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**MODULAR INVARIANT THEORY AND COHOMOLOGY
ALGEBRAS OF EXTRA-SPECIAL p -GROUPS**

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Let W_n be the group of all translations on the vector space \mathbf{Z}_p^{n-1} . Every element of W_n is considered as a linear transformation on \mathbf{Z}_p^n , i.e. W_n is identified to a subgroup of $\text{GL}(n, \mathbf{Z}_p)$. We have then a natural action of W_n on $E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$. The purpose of this paper is to determine a full system of invariants of W_n in this algebra. Using this result, we determine the image $\text{Im Res}(A, G)$, for every maximal elementary abelian p -subgroup A of an extra-special p -group G .

Introduction. Let G be a finite group and \mathbf{Z}_p be the prime field of p elements. Let us write $H^*(G) = H^*(G, \mathbf{Z}_p)$ (the mod p cohomology algebra of G).

If $p = 2$, the cohomology algebras of all extra-special p -groups were determined by Quillen [7]. We are interested in the case $p > 2$. So from now on, we shall assume this condition through the paper. For the extra-special p -groups of order p^3 , their integral cohomology rings have been computed by Lewis in [3], and their mod p cohomology algebras are determined recently in Phạm Anh Minh-Huỳnh Mùi [4] and Huỳnh Mùi [6]. For an arbitrary extra-special p -group, Tezuka and Yagita had computed $H^*(G)/\sqrt{0}$ in [9]. As observed in [6], the ideal $\sqrt{0}$ of the nilpotents in this algebra is quite complicated, so it seems difficult to determine their nilpotent elements.

Let A be a maximal elementary abelian p -subgroup of an extra-special p -group G . The inclusion map $A \hookrightarrow G$ induces the restriction homomorphism $\text{Res}(A, G): H^*(G) \rightarrow H^*(A)^{W_G(A)}$, where $W_G(A) = N_G(A)/C_G(A)$, the quotient of the normalizer by the centralizer of A in G . The purpose of this paper is to determine the image $\text{Im Res}(A, G)$ for every A . We shall see that the nilideal of $\text{Im Res}(A, G)$ is complicated, so our results will be needed in the study of the ideal $\sqrt{0}$ of $H^*(G)$.

This paper contains 3 sections. In §1, we consider maximal elementary abelian p -subgroups of an extra-special p -group following Quillen [7] and Tezuka-Yagita [9]. By means of the modular invariant theory developed by Huỳnh Mùi [5], we determine in §2 a full system for the

invariants of $W_G(A)$ in $H^*(A)$. Using the results in §2, we determine $\text{Im Res}(A, G)$ in §3. The main results of this paper are Theorem 2.4 and Theorem 3.1.

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1. Extra-special p -groups and maximal elementary abelian p -subgroups. Let G be a p -group. As usual, let $[G, G]$, $Z(G)$ $\Phi(G) = G^p \cdot [G, G]$ denote the commutator subgroup, the center and the Frattini group of G respectively. G is called an extra-special p -group if it satisfies the following condition

$$(1.1) \quad [G, G] = \Phi(G) = Z(G) \cong \mathbf{Z}_p.$$

Equivalently, G is an extra-special p -group if we have the group extension

$$(1.2) \quad 0 \rightarrow \mathbf{Z}_p \xrightarrow{i} G \xrightarrow{\pi} V \rightarrow 0$$

where V is a vector space of finite dimension over \mathbf{Z}_p and i is an isomorphism from \mathbf{Z}_p onto the center of G . (For details on extra-special p -groups see D. Gorenstein, *Finite Groups*, Harper & Row, New York, 1968, especially §5.5.)

As well known, the dimension of $V \cong G/Z(G)$ is even. If $\dim V = 2$, G is isomorphic to one of the following groups

$$E = \langle a, b \mid a^p = b^p = [a, b]^p = [a, [a, b]] = [b, [a, b]] = 1 \rangle,$$

$$M = \langle a, b \mid a^{p^2} = b^p = 1, b^{-1} \cdot ab = a^{1+p} \rangle.$$

Generally, if $\dim V = 2n - 2$ ($n \geq 2$), then G is isomorphic to one of the following central products

$$(1.3) \quad \begin{aligned} E_{n-1} &= E \cdot \cdots \cdot E && (n-1 \text{ times}) \\ M_{n-1} &= E_{n-2} \cdot M. \end{aligned}$$

Let $B: G/Z(G) \times G/Z(G) \rightarrow [G, G]$ be the map defined by

$$B(u, v) = [u', v'] \quad \text{for } u, v \in G/Z(G)$$

where u', v' mean representatives of u and v respectively. One can easily see that B is well-defined. Identifying $G/Z(G) = V = \mathbf{Z}_p^{2n-2}$ and $[G, G] = \mathbf{Z}_p$, B becomes the alternating form $V \times V \rightarrow \mathbf{Z}_p$ defined by

$$(1.4) \quad B(u, v) = \sum_{i=1}^{n-1} u_{2i-1} \cdot v_{2i} - u_{2i} \cdot v_{2i-1}$$

for

$$u = (u_1, \dots, u_{2n-2}), \quad v = (v_1, \dots, v_{2n-2}) \in V.$$

A subspace W of V is said to be B -isotropic if $B(u, v) = 0$ for all $u, v \in W$.

In Quillen [7; §4] and Tezuka-Yagita [9; 1.7 and 3.4], we have

LEMMA 1.5. *There is a 1-1 correspondence between maximal abelian p -subgroups A of G and maximal B -isotropic subspaces W of V . The dimension of any maximal B -isotropic subspaces W of V is just $n - 1$.*

From this lemma, we have

LEMMA 1.6. *Any maximal elementary abelian p -subgroup A of G is of rank n , i.e. $A \cong \mathbf{Z}_p^n$.*

Proof. It suffices to prove that A is also a maximal abelian subgroup of G , and the result is implied from (1.5). Assume that A is not a maximal abelian subgroup of G , then $A \not\subseteq A'$, where A' is a maximal abelian subgroup but not elementary of G . Let $a \in A'$ with $\text{ord}(a) = p^2$. Let $\Omega_1(G)$, $\mathfrak{U}_1(G)$ denote the subgroups of G defined by $\Omega_1(G) = \{x \in G / \text{ord}(x) \leq p\}$ and $\mathfrak{U}_1(G) = \{y^p | y \in G\}$. Since $|\mathfrak{U}_1(G)| = p$, we have $|\Omega_1(G)| = p^{2n-2}$ and $\Omega_1(G)$ is not an extra-special p -group. Hence $Z(\Omega_1(G)) \not\subseteq Z(G)$. Let b be an element of $Z(\Omega_1(G)) \setminus Z(G)$, we have $[b, a] \neq 1$, hence $b \notin A$ and $\langle A, b \rangle$ is then an elementary abelian p -subgroup of G which contains strictly A , a contradiction. The lemma is proved.

