ON INCLUSION RELATIONS FOR ABSOLUTE NÖRLUND SUMMABILITY

IKUKO MIYAMOTO
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Recently Das gives sufficient conditions for \((N, r_n) \subseteq (N, p_n)(N, q_n)\)
or \((N, p_n)(N, q_n) \subseteq (N, r_n)\), and for \(|N, P_n| \sim |N, p_n)(C, 1)|\). The purpose of this paper is to give sufficient conditions for \(|N, r_n| \subseteq |(N, p_n)(N, q_n)| \) or \(|(N, p_n)(N, q_n)| \subseteq |N, r_n|\). The results obtained here are also absolute summability analogues of Das' theorems.

1. Let \(\{p_n\}\) and \(\{q_n\}\) be real or complex sequences such that \(P_n = \sum_{k=0}^{n} p_k \neq 0\) and \(Q_n = \sum_{k=0}^{n} q_k \neq 0\). A sequence \(\{s_n\}\) is said to be summable \((N, p_n)\) to \(s\), if \(t^n = \sum_{k=0}^{n} p_{n-k}s_k/P_n \to s(n \to \infty)\), and summable \((N, p_n)(N, q_n)\) to \(s\), if \(t^{p,q} = \sum_{k=0}^{n} p_{n-k}t^q/P_n \to s(n \to \infty)\). It is said to be absolutely summable \((N, p_n)\), or summable \((N, p_n)\), if \(\sum|t^n - t^{n+1}| < \infty\).

Given two summability methods \(A\) and \(B\), we write \(A \subseteq B\) if each sequence summable \(A\) is summable \(B\). If each includes the other, we write \(A \sim B\).

We define the sequence \(\{r_n\}\) by \(r_n = \sum_{k=0}^{n} p_{n-k}q_k\) and define the sequence \(\{c_n\}\) formally by \(1/\sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} c_n x^n\). We write \(\{p_n\} \in \mathcal{M}\) if \(p_n > 0, p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1\), and also write, for any sequence \(\{f_n\}\), \(f_n^{(1)} = \sum_{k=0}^{n} f_k, f_n^{(2)} = \sum_{k=0}^{n} f_k^{(1)}\). And \(K\) denotes an absolute constant, not necessarily the same at each occurrence.

On inclusion relations between two summability methods Das gives the following theorems.

**Theorem A** [1, Theorem 2]. If \(\{p_n\} \in \mathcal{M}\) and \(\{q_n\}\) is positive, then \((N, r_n) \subseteq (N, p_n)(N, q_n)\).

**Theorem B** [1, Theorem 5]. If \(\{p_n\} \in \mathcal{M}\) and \(\{q_n\}\) is positive and \((n + 1)q_n = O(Q_n)\), then \((N, p_n)(N, q_n) \subseteq (N, r_n)\).

**Theorem C** [2, Theorem 5]. If \(\{p_n\} \in \mathcal{M}\), then \(|N, P_n| \sim |(N, p_n)(C, 1)|\).

The purpose of this paper is to prove the following theorems.
THEOREM 1. If \( \{ p_n \} \in \mathfrak{M} \) and if \( \{ q_n \} \) is positive and nonincreasing, then \(|N, r_n| \leq |(N, p_n)(N, q_n)|\).

This is an absolute summability analogue of Theorem A.

THEOREM 2. If \( \{ p_n \} \in \mathfrak{M} \) and if \( \{ q_n \} \) is positive and nonincreasing and if \( R_n = \sum_{k=0}^{n} r_k \to \infty (n \to \infty) \), then \(|(N, p_n)(N, q_n)| \leq |N, r_n|\).

This is an absolute summability analogue of Theorem B. Combining Theorem 1 and Theorem 2, we have the following

THEOREM 3. Under the assumptions of Theorem 2, the relation \(|(N, p_n)(N, q_n)| \sim |N, r_n|\) holds.

In this Theorem, if we put \( q_n = 1 \), then we obtain Theorem C.

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2. We require the following lemmas.

LEMMA 1. Let \( y_n = \sum_{\nu=0}^{n} a_{\nu} x_{\nu} \). If
\[
\sum_{n=\rho}^{\infty} \left| \sum_{\nu=\rho}^{\infty} (a_{\nu} - a_{\nu-1}) \right| \leq c < \infty \quad \text{for all } \rho,
\]
then \( \sum_{n=0}^{\infty} |\Delta y_n| < \infty \) whenever \( \sum_{n=0}^{\infty} |\Delta x_n| < \infty \).

