THE SPACING OF THE MINIMA IN CERTAIN CUBIC LATTICES

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Let \( \mathcal{X} \) be a cubic field with negative discriminant; let \( \mu, \nu \in \mathcal{X} \); and let \( \mathcal{R} \) be a lattice with basis \( \{1, \mu, \nu\} \) such that 1 is a minimum of \( \mathcal{R} \). If

\[
1 = \theta_1, \theta_2, \theta_3, \ldots, \theta_n, \ldots
\]

is a chain of adjacent minima of \( \mathcal{R} \) with \( \theta_{i+1} > \theta_i \) (\( i = 1, 2, 3, \ldots \)), then

\[
\theta_{n+5} \geq \theta_{n+3} + \theta_n.
\]

This result can be used to prove that if \( p \) is the period of Voronoi's continued fraction algorithm for finding the fundamental unit \( \varepsilon_0 \) of \( \mathcal{X} \), then

\[
\varepsilon_0 > \tau^{p/2},
\]

where \( \tau = (1 + \sqrt{5})/2 \). It is also shown that

\[
\theta_n > 4^{(n-1)/7}.
\]

1. Introduction. In order to discuss the problems considered in this paper, it is necessary to give a brief description of the properties of cubic lattices. For a more extensive and more general treatment of these topics we refer the reader to Delone and Faddeev [1].

Let \( f(x) \in \mathbb{Z}[x] \) be a cubic polynomial, irreducible over the rationals \( \mathbb{Q} \) and having a negative discriminant. Let \( \delta \) be the real zero of \( f(x) \) and denote by \( \mathcal{X} = \mathbb{Q}(\delta) \) the complex cubic field formed by adjoining \( \delta \) to \( \mathbb{Q} \). If \( \mathbb{E}_3 \) denotes Euclidean 3-space, we can associate with each \( \alpha \in \mathcal{X} \) a point \( A \in \mathbb{E}_3 \), where

\[
A = \left( \alpha, \left( \alpha' - \alpha'' \right)/2i, \left( \alpha' + \alpha'' \right)/2 \right),
\]

\( i^2 + 1 = 0 \), and \( \alpha', \alpha'' \) are the conjugates of \( \alpha \). Since \( f(x) \) has a negative discriminant, all three components of \( A \) must be real. If \( \lambda, \mu, \nu \in \mathcal{X} \) and \( \lambda, \mu, \nu \) are rationally independent, we define the cubic lattice \( \mathcal{L} \) by

\[
\mathcal{L} = \{ u\Lambda + vM + wN | (u, v, w) \in \mathbb{Z}^3 \}.
\]

We say that \( \mathcal{L} \) has a basis \( \{ \lambda, \mu, \nu \} \) and denote \( \mathcal{L} \) by \( \langle \lambda, \mu, \nu \rangle \). For the sake of convenience we will often use the expression \( \alpha \in \mathcal{L} \) to denote that it is the corresponding point \( A \in \mathbb{E}_3 \) that is actually in \( \mathcal{L} \). Also, if \( \mathcal{L} = \langle \lambda, \mu, \nu \rangle \), we define \( \alpha \mathcal{L} \) (\( \alpha \in \mathcal{X}^* \)) to be the lattice \( \langle \alpha \lambda, \alpha \mu, \alpha \nu \rangle \).
If $A$ is any point of $\mathcal{L}$, we define the normed body of $A$ to be

$$
\mathcal{N}(A) = \mathcal{N}(\alpha) = \{(x, y, z) | (x, y, z) \in \mathbb{R}^3, |x| < |\alpha|, y^2 + z^2 \leq |\alpha'|^2\}.
$$

This is a semi-open right circular cylinder, symmetric about the origin $O$ of $\mathbb{R}^3$, with axis the $x$-axis of $\mathbb{R}^3$. It should be mentioned at this point that if $\alpha, \beta \in \mathcal{X}$ and $|\alpha'| = |\beta'|$, then $\alpha = \pm \beta$ (see [1], p. 274). Thus, if $|\beta'| = |\alpha'|$, then $B \not\in \mathcal{N}(\alpha)$.