PROPOSITION 1.7. *Let A be a maximal elementary abelian p -subgroup of G . Then there exist the elements $a_1, \dots, a_n, b_1, \dots, b_{n-1}$ of G such that*

- (a) $A = \langle a_1, \dots, a_n \rangle$ and $a_n = c$ is a generator of $Z(G)$
- (b) $W_G(A) = \langle \underline{b}_1, \dots, \underline{b}_{n-1} \rangle$ where $\underline{b}_i = b_i A, 1 \leq i \leq n - 1$
- (c) $a_i^{b_j} = a_i$ if $i \neq j, a_i \cdot a_n$ if $i = j$ for $1 \leq i, j \leq n - 1$.

Proof. It suffices to prove that: (*) there exist the elements $a_1, \dots, a_n, b_1, \dots, b_{n-1}$ of G satisfying the conditions:

- (a') $A = \langle a_1, \dots, a_n \rangle$, where $a_n = c$,
- (b') for each $i, 1 \leq i \leq n - 1, \langle a_i, b_i \rangle$ is an extra-special p -subgroup of G of order p^3 ,

(c') $[b_i, a_j] = 1$ if $i \neq j$, and the proposition can be obtained by noting that $W_G(A) = G/A$ and $a_i \in C_G(\langle a_j, b_j \rangle)$ if $i \neq j$.

First, let $c_1, \dots, c_{n-1}, c_n = c$, be a basis of A . Clearly, for $1 \leq i \leq n - 1$, $c_i \in G \setminus Z(G)$, so there exists an element d_i of G such that $[c_i, d_i] \neq 1$. Hence $E_i = \langle c_i, d_i \rangle \supset Z(G) = \Phi(G)$ and $\langle c_i \Phi(G), d_i \Phi(G) \rangle$ is a subgroup of $G/\Phi(G)$ of order less than p^2 . Then $|E_i| \leq p^3$. Since E_i is not abelian, we have $|E_i| = p^3$. Thus E_i is an extra-special p -group of order p^3 . By [8, 4.17 Chap. 4], we have $G = E_i \cdot C_G(E_i)$.

Since $G = E_1 \cdot C_G(E_1)$, each c_i ($i \neq n$) has the form

$$c_i = c_1^{r_i} \cdot d_1^{s_i} \cdot a_i^{(1)}$$

with $0 \leq r_i, s_i \leq p - 1$ and $a_i^{(1)} \in C_G(E_1)$. Since $[c_i, c_1] = 1$, s_i is then equal zero. Set $a_1^{(1)} = c_1, b_1^{(1)} = d_1$. We have $A = \langle a_1^{(1)}, \dots, a_{n-1}^{(1)}, c \rangle$ and there exist the elements $b_2^{(1)}, \dots, b_{n-1}^{(1)}$ of G such that $\langle a_i^{(1)}, b_i^{(1)} \rangle$ is an extra special p -group of order p^3 , and $[b_1^{(1)}, a_i^{(1)}] = 1$ for $i \neq 1$.

Assume that there exists the elements $a_1^{(k)}, \dots, a_{n-1}^{(k)}, b_1^{(k)}, \dots, b_{n-1}^{(k)}$ ($1 \leq k < n - 1$) of G such that

- (i) $A = \langle a_1^{(k)}, \dots, a_{n-1}^{(k)}, c \rangle$,
- (ii) $\langle a_i^{(k)}, b_i^{(k)} \rangle$ is an extra-special p -group of order p^3 ,
- (iii) $[b_j^{(k)}, a_i^{(k)}] = 1$ for $i \neq j$ and $j \leq k$.

For $i \neq k + 1$, $a_i^{(k)}$ has the form $a_i^{(k)} = a_{k+1}^{(k)m_i} \cdot a_i^{(k+1)}$ with $0 \leq m_i < p$ and $a_i^{(k+1)} \in C_G(\langle a_{k+1}^{(k)}, b_{k+1}^{(k)} \rangle)$. Set $a_{k+1}^{(k+1)} = a_{k+1}^{(k)}, b_j^{(k+1)} = b_j^{(k)}$ for $j \leq k + 1$. Let $b_i^{(k+1)}$ ($k - 2 \leq i \leq n - 1$) be the elements of G such that $\langle a_i^{(k+1)}, b_i^{(k+1)} \rangle$ is an extra-special p -group of order p^3 . We have then

- (i) $A = \langle a_1^{(k+1)}, \dots, a_{n-1}^{(k+1)}, c \rangle$,
- (ii) $\langle a_i^{(k+1)}, b_i^{(k+1)} \rangle$ is an extra-special p -group of order p^3 , for $i \neq n$,
- (iii) $[b_j^{(k+1)}, a_i^{(k+1)}] = 1$ for $j \neq i$ and $j \leq k + 1$. Finally, put $a_i = a_i^{(n-1)}, b_i = b_i^{(n-1)}, 1 \leq i \leq n - 1$. We obtain (*). The proposition is then proved.

(1.8) From now on, suppose that we are given a maximal elementary abelian p -subgroup A of G . Let us identify A with the vector space \mathbf{Z}_p^n by the correspondence

$$a_i \mapsto e_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} < i,$$

where a_1, \dots, a_n satisfy (1.7a). Then $W_G(a)$ is the group

$$W_G(A) = \left\{ \begin{bmatrix} 1 & & & & \\ & 1 & & & 0 \\ & 0 & \dots & & \\ * & * & \dots & * & 1 \end{bmatrix} \in \text{GL}(n, \mathbf{Z}_p) \right\}.$$

Let $x_1, \dots, x_n \in H^1(A) = \text{Hom}(A, \mathbb{Z}_p)$ be the duals of c_1, \dots, c_n . Let $y_i = \beta x_i$, where β denotes the Bockstein operator. As it is well known, we have

$$H^*(A) = E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$$

where $E(x_1, \dots, x_n; 1)$ (resp. $P(y_1, \dots, y_n; 2)$) denotes the exterior (resp. polynomial) algebra of n generators x_1, \dots, x_n (resp. y_1, \dots, y_n) of order 1 (resp. 2) over \mathbb{Z}_p .