This is due to F. M. Mears ([3, p. 595]).

LEMMA 2. Let \( \{ p_n \} \in \mathfrak{M} \). Then
\begin{align*}
(1) & \quad \sum_{\rho=0}^{r} P_{\rho} \sum_{n=r+1}^{\infty} |c_{n-\rho}| \leq r + 1, \\
(2) & \quad \{ c_n^{(1)} \} \text{ is nonnegative and nonincreasing and} \\
(3) & \quad c_n^{(2)} p_n \leq 1.
\end{align*}

This is Lemmas 3 and 4 in [2].
Lemma 3. If \( \{p_n\} \) and \( \{q_n\} \) are nonnegative, then

\[
P_n^{(1)} \leq K(n + 1)P_n \quad \text{and}
\]

\[
R_n \leq P_nQ_n.
\]

Further, if \( \{p_n\} \) and \( \{q_n\} \) are nonincreasing, then

\[
(n + 1)P_n \leq KP_n^{(1)} \quad \text{and}
\]

\[
R_n \geq KP_nQ_n.
\]

Proof. The inequalities (4) and (6) are Lemma 5 in [2]. The inequality (5) is easily established. So we shall prove the inequality (7). Since the sequence \( \{P_n/(n + 1)\} \) is nonincreasing, and \( KQ_n^{(1)} \geq (n + 1)Q_n \),

\[
R_n = P_0q_n + P_1q_{n-1} + \cdots + P_nq_0
\]

\[
= P_0q_n + \frac{P_1}{2}q_{n-1} + \cdots + (n + 1)\frac{P_n}{n + 1}q_0
\]

\[
\geq \frac{P_n}{n + 1}(q_n + 2q_{n-1} + \cdots + (n + 1)q_0)
\]

\[
= P_nQ_n^{(1)}/(n + 1) \geq P_nQ_n/K.
\]

Lemma 4. If \( \{p_n\} \in \mathbb{N} \) and if \( \{q_n\} \) is positive and nonincreasing, then

\[
0 \leq \sum_{\mu}^{v} p_{n-\rho}c_{\rho-\mu} \leq p_{n-\mu}c_{v-\mu} \quad (\mu \leq v \leq n),
\]

\[
0 \leq \sum_{\mu}^{n-1} (Q_n - Q_{v})p_{n-\nu}c_{v-\mu} \leq q_{\mu},
\]

\[
\frac{1}{Q_n} \sum_{v=\mu}^{n-1} \frac{Q_n - Q_{v}}{Q_{v}}p_{n-\nu}c_{v-\mu} \leq p_{n-\mu} \sum_{\mu}^{n-1} \frac{q_{v+1}c_{v-\mu}}{Q_vQ_{v+1}}
\]

and uniformly in \( v \leq \rho \),

\[
\sum_{n=\rho+1}^{\infty} \frac{(Q_n - Q_{v})(p_{n-\nu - 1} - p_{n-\nu})}{Q_nP_{n-1}} = O\left( \frac{1}{\rho + 1} \right).
\]

Proof. The inequality (8) is Lemma 6(11) in [2].
The inequality (9); Using Abel's transformation, from (3) and (8), we have
\[
\sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu} c_{\nu-\mu} = \sum_{\nu=\mu}^{n-1} q_{\nu+1} \sum_{\rho=\mu}^{\nu} p_{n-\rho} c_{\rho-\mu}
\]
\[
\leq \sum_{\nu=\mu}^{n-1} q_{\nu+1} p_{n-\mu} c_{\nu-\mu} \leq q_{\mu} p_{n-\mu} \sum_{\nu=\mu}^{n-1} c_{\nu-\mu}
\]
\[
= q_{\mu} p_{n-\mu} c_{n-\mu}^{(2)} \leq q_{\mu}.
\]