We say that $\phi \neq 0 \in \mathcal{X}$ or the point $\Phi$ corresponding to $\phi$ is a minimum of $\mathcal{L}$ if $\mathcal{N}(\phi) \cap \mathcal{L} = \{0\}$. If $\psi$ and $\phi$ are minima of $\mathcal{L}$ and $\psi > \phi$, we say that $\psi$ and $\phi$ are adjacent minima when there does not exist a non-zero $\chi \in \mathcal{L}$ such that

$$\phi < \chi < \psi \quad \text{and} \quad |\chi'| < |\phi'|.$$ 

If

$$\theta_1, \theta_2, \theta_3, \ldots, \theta_n, \ldots$$

is a sequence of minima of $\mathcal{L}$ such that $\theta_{i+1} > \theta_i$ and $\theta_{i+1}, \theta_i$ are adjacent ($i = 1, 2, 3, \ldots, n, \ldots$), we call (1.1) a chain of minima of $\mathcal{L}$. By using Minkowski's theorem (see [1]) we can prove that such chains always exist in $\mathcal{L}$.

If $\mathcal{R} = \langle 1, \mu, \nu \rangle$ and 1 is a minimum of $\mathcal{R}$, we say that $\mathcal{R}$ is a reduced lattice. In this paper we shall be concerned with the problem of how closely spaced the minima of $\mathcal{R}$ can be. We will show that if $\theta_1 = 1$ and $\theta_4 < \theta_2 + 1$, then $\theta_2 + \theta_3 = \theta_4 + 1$. We can use this result to prove that if $\varepsilon_0$ is the fundamental unit of $\mathcal{X}$, then

$$\varepsilon_0 > \tau^{p/2},$$

where $p$ is the period of Voronoi's continued fraction algorithm for finding $\varepsilon_0$ and $\tau = (1 + \sqrt{5})/2$. We will also show that $\theta_5 \geq \theta_3 + 1 > 2$ and $\theta_8 > 4$. The methods used to prove these results are completely elementary.

2. Preliminary results. From [1] or Williams and Dueck [3] we see that if $\mathcal{R}_1 = \mathcal{R}$ (a reduced lattice), $\theta_{m}^{(m)}$ is the minimum of $\mathcal{R}_m$ adjacent to 1 and $\mathcal{R}_{m+1}$ is defined to be $(1/\theta_{m}^{(m)}) \mathcal{R}_m$, then $\theta_n \mathcal{R}_n = \mathcal{R}_1$, where $\mathcal{R}_n$ is a reduced lattice and

$$\theta_n = \prod_{i=1}^{n-1} \theta_{g}^{(i)}.$$
We shall need to make use of these results together with several others established in [3]; however, we first give some simple lemmas concerning points of $\mathcal{R}$. Throughout this work we will use $\theta$ to denote the minimum of $\mathcal{R}$ adjacent to 1, $\omega$ to denote the minimum of $\mathcal{R}$ adjacent to $\theta$, and $\chi$ to denote the minimum of $\mathcal{R}$ adjacent to $\omega$. That is, $\theta = \theta_2$, $\omega = \theta_3$, $\chi = \theta_4$. Note that if $\gamma \in \mathcal{R}$, $|\gamma| < \theta$, and $|\gamma'| \leq 1$, we must have $\gamma = 0$ or $\gamma = \pm 1$. We also have

**Lemma 2.1.** If $\alpha \in \mathcal{R}$ and $0 < \alpha < \theta + 1$, then either $\alpha = 1, 2$ or $|\alpha' - 1| > 1$. Further, if $\alpha, \beta \in \mathcal{R}$, $\alpha \neq \beta$, and $\beta < \alpha$, $\beta < \theta + 1$, then $|\alpha' - \beta'| > 1$.