As in Huynh Mui [3, Chap. 2, §1], we have

$$(1.9) \quad (H^*(A))^{W_G(A)} = (E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2))^{W_n}$$

where W_n is the subgroup of $\text{GL}(n, \mathbb{Z}_p)$ given by

$$W_n = \left\{ \begin{bmatrix} 1 & & & * \\ & 1 & & * \\ & & \ddots & \vdots \\ 0 & & & 1 \\ & & & & * \\ & & & & & 1 \end{bmatrix} \in \text{GL}(n, \mathbb{Z}_p) \right\}$$

and $(E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2))^{W_n}$ denotes the invariants of W_n in $E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$.

2. A full system for the invariants of W_n in $E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$. We shall determine a full system for the invariants $(E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2))^{W_n}$ by use of Huynh Mui's invariants in [5].

Let $1 \leq k \leq n$ be an integer. Following Huynh Mui [5], we let

$$(2.1) \quad V_k = \prod_{\lambda_i \in \mathbb{Z}_p} (\lambda_1 y_1 + \lambda_2 y_2 + \dots + \lambda_{k-1} y_{k-1} + y_k).$$

Let (s_1, \dots, s_k) be a sequence of integers with $0 \leq s_1 < \dots < s_k < n$. For $1 \leq i \leq k$, define

$$(2.2) \quad M_{n,s_i} = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ \dots & \dots & \dots & \dots \\ y_1^{p^{s_{i-1}}} & y_2^{p^{s_{i-1}}} & \dots & y_n^{p^{s_{i-1}}} \\ y_1^{p^{s_i+1}} & y_2^{p^{s_i+1}} & \dots & y_n^{p^{s_i+1}} \\ \dots & \dots & \dots & \dots \\ y_1^{p^{n-1}} & y_2^{p^{n-1}} & \dots & y_n^{p^{n-1}} \end{vmatrix}.$$

As in [5, Prop. I4.5], the product $M_{n,s_1} \cdot M_{n,s_2} \cdot \dots \cdot M_{n,s_k}$ has the factor L_n^{k-1} . Here

$$L_n = V_1 \cdot V_2 \cdot \dots \cdot V_n$$

is Dickson's invariant (see e.g. [5]). Hence we have Huÿmh Mui's invariants

$$(2.3) \quad \begin{aligned} M_{n,s_1,s_2,\dots,s_k} &= M_{n,s_1,\dots,s_k}(x_1, \dots, x_n, y_1, \dots, y_n) \\ &= (-1)^{k(k-1)/2} M_{n,s_1} \dots M_{n,s_k} / L_n^{k-1}. \end{aligned}$$

We have the following theorem

THEOREM 2.4. *There is a direct sum decomposition of modules*

$$\begin{aligned} &(E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2))^{W_n} \\ &= E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n) \\ &\quad \oplus \sum_{k=1}^n \oplus \sum_{0 \leq s_1 < \dots < s_k = n-1} \oplus M_{n,s_1,\dots,s_k} P(y_1, \dots, y_{n-1}, V_n). \end{aligned}$$

Therefore the invariants $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, V_n, M_{n,s_1,\dots,s_k}, 1 \leq k \leq n, 0 \leq s_1 < \dots < s_k = n - 1$ form a full system for the invariants of W_n in

$$E(x_1, x_2, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2).$$

Let $1 \leq k \leq n$ and let $W_{n,k}$ denote the subgroup of $GL(n, \mathbb{Z}_p)$ consisting of all elements

$$\left[\begin{array}{cccc} \boxed{\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{array}} & & & \\ & 0 & 1 & \\ & & & \ddots \\ & & & & 1 \end{array} \right] \left. \vphantom{\begin{array}{c} \boxed{\dots} \\ \dots \\ \dots \\ \dots \end{array}} \right\} k \text{ lines} \in GL(n, \mathbb{Z}_p).$$

Particularly, $W_{n,n-1} = W_n$ and $W_{n,1} = GL_{n,p}$.

As a corollary of Theorem 2.4, we have

COROLLARY 2.6.

$$\begin{aligned} &(E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2))^{W_{n,k}} \\ &= E(x_1, \dots, x_k) \otimes P(y_1, \dots, y_k, V_{k+1}, \dots, V_n) \\ &\quad \oplus \sum_{l=1}^n \oplus \sum_{s=k+1}^n \oplus \sum_{0 \leq s_1 < \dots < s_l = s-1} \oplus M_{s,s_1,\dots,s_l} P(y_1, \dots, y_k, V_{k+1}, \dots, V_n). \end{aligned}$$

(2.8) For later use, we need some notations. Consider $V_n = V_n(y_1, \dots, y_n)$, we set

$$V'_n = V_n(y_2, \dots, y_n, y_1)$$

$$V''_{n-1} = V_{n-1}(y_2, \dots, y_{n-1}, y_n).$$

Let $0 \leq s \leq n$ be an integer. Then we have inductively the Dickson invariants

$$Q_{n,0} = (V_1 \cdot \dots \cdot V_n)^{p-1}$$

$$Q_{n,s} = Q_{n-1,s} \cdot V_n^{p-1} + Q_{n-1,s-1}^p, \quad 0 < s \leq n$$

where $Q_{s,s} = 1$. By a similar way as in 2.8, we set

$$Q'_{n-1,s} = Q_{n-1,s}(y_2, \dots, y_n)$$

and

$$M'_{n-1,s_1, \dots, s_k} = M_{n-1,s_1, \dots, s_k}(x_2, \dots, x_n; y_2, \dots, y_n).$$

Let $I = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$ be a subset of $\{1, \dots, n\}$. We set

$$x_I = x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_k}.$$

Further, we denote

$$W'_{n-1} = \left\{ \begin{bmatrix} 1 & & & 0 \\ & 1 & & * \\ & & \ddots & \vdots \\ & 0 & & 1 \\ & & & & 1 \end{bmatrix} \in \text{GL}(n, \mathbf{Z}_p) \right\}.$$

LEMMA 2.9. Let $1 \leq k \leq n$ and let f be an element of

$$E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$$

having the form

$$f = \sum_I x_I f_I(y_1, \dots, y_n)$$

where I runs over the subsets of order k in $\{1, \dots, n\}$. If f is an invariant of W_n , then

(a) f_I is an invariant of W_n , for all I such that $n \in I$. Furthermore, if $k = 1$, then $f_{\{n\}}$ contains L_{n-1} as a factor.