The inequality (10); Using Abel's transformation, from (8), we get
\[
\frac{1}{Q_n} \sum_{\nu=\mu}^{n-1} \frac{Q_n - Q_\nu}{Q_\nu} p_{n-\nu} c_{\nu-\mu}
\]
\[
= \frac{1}{Q_n} \sum_{\nu=\mu}^{n-2} \left( \frac{Q_n - Q_\nu}{Q_\nu} - \frac{Q_n - Q_{\nu+1}}{Q_{\nu+1}} \right) \sum_{r=\mu}^{\nu} p_{n-r} c_{r-\mu}
\]
\[
+ \frac{q_n}{Q_{n-1} Q_n} \sum_{r=\mu}^{n-1} p_{n-r} c_{r-\mu}
\]
\[
= \sum_{\nu=\mu}^{n-1} \frac{q_{\nu+1}}{Q_\nu Q_{\nu+1}} \sum_{r=\mu}^{\nu} p_{n-r} c_{r-\mu} \leq p_{n-\mu} \sum_{\nu=\mu}^{n-1} \frac{q_{\nu+1} c_{\nu-\mu}}{Q_\nu Q_{\nu+1}}.
\]

The inequality (11); Since \{q_n\} is nonincreasing, we have
\[
\frac{Q_n}{Q_\nu} = 1 + \frac{Q_n - Q_\nu}{Q_\nu} \leq 1 + \frac{(n - \nu) q_\nu}{\nu q_\nu} = \frac{n}{\nu}.
\]
Hence, \((Q_n - Q_\nu)/Q_n \leq (n - \nu)/n \). Therefore using Das' Lemma 7 in [2], we obtain the inequality (11).

**Lemma 6.** If \{p_n\} is positive and nonincreasing, then uniformly in \(0 \leq \mu \leq \nu\),
\[
\sum_{n=\nu}^{\infty} \frac{p_n}{P_n} p_{n-\mu} \leq O\left(\frac{1}{\nu + 1}\right).
\]

This is Lemma 8 in [2].

**3. Proof of Theorem 1.** Let us write
\[
t'_n = \frac{1}{R_n} \sum_{\nu=0}^{n} r_{n-\nu} s_{\nu} \quad \text{and} \quad t^{p,q}_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} t^{q}_{\nu}.
\]
Then, following Das’ [1, pp. 32–33], we have

\[ t_{n}^{p,q} = \sum_{\mu=0}^{n} \lambda_{n\mu} t_{\mu}^{p}, \]

where

\[ \lambda_{n\mu} = \begin{cases} \frac{R_{\mu}}{P_{n}} \sum_{\nu=\mu}^{n} \frac{p_{n-\nu}c_{\nu-\mu}}{Q_{\nu}} & (\mu \leq n) \\ 0 & (\mu > n). \end{cases} \]

By Lemma 1, it is sufficient to show that

\[ J_{\rho} = \sum_{n=\rho}^{\infty} \left| \sum_{\mu=\rho}^{n} (\lambda_{n\mu} - \lambda_{n-1,\mu}) \right| = O(1) \quad (\rho = 0, 1, 2, \ldots). \]

Noting that

\[ \sum_{\nu=\mu}^{n} p_{n-\nu}c_{\nu-\mu} = \begin{cases} 1 & (n = \mu) \\ 0 & (n > \mu), \end{cases} \]

for \( n > \mu \), we get

\[ \lambda_{n\mu} = \frac{R_{\mu}}{P_{n}} \sum_{\nu=\mu}^{n-1} \frac{Q_{n} - Q_{\nu}}{Q_{n}Q_{\nu}} p_{n-\nu}c_{\nu-\mu}, \]

and for \( n > \mu + 1 \),

\[ \lambda_{n-1,\mu} = \frac{R_{\mu}}{P_{n-1}} \sum_{\nu=\mu}^{n-1} \frac{Q_{n} - Q_{\nu}}{Q_{n}Q_{\nu}} p_{n-\nu-1}c_{\nu-\mu}. \]

Also it is easily seen that \( \sum_{\mu=0}^{n} \lambda_{n\mu} = 1 \). Hence, for \( n > \rho \),

\[ \sum_{\mu=\rho}^{n} (\lambda_{n\mu} - \lambda_{n-1,\mu}) = \sum_{\mu=0}^{\rho-1} (\lambda_{n-1,\mu} - \lambda_{n\mu}) \]

\[ = \sum_{\mu=0}^{\rho-1} R_{\mu} \sum_{\nu=\mu}^{n-1} \frac{Q_{n} - Q_{\nu}}{Q_{n}Q_{\nu}} \left( \frac{p_{n-\nu-1}}{P_{n-1}} - \frac{p_{n-\nu}}{P_{n}} \right) c_{\nu-\mu}. \]