**Proof.** We have $-1 < \alpha - 1 < \theta$; thus, if $|\alpha' - 1| \leq 1$, we get $\alpha - 1 = 0, 1$. Since $\theta < \alpha$, $\beta < \theta + 1$, we have $|\alpha - \beta| < 1$. It follows that if $|\alpha' - \beta'| < 1$, then $\alpha = \beta$. If $|\alpha' - \beta'| = 1$, then $\alpha = \beta \pm 1$, which is also impossible. □

From this result we see that $|\theta' - 1| > 1$ and if $\chi < \theta + 1$, then $|\omega' - 1| > 1$ and $|\chi' - 1| > 1$.

In order to develop further results we define

\[(2.2) \quad \eta_\alpha = (\alpha' - \alpha'')/2i, \quad \xi_\alpha = (\alpha' + \alpha'')/2\]

for any $\alpha \in \mathcal{R}$. Note that

\[(2.3) \quad |\alpha'|^2 = |\alpha''|^2 = \alpha'\alpha'' = \eta_\alpha^2 + \xi_\alpha^2.\]

Also, if $\alpha \in \mathcal{R}$ and $\eta_\alpha \in \mathbb{Z}$, then $\alpha \in \mathbb{Z}$ and $\eta_\alpha = 0$ (see [3]). Hence, $\eta_\alpha \neq 0$ if $\alpha = \theta_i (i > 1)$.

**Lemma 2.2.** If $\alpha, \beta \in \mathcal{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $|\beta' - \alpha' + 1| > 1$, then $|\beta' - \alpha' + 2| > 1$.

**Proof.** Since $|\beta'| < 1$, we have $\xi_\beta > -1$ by (2.3). Further, since $|\alpha'| < 1$ and $|\alpha' - 1| > 1$, we must have $\xi_\alpha < 1/2$; thus, $\xi_\beta - \xi_\alpha + 1 > -1/2$ and

\[|\beta' - \alpha' + 2|^2 = |\beta' - \alpha' + 1|^2 + 2(\xi_\beta - \xi_\alpha + 1) + 1 > 1. \quad \Box\]

**Lemma 2.3.** If $\alpha, \beta \in \mathcal{R}$, $|\alpha' - 1| > 1$, $|\alpha' + 1| > 1$, $|\alpha'| < 1$, $|\beta'| < 1$, $\eta_\beta \eta_\alpha > 0$, and $|\beta' - \alpha'| > 1$, then $|\eta_\alpha| > |\eta_\beta|$.

**Proof.** Suppose $|\eta_\alpha| \leq |\eta_\beta|$ and consider Figure 1.
Here $P = (|\eta_\beta|, \xi_\beta)$, $Q = (|\eta_\beta|, \xi_\beta)$. Let the chord through $P$ parallel to $AC$ meet the circle $ABCD$ (radius 1, centre $O$) at $L$ and $M$. Since $|\alpha' + 1| > 1$, we have $PL < 1$; also, since $|\alpha' - 1| > 1$, we have $PM < 1$. Since $PQ < \max(PL, PM)$, we get $PQ = |\beta' - \alpha'| < 1$, a contradiction. \hfill \Box

In the next sequence of lemmas we prove a number of results concerning points $\alpha \in \mathcal{R}$ such that $|\alpha'| < 1$. We first define $\kappa(\alpha)$ for $\alpha \in \mathcal{R}$ by

\begin{equation}
(2.4) \quad \kappa(\alpha) = (\xi_\alpha - 1/2)^2 + (\sqrt{3}/2 - |\eta_\alpha|)^2
= \xi_\alpha^2 - \eta_\alpha + \eta_\alpha^2 - \sqrt{3}|\eta_\alpha| + 1.
\end{equation}

**Lemma 2.4.** If $\alpha \in \mathcal{R}$, $|\alpha'| < 1$, and $\kappa(\alpha) \geq 1$, then $\xi_\alpha \leq 0$, $|\eta_\alpha| \leq \sqrt{3}/2$, and $|\xi_\alpha| \geq |\eta_\alpha|/\sqrt{3}$.  