(b) If $f_I = 0$ for all I such that $n \in I$, then f_I is an invariant of W_n , for all I .

Proof. Let $\omega = (\omega_{ij})$ be an element of W_n , we have

$$\omega x_i = \begin{cases} x_i & 1 \leq i < n, \\ \omega_{1n}x_1 + \cdots + \omega_{n-1n}x_{n-1} + x_n, & i = n. \end{cases}$$

Then f has the form

$$f = \sum_{I \neq j} x_I (\omega f_I) + x_j (\omega f_j).$$

This implies that $\omega f_j = f_j$, hence f_j is an invariant of W_n .

For the case $k = 1$, let $1 \leq m \leq n - 1$ be an integer and $\omega = 1 + \lambda_1 \varepsilon_{1n} + \cdots + \lambda_{m-1} \varepsilon_{m-1n} + \varepsilon_m$ be an element of W_n , where $\lambda_i \in \mathbb{Z}_p$ and ε_{ij} denote the matrix with 1 in the (i, j) -position and 0 elsewhere. By comparing the coefficients of x_m , we have

$$\begin{aligned} & f_m(y_1, \dots, y_{n-1}, y_n + \lambda_1 y_1 + \cdots + \lambda_{m-1} y_{m-1} + y_m) \\ & \quad + f_n(y_1, \dots, y_{n-1}, y_n + \lambda_1 y_1 + \cdots + \lambda_{m-1} y_{m-1} + y_m) \\ & = f_m(y_1, \dots, y_{n-1}, y_n). \end{aligned}$$

Put $y_m = -(\lambda_1 y_1 + \cdots + \lambda_{m-1} y_{m-1})$, we have

$$f_n(y_1, \dots, y_{m-1}, -(\lambda_1 y_1 + \cdots + \lambda_{m-1} y_{m-1}), y_{m+1}, \dots, y_n) = 0$$

hence f_n contains $y_m + \lambda_1 y_1 + \cdots + \lambda_{m-1} y_{m-1}$ as a factor. Consequently f_n contains L_{n-1} as a factor. The lemma is proved.

LEMMA 2.10. *If $0 \leq s_1 < \cdots < s_k \leq n - 2$, we have*

$$M_{n-1, s_1, \dots, s_k} \cdot V_n = M_{n, s_1, \dots, s_k} - \sum_{i=1}^k (-1)^{k+i} M_{n, s_1, \dots, \hat{s}_i, \dots, s_k, n-1} \cdot Q_{n-1, s_i}$$

and

$$M'_{n-1, s_1, \dots, s_k} \cdot V'_n = M_{n, s_1, \dots, s_k} - \sum_{i=1}^k (-1)^{k+i} M_{n, s_1, \dots, \hat{s}_i, \dots, s_k, n-1} \cdot Q'_{n-1, s_i}$$

up to a sign.

Proof. The first relation was proved in [5, Lemma I 4.12]. The second is a direct consequence of the first by permuting 1 and n .

LEMMA 2.11. *If $0 \leq s_1 < \cdots < s_k \leq n - 2$, we have*

$$M'_{n-1, s_1, \dots, s_k} \cdot V'_{n-1} = \sum_{0 \leq t_1 < \cdots < t_k = n-1} M_{n, t_1, \dots, t_k} \cdot F_{(t_1, \dots, t_k) + h}$$

where $F_{(t_1, \dots, t_k)}$ are elements of $P(y_1, \dots, y_n)$ and $h \in E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_n)$.

Proof. Put

$$U = \prod_{\substack{\lambda_i \in \mathbb{Z}_p \\ \lambda_n \neq 0}} (\lambda_2 y_2 + \dots + \lambda_{n-1} y_{n-1} + \lambda_n y_n + y_1)$$

then $V'_n = V'_{n-1} \cdot U$. By Lemma 2.10, we have

$$\begin{aligned} M'_{n-1, s_1, \dots, s_k} \cdot V'_{n-1} \cdot U &= M_{n, s_1, \dots, s_k} \\ &\quad - \sum_{i=1}^k (-1)^{k+i} M_{n, s_1, \dots, \hat{s}_i, \dots, s_k, n-1} \cdot Q'_{n-1, s_i} \\ &= M_{n, s_1, \dots, s_k} \cdot V_n \\ &\quad + \sum_{i=1}^k (-1)^{k+i} M_{n, s_1, \dots, \hat{s}_i, \dots, s_k, n-1} (Q_{n-1, s_i} - Q'_{n-1, s_i}) \end{aligned}$$

up to a sign.

Since V_n contains U as a factor, it remains to prove that $Q_{n-1, s_i} - Q'_{n-1, s_i}$ has U as a factor. This is the fact by noting that

$$Q_{n-1, s_i} (\lambda_2 y_2 + \dots + \lambda_n y_n, y_2, \dots, y_{n-1}) = Q'_{n-1, s_i} (y_2, \dots, y_n)$$

for any $\lambda_i \in \mathbb{Z}_p, \lambda_n \neq 0$. The lemma is proved.

LEMMA 2.12. Let $1 \leq k \leq n$ and f be an element of

$$E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$$

given by

$$f = \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n, s_1, \dots, s_k} f_{(s_1, \dots, s_k)}(y_1, \dots, y_k)$$

then f contains V_n as a vector if and only if so does every $f_{(s_1, \dots, s_k)}$.