Thus

\[ J_{\rho} = \left| \lambda_{\rho\rho} \right| + \sum_{n=\rho+1}^{\infty} \left| \sum_{\mu=0}^{\rho-1} (\lambda_{n-1,\mu} - \lambda_{n\mu}) \right| \]

\[ \leq \left| \lambda_{\rho\rho} \right| + \sum_{n=\rho+1}^{\infty} \frac{p_{n}}{Q_{n}P_{n}P_{n-1}} \left| \sum_{\mu=0}^{\rho-1} R_{\mu} \sum_{\nu=\mu}^{n-1} \frac{Q_{n} - Q_{\nu}}{Q_{n}Q_{\nu}} \left( \frac{p_{n-\nu-1}}{P_{n-1}} - \frac{p_{n-\nu}}{P_{n}} \right) c_{\nu-\mu} \right| \]

\[ + \sum_{n=\rho+1}^{\infty} \frac{1}{Q_{n}P_{n-1}} \left| \sum_{\mu=0}^{\rho-1} R_{\mu} \sum_{\nu=\mu}^{n-1} \frac{Q_{n} - Q_{\nu}}{Q_{\nu}} \left( p_{n-\nu-1} - p_{n-\nu} \right) c_{\nu-\mu} \right| \]

\[ = J_{\rho}^{(1)} + J_{\rho}^{(2)} + J_{\rho}^{(3)}, \quad \text{say.} \]
From (5),

\[ J_{\rho}^{(1)} = |\lambda_{\rho\rho}| = \frac{R_{\rho}p_{0}c_{0}}{P_{\rho}Q_{\rho}} \leq 1. \]

By Lemma 4(10),

\[ J_{\rho}^{(2)} \leq \sum_{n=\rho+1}^{\infty} \frac{p_{n}}{P_{n}P_{n-1}} \sum_{\mu=0}^{\rho-1} R_{\mu} p_{n-\mu} \sum_{\nu=\mu}^{n-1} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu}Q_{\nu+1}} \]

\[ = \sum_{\mu=0}^{\rho-1} R_{\mu} \sum_{n=\rho+1}^{\infty} \frac{p_{n}p_{n-\mu}}{P_{n}P_{n-1}} \left( \sum_{\nu=\mu}^{\rho-1} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu}Q_{\nu+1}} + \sum_{\nu=\rho}^{n-1} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu}Q_{\nu+1}} \right) \]

\[ = J_{\rho}^{(2)} + J_{\rho}^{(2)}', \text{ say.} \]

Using the identity

\[ \sum_{\mu=0}^{\nu} P_{\mu}c_{\nu-\mu}^{(1)} = \nu + 1, \]

(5), (12) and the monotonicity of \{p_{n}\}, \{q_{n}\} and \{Q_{n}\}, we have

\[ J_{\rho}^{(2)} \leq \sum_{n=\rho+1}^{\infty} \frac{p_{n}p_{n-\rho}}{P_{n}P_{n-1}} \sum_{\nu=0}^{\rho-1} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu}Q_{\nu+1}} \sum_{\mu=0}^{\nu} R_{\mu}c_{\nu-\mu}^{(1)} \]

\[ \leq \sum_{n=\rho+1}^{\infty} \frac{p_{n}p_{n-\rho}}{P_{n}P_{n-1}} \sum_{\nu=0}^{\rho-1} \frac{q_{\nu+1}Q_{\nu}}{Q_{\nu}Q_{\nu+1}} \sum_{\mu=0}^{\nu} P_{\mu}c_{\nu-\mu}^{(1)} \]

\[ = \sum_{n=\rho+1}^{\infty} \frac{p_{n}p_{n-\rho}}{P_{n}P_{n-1}} \sum_{\nu=0}^{\rho-1} \left( \nu + 1 \right) q_{\nu+1}Q_{\nu} \]

\[ = O(\rho + 1) \sum_{n=\rho+1}^{\infty} \frac{p_{n}p_{n-\rho}}{P_{n}P_{n-1}} = O(1). \]

