nörlund summability .....	365
<b>A. Papadopoulos</b> , Geometric intersection functions and Hamiltonian flows on the space of measured foliations on a surface .....	375
<b>Richard Dean Resco, J. Toby Stafford and Robert Breckenridge Warfield, Jr.</b> , Fully bounded $G$ -rings .....	403
<b>Haskell Paul Rosenthal</b> , Functional Hilbertian sums .....	417
<b>Luen-Fai Tam</b> , Regularity of capillary surfaces over domains with corners: borderline case .....	469
<b>Hugh C. Williams</b> , The spacing of the minima in certain cubic lattices .....	483