Proof. By definition of M_{n, s_1, \dots, s_k} , we have

$$\begin{aligned} f &= (-1)^{n-1} x_n \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n-1, s_1, \dots, s_{k-1}} f_{s_1, \dots, s_k} \\ &\quad + \sum_I x_I f_I(y_1, \dots, y_n) \end{aligned}$$

where \sum_I denotes the summation over the subsets I of order k in $\{1, \dots, n - 1\}$. Put $y_n = \lambda_1 y_1 + \dots + \lambda_{n-1} y_{n-1}$. For each I , $f_I(y_1, \dots, y_{n-1}, \lambda_1 y_1 + \dots + \lambda_{n-1} y_{n-1})$ must be equal zero. Then f_I has V_n as a factor. Consequently

$$F = \sum_{0 \leq s_1 < \dots < s_k} M_{n-1, s_1, \dots, s_{k-1}} f_{s_1, \dots, s_k}$$

also contains V_n as a factor.

Let $0 \leq s_1 < \dots < s_k = n - 1$ and $s_{k+1} < \dots < s_{n-1}$ be its complement in $\{0, \dots, n - 2\}$, we have

$$F \cdot M_{n-1, s_{k+1}, \dots, s_{n-1}} = \pm x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} I_{n-1} f_{s_1, \dots, s_k}$$

by (2.3). Since the left side is equal zero for $y_n = \lambda_1 y_1 + \dots + \lambda_{n-1} y_{n-1}$, so is f_{s_1, \dots, s_k} . Hence f_{s_1, \dots, s_k} contains V_n as a factor. The lemma is proved.

LEMMA 2.13. *Let $1 \leq k \leq n$ and*

$$f = \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n, s_1, \dots, s_k} f_{s_1, \dots, s_k}(y_1, \dots, y_k) + g$$

be an element of $E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$, where g is an element of $E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_n)$. If $f = 0$ then $g = 0$ and $f_{s_1, \dots, s_k} = 0$ for each $0 \leq s_1 \dots s_k = n - 1$.

Proof. Let $g = \sum_I x_I g_I(y_1, \dots, y_n)$, where I runs over the subsets of order k of $\{1, \dots, n - 1\}$. We have

$$f \cdot M_{n, n-1} = 0 = g \cdot M_{n, n-1}.$$

For each I , the coefficient of $x_I \cdot x_n$ in $g \cdot M_{n, n-1}$ is $(-1)^{n-1} g_I \cdot L_{n-1}$. Hence $g_I = 0$. Then $g = 0$ and

$$f = \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n, s_1, \dots, s_k} f_{s_1, \dots, s_k} = 0.$$

For $0 \leq s_1 < \dots < s_k = n - 1$, let $s_{k+1} < \dots < s_n$ be its complement in $\{0, \dots, n - 1\}$, we have

$$f \cdot M_{n, s_{k+1}, \dots, s_n} = \pm x_1 \cdot x_2 \cdot \dots \cdot x_n \cdot L_n \cdot f_{s_1, \dots, s_k}$$

then $f_{s_1, \dots, s_k} = 0$. The lemma is proved.

Let k be an integer with $2 \leq k \leq n$ and let f be an invariant of W_n having the form

$$f = \sum_I x_I f_I(y_1, \dots, y_n)$$

where I runs over the subsets of order k in $\{1, \dots, n\}$. We write

$$(2.14) \quad f = x_1 \left(\sum' x_J f_I \right) + \sum'' x_I f_I$$

where Σ' (resp. Σ'') denotes the summation over the subsets of order $k - 1$ (resp. k) in $\{2, \dots, n - 1, n\}$, and in the first summation J is given by $I = J \cup \{1\}$ for each I containing 1. We set

$$G = \sum' x_J f_I$$

then G is an invariant of W'_{n-1} .

Now, we suppose that Theorem 2.4 is true for W_{n-1} . We have then

$$(2.15) \quad G = \sum_{0 \leq s_1 < \dots < s_{k-1} = n-2} M'_{n-1, s_1, \dots, s_{k-1}} g_{s_1, \dots, s_{k-1}} + \sum_J x_J g_J$$

where $g_{s_1, \dots, s_{k-1}}$ and g_J are the invariants of W'_{n-1} in $P(y_1, \dots, y_n)$ and \sum_J denotes the summation over the subsets of order $k - 1$ in $\{2, \dots, n - 1\}$.

LEMMA 2.16. All $g_{s_1, \dots, s_{k-1}}$ in (2.15) are invariants of W_n .

Proof. Clearly all $g_{s_1, \dots, s_{k-1}}$ are invariants of W'_{n-1} . We need only prove that $g_{s_1, \dots, s_{k-1}} = \alpha_1 g_{s_1, \dots, s_{k-1}}$ with $\omega_1 = 1 + \epsilon_{1n}$. We have

$$\begin{aligned} f &= x_1 G + \sum'' x_I f_I \\ &= \sum_{0 \leq s_1 < \dots < s_{k-1} = n-2} x_1 M'_{n-1, s_1, \dots, s_{k-1}} g_{s_1, \dots, s_{k-1}} \\ &\quad + \sum^{(1)} x_I h_I(y_1, \dots, y_n) + \sum^{(2)} x_I 1_I(y_1, \dots, y_n) \end{aligned}$$

where $\Sigma^{(1)}$ (resp. $\Sigma^{(2)}$) denotes the summation over the subsets of order k in $\{1, \dots, n - 1\}$ (resp. $\{2, \dots, n - 1, n\}$ such that $n \in I$). By Lemma 2.9, each 1_I with I in $\Sigma^{(2)}$ is an invariant of W_n . Hence

$$\begin{aligned} \omega_1 f &= \sum_{0 \leq s_1 < \dots < s_{k-1} = n-2} \left(x_1 M'_{n-1, s_1, \dots, s_{k-1}} \omega_1 g_{(s_1, \dots, s_{k-1})} \right. \\ &\quad \left. \pm x_1 M_{n-1, s_1, \dots, s_{k-1}} \omega_1 g_{(s_1, \dots, s_{k-1})} \right) \\ &\quad + \sum^{(1)} x_1 \omega_1 h_I \pm \sum^{(2)} x_1 \cdot x_{I \setminus \{n\}} \cdot 1_I + \sum^{(2)} x_I 1_I. \end{aligned}$$