Using (2), (5), (12) and (13), since \{q_{n}\} and \{Q_{n}\} are monotone, we get

\[ J_{\rho}^{(2)} = \sum_{\mu=0}^{\rho-1} R_{\mu} \sum_{\nu=\rho}^{\infty} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu}Q_{\nu+1}} \sum_{n=\nu+1}^{\infty} \frac{p_{n}p_{n-\mu}}{P_{n}P_{n-1}} \]

\[ \leq K \sum_{\mu=0}^{\rho-1} Q_{\mu}P_{\mu} \sum_{\nu=\rho}^{\infty} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu}Q_{\nu+1}(\nu + 1)} \]

\[ \leq K \sum_{\mu=0}^{\rho-1} P_{\mu} \sum_{\nu=\rho}^{\infty} \frac{q_{\nu+1}c_{\nu+1}^{(1)}}{Q_{\nu+1}(\nu + 1)} \]

\[ \leq K \sum_{\mu=0}^{\rho-1} P_{\mu}c_{\nu-\mu}^{(1)} \sum_{\nu=\rho}^{\infty} \frac{1}{(\nu + 1)^{2}} = O(1). \]
Next,

\[
J_{\rho_2}^{(3)} \leq \sum_{n=\rho+1}^{\infty} R_\mu \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \left| \sum_{v=\rho}^{n-1} \frac{Q_{n} - Q_{v}}{Q_{v}} (p_{n-v-1} - p_{n-v}) c_{v-\mu} \right|
\]

\[
= \sum_{n=\rho+1}^{\infty} \sum_{v=\rho}^{n-1} \frac{Q_{n} - Q_{v}}{Q_{n} P_{n-1}} (p_{n-v-1} - p_{n-v})
\]

\[
= \rho \sum_{n=\rho+1}^{\infty} \frac{(Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})}{Q_{n} P_{n-1}} = O(1).
\]

Next, using (5), we get

\[
J_{\rho_2}^{(3)} \leq \sum_{n=\rho+1}^{\infty} R_\mu \sum_{n=\rho+1}^{\infty} \frac{1}{Q_n P_{n-1}} \left| \sum_{v=\rho}^{n-1} \frac{Q_{n} - Q_{v}}{Q_{v}} (p_{n-v-1} - p_{n-v}) c_{v-\mu} \right|
\]

\[
= \sum_{n=\rho+1}^{\infty} R_\mu \sum_{v=\rho}^{\infty} \frac{|c_{v-\mu}|}{Q_{v}} \sum_{n=v+1}^{n-1} \frac{(Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})}{Q_{n} P_{n-1}}
\]

\[
\leq K \sum_{n=\rho+1}^{\infty} R_\mu \sum_{v=\rho}^{\infty} \frac{|c_{v-\mu}|}{Q_{v}(v+1)}
\]

\[
= O\left(\frac{1}{\rho + 1}\right) \sum_{n=\rho}^{\infty} P_{\mu} \sum_{v=\rho}^{\infty} |c_{v-\mu}| = O(1),
\]

by Lemma 4(11) and Lemma 2(1).

This completes the proof of Theorem 1.

4. Proof of Theorem 2. First, we have, following Das' [1, p. 37],

\[
t_n' = \frac{1}{R_n} \sum_{\mu=0}^{n} P_{\mu} \left( \sum_{v=\mu}^{n} p_{n-v} Q_{v} c_{v-\mu} \right) t_{\mu}^{p,q} = \sum_{\mu=0}^{n} \alpha_{\mu} t_{\mu}^{p,q},
\]
where

\[ \alpha_{n\mu} = \begin{cases} \frac{P_\mu}{R_n} \sum_{\nu=\mu}^{\mu} p_{n-\nu} Q_\nu c_{\nu-\mu} & (\mu \leq n) \\ 0 & (\mu > n) \end{cases} \]

By Lemma 1, it is sufficient to show that

\[ J_\rho = \sum_{n=\rho}^{\infty} \left| \sum_{\mu=\rho}^{n} (\alpha_{n\mu} - \alpha_{n-1,\mu}) \right| = O(1) \quad (\rho = 0, 1, 2, \ldots). \]