**Proof.** Since $|\alpha'| < 1$, we have $|\eta_\alpha| < 1$ and $|\xi_\alpha| < 1$; thus,

$$-\sqrt{3}/2 < \sqrt{3}/2 - 1 < \sqrt{3}/2 - |\eta_\alpha| < \sqrt{3}/2,$$

and $(\xi_\alpha - 1/2)^2 \geq 1/4$ by (2.4). If $0 < \xi_\alpha < 1$, this latter result is not possible; hence, $\xi_\alpha \leq 0$. If $|\eta_\alpha| > \sqrt{3}/2$, then $|\xi_\alpha| < 1/2$ by (2.3) and the fact that $|\alpha'| < 1$; thus, by (2.4)

$$\kappa(\alpha) < -1/2 + \xi_\alpha^2 + \eta_\alpha^2 - \xi_\alpha < 1/2 - \xi_\alpha < 1,$$
which is also not possible. Since $|\eta_a| < \sqrt{3}/2$, we have $|\eta_a| < \frac{3(\sqrt{3} - 1/\sqrt{3})}{4}$ and
\[
\left( \frac{|\eta_a|}{\sqrt{3}} + 1/2 \right)^2 + \left( \frac{\sqrt{3}}{2} - |\eta_a| \right)^2 \leq 1.
\]
It follows that since $\kappa(\alpha) \geq 1$, we must have $|\xi_a| \geq |\eta_a|/\sqrt{3}$ by (2.4).

**Corollary 2.4.1.** If $\alpha \in \mathcal{R}$, $|\alpha'| < 1$, and $\kappa(\alpha) \geq 1$, then $|\alpha' + 1| \leq 1$.

**Proof.** By the lemma, $1 - |\eta_a|/\sqrt{3} > 0$ and $0 < \xi_a + 1 \leq 1 - |\eta_a|/\sqrt{3}$. Thus,
\[
|\alpha' + 1|^2 = (\xi_a + 1)^2 + \eta_a^2 \leq (1 - |\eta_a|/\sqrt{3})^2 + \eta_a^2 \leq 1
\]
as $|\eta_a| < \sqrt{3}/2$.

**Lemma 2.5.** If $\alpha, \beta \in \mathcal{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $\eta_a \eta_\beta > 0$, $\kappa(\alpha) < 1$, and $|\alpha' - \beta'| > 1$, then $\kappa(\beta) > 1$.

**Proof.** The point $(\eta_a, \xi_a)$ must lie in the Reuleaux triangle (see [3]) with vertices $O$ (the origin), $(\sigma\sqrt{3}/2, 1/2)$, $(\sigma\sqrt{3}/2, -1/2)$, where $\sigma = \text{sgn}(\eta_a)$. If $\kappa(\beta) \leq 1$, then $(\eta_\beta, \xi_\beta)$ is in the same Reuleaux triangle as $(\eta_a, \xi_a)$; hence, $|\alpha' - \beta'| \leq 1$, which is impossible.

**Lemma 2.6.** If $\alpha, \beta \in \mathcal{R}$, $|\alpha'| < 1$, $|\beta'| < 1$, $|\alpha' - 1| > 1$, $|\alpha' + 1| > 1$, and $\kappa(\beta) \geq 1$, then $|1 - \alpha' - \beta'| > 1$.