Then

$$\begin{aligned} (1) \quad 0 &= f - \omega_1 f \\ &= \sum_{0 \leq s_1 < \dots < s_{k-1} = n-2} x_1 M'_{n-1, s_1, \dots, s_{k-1}} (g_{s_1, \dots, s_{k-1}} - \omega_1 g_{s_1, \dots, s_{k-1}}) \\ &\quad \times \sum_K x_K f'_K(y_1, \dots, y_n) \end{aligned}$$

where \sum_K denotes the summation over the subsets of order k in $\{1, \dots, n - 1\}$. By multiplying (1) by $M'_{n-1, n-2}$, we obtain $f'_K = 0$ for each K , by (2.3). Let $0 \leq s_1 < \dots < s_{k-1} = n - 2$ and $s_k < s_{k+1} < \dots < s_{n-1}$ be the ordered complement in $\{0, 1, \dots, n - 2\}$. By multiplying (1) by $M'_{n-1, s_k, \dots, s_{n-1}}$, according to (2.3), we obtain

$$g_{s_1, \dots, s_{k-1}} - \omega_1 g_{s_1, \dots, s_{k-1}} = 0.$$

The lemma is proved.

Lemmata 2.7–2.13 are obtained by a similar way as in [5]. The following is crucial in the determination of the invariants of W_n .

LEMMA 2.17. *Let k be an integer with $2 \leq k \leq n$ and f be an element of $E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$ given by*

$$f = \sum_I x_I f_I(y_1, \dots, y_n)$$

where I runs over the subsets of order k in $\{1, \dots, n\}$ such that $\{1, n\} \not\subset I$. If f is an invariant of W_n , then f can be decomposed into the form

$$f = \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n, s_1, \dots, s_k} f_{s_1, \dots, s_k} + h$$

where all f_{s_1, \dots, s_k} are invariants of W_n in $P(y_1, \dots, y_n)$ and h is an element of $E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n)$.

Proof. We write

$$f = \sum'' x_I f_I + x_1 \left(\sum' x_J f_J \right)$$

as in (2.14). Set $F = \sum'' x_I f_I$, then F is an invariant of W'_{n-1} , and we have the decomposition

$$F = \sum_{0 \leq s_1 < \dots < s_k = n-2} M'_{n-1, s_1, \dots, s_k} F_{s_1, \dots, s_k}(y_1, \dots, y_n) + \sum_I x_I F_I(y_1, \dots, y_n)$$

where F_{s_1, \dots, s_k} and F_I are invariants of W'_{n-1} , and \sum_I denotes the summation over the subsets of order k in $\{2, \dots, n - 1\}$. As in the proof of Lemma 2.16, one can see that all F_{s_1, \dots, s_k} are invariants of W_n . Furthermore, we can assume that all F_I , where I occurs in \sum_I , and all f_I , with $1 \in I$, have y_n as a factor. Hence they obtain V''_{n-1} as a factor.

Let $\omega_1 = 1 + \varepsilon_{1n}$. We have

$$\begin{aligned} \omega_1 f &= \sum_{0 \leq s_1 < \dots < s_k = n-2} M'_{n-1, s_1, \dots, s_k} F_{s_1, \dots, s_k} \\ &\pm \sum_{0 \leq s_1 < \dots < s_k = n-2} M_{n-1, s_1, \dots, s_k} F_{s_1, \dots, s_k} \\ &+ \sum_I x_I \omega_1 F_I + x_1 \left(\sum' x_J \omega_1 f_J \right). \end{aligned}$$

Hence

$$\begin{aligned} 0 = f - \omega_1 f &= \pm \sum_{0 \leq s_1 < \dots < s_k = n-2} M_{n-1, s_1, \dots, s_k} F_{s_1, \dots, s_k} \\ &+ \sum_I x_I (F_I - \omega_1 F_I) + x_1 \left(\sum' x_J (f_J - \omega_1 f_J) \right). \end{aligned}$$

Since F_I and f_J have V''_{n-1} as a factor, $F_I - \omega_1 F_I$ and $f_J - \omega_1 f_J$ contain V'_{n-1} as a factor. By Lemma 2.12, F_{s_1, \dots, s_k} also contains V'_{n-1} as a factor. Then we have

$$\begin{aligned} f &= \sum_{0 \leq s_1 < \dots < s_k = n-2} M'_{n-1, s_1, \dots, s_k} V'_{n-1} F'_{s_1, \dots, s_k} \\ &+ \sum_I x_I F_I + x_1 \left(\sum' x_J f_J \right). \end{aligned}$$

By Lemma 2.11, f has then the form

$$f = \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n, s_1, \dots, s_k} f_{s_1, \dots, s_k}(y_1^2, y_n) + \sum''' x_I h_I$$

where \sum''' denotes the summation over the subsets of order k in $\{1, \dots, n-1\}$.

Let ω be an element of W_n . We have

$$\begin{aligned} 0 = f - \omega f &= \sum''' x_I (h_I - \omega h_I) \\ &+ \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n, s_1, \dots, s_k} (f_{s_1, \dots, s_k} - \omega f_{s_1, \dots, s_k}). \end{aligned}$$

By Lemma 2.13, we have $f_{s_1, \dots, s_k} - \omega f_{s_1, \dots, s_k} = 0$ and $h_I - \omega h_I = 0$. Hence f_{s_1, \dots, s_k} and h_I are invariants of W_n . The lemma is proved.

The proof of Theorem 2.4 will be completed by the following

LEMMA 2.18. *Let k be an integer with $1 \leq k \leq n$ and f be an element of $E(x_1, \dots, x_n; 1) \otimes P(y_1, \dots, y_n; 2)$ given by*

$$f = \sum_I x_I f_I(y_1, \dots, y_n)$$

where I runs over the subsets of order k in $\{1, \dots, n\}$. If f is an invariant of W_n , then f can be decomposed into the form

$$f = \sum_{0 \leq s_1 < \dots < s_k = n-1} M_{n,s_1,\dots,s_k} f_{s_1,\dots,s_k}(y_1, \dots, y_n) + h$$

where all f_{s_1,\dots,s_k} are invariants of W_n , and h is an element of $E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n)$.