By (13), we get for \( n > \mu \),

\[ \alpha_{n\mu} = -\frac{P_\mu}{R_n} \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu} c_{\nu-\mu} \]

and for \( n > \mu + 1 \),

\[ \alpha_{n-1,\mu} = -\frac{P_\mu}{R_n-1} \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu-1} c_{\nu-\mu}. \]

Hence, for \( n > \rho \),

\[ \sum_{\mu=\rho}^{n} (\alpha_{n\mu} - \alpha_{n-1,\mu}) = \sum_{\mu=0}^{\rho-1} (\alpha_{n-1,\mu} - \alpha_{n\mu}) \]

\[ = \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) \left( \frac{p_{n-\nu}}{R_n} - \frac{p_{n-\nu-1}}{R_n-1} \right) c_{\nu-\mu}. \]

Thus,

\[ J_\rho \leq |\alpha_{\rho\rho}| + \sum_{n=\rho+1}^{\infty} \frac{1}{R_n-1} \left| \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) (p_{n-\nu-1} - p_{n-\nu}) c_{\nu-\mu} \right| \]

\[ + \sum_{n=\rho+1}^{\infty} \frac{r_n}{R_n R_n-1} \left| \sum_{\mu=0}^{\rho-1} P_\mu \sum_{\nu=\mu}^{n-1} (Q_n - Q_\nu) p_{n-\nu} c_{\nu-\mu} \right| \]

\[ = J_\rho^{(1)} + J_\rho^{(2)} + J_\rho^{(3)}, \quad \text{say.} \]

Using (7),

\[ J_\rho^{(1)} = |\alpha_{\rho\rho}| = \frac{P_\rho}{R_\rho} p_0 Q_\rho c_0 \leq K. \]
By Lemma 4(9), we have

\[ J_{p}^{(3)} \leq \sum_{n=\rho+1}^{\infty} \frac{r_{n}}{R_{n}R_{n-1}} \sum_{\mu=0}^{\rho-1} P_{\mu}Q_{\mu} \]

\[ \leq P_{\rho}Q_{\rho} \sum_{n=\rho+1}^{\infty} \left( \frac{1}{R_{n-1}} - \frac{1}{R_{n}} \right) \leq \frac{P_{\rho}Q_{\rho}}{R_{\rho}} = O(1). \]

Next,

\[ J_{p}^{(2)} \leq \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{\mu=0}^{\rho-1} P_{\mu} \sum_{v=\mu}^{\rho-1} (Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})c_{v-\mu} \right| \]

\[ + \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{\mu=0}^{\rho-1} P_{\mu} \sum_{v=\rho}^{n-1} (Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})c_{v-\mu} \right| \]

\[ = J_{p1}^{(2)} + J_{p2}^{(2)}, \quad \text{say.} \]

Since \( \sum_{\mu=0}^{\rho} P_{\mu}c_{\rho-\mu} = 1 \), using (7) and (11), we get

\[ J_{p1}^{(2)} = \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \left| \sum_{v=0}^{\rho-1} (Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v}) \sum_{\mu=0}^{\rho-1} P_{\mu}c_{\rho-\mu} \right| \]

\[ = \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \sum_{v=0}^{\rho-1} (Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v}) \]

\[ \leq K \sum_{\nu=0}^{\rho-1} \sum_{n=\rho+1}^{\infty} \frac{(Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})}{Q_{n}P_{n-1}} = O(1). \]

Lastly, using (1), (7) and (11), we obtain

\[ J_{p2}^{(2)} \leq \sum_{n=\rho+1}^{\infty} \frac{1}{R_{n-1}} \sum_{\mu=0}^{\rho-1} P_{\mu} \sum_{v=\rho}^{n-1} (Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})|c_{\rho-\mu}| \]

\[ = \sum_{\mu=0}^{\rho-1} P_{\mu} \sum_{v=\rho}^{\infty} |c_{\rho-\mu}| \sum_{n=\rho+1}^{\infty} \frac{(Q_{n} - Q_{v})(p_{n-v-1} - p_{n-v})}{R_{n-1}} \]

\[ = O\left( \frac{1}{\rho + 1} \right) \sum_{\mu=0}^{\rho-1} P_{\mu} \sum_{v=\rho}^{\infty} |c_{\rho-\mu}| = O(1). \]

This completes the proof of Theorem 2.
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