Proof. If $k = 1$, we have $f = x_1 f_1 + \dots + x_n f_n$. By Lemma 2.9, f_n contains L_{n-1} as a factor

$$f_n = L_{n-1} g$$

with some invariant g of W_n in $P(y_1, \dots, y_n)$. Then

$$\begin{aligned} f &= x_1 f_1 + \dots + x_{n-1} f_{n-1} + x_n L_{n-1} g \\ &= (-1)^{n-1} M_{n,n-1} g + h. \end{aligned}$$

Hence the lemma is proved for the case $k = 1$.

Next we consider the case $2 \leq k \leq n$. As in the proof of Lemma 2.16, f has the form

$$\begin{aligned} f &= x_1 \sum_{0 \leq s_1 < \dots < s_{k-1} = n-2} M'_{n-1,s_1,\dots,s_{k-1}} g_{s_1,\dots,s_{k-1}} \\ &\quad + \sum^{(1)} x_I h_I(y_1, \dots, y_n) + \sum^{(2)} x_I g_I(y_1, \dots, y_n) \end{aligned}$$

and all $g_{s_1,\dots,s_{k-1}}$ are invariants of W_n by Lemma 2.16. Let $0 \leq s_1 < \dots < s_{k-1} = n - 2$. By definition of $M_{n,s_1,\dots,s_{k-1},n-1}$ we have

$$x_1 M'_{n-1,s_1,\dots,s_{k-1}} = M_{n,s_1,\dots,s_{k-1},n-1} + \sum_I x_I h'_I(y_1, \dots, y_n)$$

where \sum_I denotes the summation over the subsets of order k in $\{2, \dots, n\}$. Then f has the form

$$\begin{aligned} f &= \sum_{0 \leq s_1 < \dots < s_{k-1} = n-2} M_{n,s_1,\dots,s_{k-1},n-1} g_{s_1,\dots,s_{k-1}} \\ &\quad + \sum^{(3)} x_I f'_I(y_1, \dots, y_n) \end{aligned}$$

where $\sum^{(3)} x_I f'_I$ satisfies the conditions of Lemma 2.17. The lemma is proved.

3. The restriction homomorphism. Let G be an extra-special p -group of order p^{2n-1} ($n \geq 2$). Let A be a maximal elementary abelian

p -subgroup of G as in (1.8). We are going to apply the invariants of W_n to prove the main theorem of this paper as follows.

THEOREM 3.1. (a) *If $G = E_{n-1}$, then*

$$\begin{aligned} \text{Im Res}(A, G) &= H^*(A)^{W_G(A)} \\ &= E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n) \\ &\quad \oplus \sum_{k=1}^n \oplus \sum_{0 \leq s_1 < \dots < s_k = n-1} \oplus M_{n, s_1, \dots, s_k} P(y_1, \dots, y_{n-1}, V_n). \end{aligned}$$

(b) *IF $G = M_{n-1}$, then*

$$\text{Im Res}(A, G) = E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n).$$

LEMMA 3.2. *The elements $x_i, y_i, 1 \leq i < n$, and V_n are in $\text{Im Res}(A, G)$.*

Proof. This lemma has been proved by Tezuka-Yagita in [9]. For V_n , Tezuka and Yagita had used the Chern class of a complex representation of G . Here we give another proof by use of the norm map in Evens [1]. Let $\mathcal{N} = \mathcal{N}_{Z(G) \rightarrow G}$ be the norm map. By [1, Th. 2], we have

$$\text{Res}(A, G) \mathcal{N}(y_n) = V_n.$$

LEMMA 3.3. *For $0 \leq s_1 < \dots < s_k = n - 1$, there exist $\varepsilon_i = 0, 1; t_i = 1, 2, 3, \dots$ such that*

$$M_{n, s_1, \dots, s_k} = \beta^{\varepsilon_0} \mathcal{P}^{t_1} \beta^{\varepsilon_1} \cdot \dots \cdot \mathcal{P}^{t_k} \beta^{\varepsilon_k} M_{n, 0, 1, \dots, n-1}$$

up to a sign, where \mathcal{P}^i are the Steenrod operations.

Proof. Let $\{i_1, \dots, i_k\}, \{i'_1, \dots, i'_{k'}\}$ be respectively two subsets of order k and k' in $\{0, 1, \dots, n - 1\}$ with $i_1 < \dots < i_k, i'_1 < \dots < i'_{k'}$. Let us define the relation

$$\{i_1, \dots, i_k\} \leq \{i'_1, \dots, i'_{k'}\}$$

if one of the following conditions is satisfied:

— $k < k'$,

— if $k = k'$, then there exists an integer $1 \leq m \leq k$ such that $i_m < i'_m$ and $i_s = i'_s$ for $m + 1 \leq s \leq k$, unless $\{i_1, \dots, i_k\} = \{i'_1, \dots, i'_{k'}\}$.

The set $\mathcal{P}(\{0, 1, \dots, n - 1\})$ is then totally ordered. The lemma will be proved by descending induction on $\{s_1, \dots, s_k\}$.

First, we have $M_{n, 1, 2, \dots, n-1} = \pm \beta M_{n, 0, 1, \dots, n-1}$ up to a sign. Assume inductively that the lemma holds with $\{s_1, \dots, s_k\}$. Let $s'_1, \dots, s'_{k'}$ be the

preceding element of $\{s_1, \dots, s_k\}$. We have

- if $k' < k$, the $M_{n,s'_1,\dots,s'_k} = \beta M_{n,0,s'_1,\dots,s'_k}$,
- if $k' = k$, then there exists $0 \leq m \leq k$ such that $s'_m < s_m$ and $s'_t = s_t$ for $m + 1 \leq t \leq k$. Hence

$$M_{n,s'_1,\dots,s'_k} = \pm \mathcal{P}^{p^m} M_{n,s_1,\dots,s_k}.$$

The lemma is proved.

Let $Z = Z(G)$. We have the following commutative diagram of group extensions

$$(3.4) \quad 1 \rightarrow Z \rightarrow G \rightarrow G/Z \rightarrow 1$$

$$(3.5) \quad \begin{array}{ccccccc} & & \parallel & & \cup & & \cup \\ 1 & \rightarrow & Z & \rightarrow & A & \rightarrow & A/Z \rightarrow 1 \end{array}$$

Let $A' = A/Z$ be identified with the subgroup \mathbf{Z}_p^{n-1} of A . The central group extension (3.5) becomes

$$(3.5)' \quad 1 \rightarrow Z \rightarrow A \rightarrow A' \rightarrow 1$$

corresponding to the trivial cohomology class.

Let $a_1, \dots, a_n, b_1, \dots, b_{n-}$ be the elements of G satisfying Prop. 1.7, such that a_1, \dots, a_n correspond to the canonical basis of A as in (1.8). Then $\{a_1Z, \dots, a_{n-1}Z, b_1Z, \dots, b_{n-1}Z\}$ form a basis of G/Z . Let us identify G/Z with \mathbf{Z}_p^{2n-2} by the correspondence

$$a_i \mapsto e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} < i, \quad b_i \mapsto e_{n+i-1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} < n + i - 1.$$

For $i \geq n$, let x_{i+1} be the dual of e_i over \mathbf{Z}_p and $y_{i+1} = \beta x_{i+1}$. For $i < n$ (resp. $i = n$), the element $x_i \in H^1(A)$ can be identified with the dual of e_i (resp. $c \in Z$) over \mathbf{Z}_p . We have then

$$H^*(G/Z) = E(x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_{2n-1}; 1) \otimes P(y_1, \dots, y_{n-1}, y_{n+1}, \dots, y_{2n-1}; 2)$$

and $H^*(Z) = E(x_n; 1) \otimes P(y_n; 2)$.

In Phạm Anh Minh–Huỳnh Mùi [4; Lemma 2.2], we have proved

LEMMA 3.6. *Let $f \in H^2(\mathbf{Z}_p^n)$ be represented by a 2-cocycle $f: \mathbf{Z}_p^n \otimes \mathbf{Z}_p^n \rightarrow \mathbf{Z}_p$. Then we have*

$$f = \sum_{i=1}^n \alpha_i y_i + \sum_{1 \leq i < j \leq n} \beta_{ij} x_i x_j$$

where

$$\alpha_i = \sum_{k=1}^{p-1} f(e_i, e_i^k) \quad \text{and} \quad \beta_{i,j} = f(e_i, e_j) - f(e_j, e_i).$$

From this lemma, one can see that the cohomology class z corresponding to the extension (3.4) is

$$(3.7) \quad \begin{aligned} & y_1 + x_1x_{n+1} + x_2x_{n+2} + \cdots + x_{n-1}x_{2n-1} && \text{if } G = M_{n-1} \\ & x_1x_{n+1} + x_2x_{n+2} + \cdots + x_{n-1}x_{2n-1} && \text{if } G = E_{n-1} \end{aligned}$$

via the isomorphism $(x_n)^*: H^2(G/Z, Z) \cong H^2(G/Z, \mathbf{Z}_p)$.

Consider the Hochschild-Serre spectral sequences of the central extensions (3.4) and (3.5)'. Let $\tau: H^*(Z) \rightarrow H^{*+1}(G/Z)$ denote the transgression as usual. From [2; Chap. III, 3], we have

$$\tau x_n = z \in H^2(G/Z).$$

LEMMA 3.8. *If $G = M_{n-1}$, then*

$$\text{Im Res}(A, G) = E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n).$$

Proof. Since $\text{Ann}_{H^*(X/G)}(\tau x_n) = 0$, we have

$$E_3(Z, G) = H^*(G/Z)|_{(\tau x_n, \beta \tau x_n)} \otimes \mathbf{Z}_p[y_n]$$

(see e.g. Phạm Anh Minh–Huỳnh Mùi [4]).

The inclusion map $A \hookrightarrow G$ gives us the corresponding map

$$E_\infty(Z, G) \rightarrow E_\infty(Z, A) = E_2(Z, A) = H^*(A) \otimes H^*(Z)$$

with image in $H^x(A') \otimes \mathbf{Z}_p[y_n]$. Then

$$\begin{aligned} \text{Im Res}(A, G) &\subset \left(H^*(A') \otimes \mathbf{Z}_p[y_n] \right)^{W_n} \\ &\subset E(x_1, \dots, x_{n-1}) \otimes P(y_1, \dots, y_{n-1}, V_n) \end{aligned}$$

by Theorem 2.4. The lemma is proved.

The above lemma concludes the part (b) of Theorem 3.1. The following completes the proof of 3.1(a).

LEMMA 3.9. *If $G = E_{n-1}$, then $M_{n,0,1,\dots,n-1} = x_1, \dots, x_n$ is an element of $\text{Im Res}(A, G)$, hence so are the elements M_{n,s_1,\dots,s_k} , $0 \leq s_1 < \cdots < s_k = n - 1$.*

Proof. Since $x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} \in \text{Ann}_{H^*(G/Z)}(\tau x_n)$, we have

$$x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} \otimes x_n \in E_\infty^{n-1,1}(Z, G)$$

(see e.g. Phạm Anh Minh–Huỳnh Mũi [4]).

Consider the morphism of spectral sequences induced by the inclusion map $(A, G) \hookrightarrow (G, Z)$. We have the commutative diagram

$$(3.10) \quad \begin{array}{ccc} F^{n-1}H^n(G) & \rightarrow & E_\infty^{n-1,1}(Z, G) \\ \downarrow & & \downarrow f \\ F^{n-1}H^n(A) & \rightarrow & E_\infty^{n-1,1}(Z, A). \end{array}$$

Here $F^iH^*(G)$ and $F^iH^*(A)$ are Hochschild-Serre filtrations corresponding to (3.4) and (3.5)'.

Let m be an element of $F^{n-1}H^n(G)$ such that

$$M \in F^{n-1}H^n(G) \mapsto x_1 \cdot x_2 \cdot \dots \cdot x_{n-1} \otimes x_n \in E_\infty^{n-1,1}(Z, G).$$

From the diagram (3.10), we have

$$\text{Res}(A, G)M = x_1 \cdot x_2 \cdot \dots \cdot x_n + F^nH^n(A).$$

Since $F^nH^n(A) = H^n(A') \subset \text{Im Res}(A, G)$ by Lemma 3.2, the element $x_1 \cdot x_2 \cdot \dots \cdot x_n$ lies to $\text{Im Res}(A, G)$.

By Lemma 3.3, all M_{n,s_1,\dots,s_k} are elements of $\text{Im Res}(A, G)$. The lemma is proved.

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