**Proof.** Since $|\alpha'| < 1$, $|\alpha' + 1| > 1$, and $|\alpha' - 1| > 1$, we have $|\xi_a| < 1/2$ and $1 - 2\xi_a > 0$. Since $|\alpha' - 1| > 1$ and $\kappa(\beta) \geq 1$, we also have
\[
\text{(2.5)} \quad |1 - \alpha' - \beta'|^2 = 1 + \xi_\beta^2 - 2\xi_\beta + \eta_\beta^2 + 2\xi_a\xi_\beta + 2\eta_a\eta_\beta + \xi_a^2 - 2\xi_a + \eta_a^2
\]
\[
> 1 + \xi_\beta(-1 + 2\xi_a) + 2\eta_a\eta_\beta + \sqrt{3}|\eta_\beta|
\]
by (2.4) and the fact that $|\alpha' - 1| > 1$. By Lemma 2.4, we have $\xi_a \leq 0$; hence, if $\eta_a \eta_\beta \geq 0$, we get $|1 - \alpha' - \beta'| > 1$. If $\eta_a \eta_\beta < 0$, then from (2.5) and Lemma 2.4 we get
\[
|1 - \alpha' - \beta'|^2 > 1 + |\eta_\beta||(1 - 2\xi_a)/\sqrt{3} + \sqrt{3} - 2|\eta_a|).
\]
Since $\xi_a < \sqrt{1 - \eta_a^2}$, we have
\[
(1 - 2\xi_a)/\sqrt{3} + \sqrt{3} - 2|\eta_a| > (1 - 2\sqrt{1 - \eta_a^2})/\sqrt{3} + \sqrt{3} - 2|\eta_a|.
\]
But \( \frac{2}{\sqrt{3}} > 1 > |\eta_\alpha| \) and \( \sqrt{1 - \eta_\alpha^2}/\sqrt{3} < 2/\sqrt{3} - |\eta_\alpha| \); hence,
\[
\left(1 - 2\sqrt{1 - \eta_\alpha^2}\right)/\sqrt{3} + \sqrt{3} - 2|\eta_\alpha| > 0.
\]

**Corollary 2.6.1.** If \( \alpha, \beta \in \mathbb{R} \), \( |\alpha'| < 1 \), \( |\beta'| < 1 \), \( |\alpha' - 1| > 1 \), \( |\beta' - 1| > 1 \), \( |\alpha' + 1| > 1 \), \( |\beta' - \alpha'| > 1 \), and \( |\beta' - \alpha' + 1| < 1 \), then \( \kappa(\gamma) < 1 \), where \( \gamma = \beta - \alpha + 1 \).

**Proof.** We have \( |\gamma'| < 1 \); thus, if \( \kappa(\gamma) \geq 1 \), then \( |1 - \alpha' - \gamma'| = |\beta'| > 1 \), which is not so.

We will also require some lemmas whose proofs have already appeared in [3]. We will only give the statements of these results here; however, we mention that the proofs of these lemmas are elementary and require, for the most part, only results from simple plane geometry.

**Lemma 2.7 (Lemma 6.1 of [3]).** If \( \alpha, \beta \in \mathbb{R} \), \( |\alpha'| < 1 \), \( |\beta'| < 1 \), and \( 2\alpha = \beta + 1 \), then \( |\alpha' - 1| \leq 1 \).

**Lemma 2.8 (Lemma 5.4 of [3]).** If \( \alpha, \beta \in \mathbb{R} \), \( |\alpha'| < 1 \), \( |\beta'| < 1 \), \( |\alpha' - 1| > 1 \), \( |\beta' - 1| > 1 \), \( \eta_\alpha\eta_\beta > 0 \), \( |\alpha' - \beta'| > 1 \), and \( |\alpha' + 1| > 1 \) \((< 1)\), then \( |\beta' + 1| < 1 \) \((> 1)\).

**Lemma 2.9 (Lemma 6.2 of [3]).** Let \( \alpha, \beta, \gamma \in \mathbb{R} \), where \( \alpha, \beta, \gamma \) are distinct, \( |\alpha'| < 1 \), \( |\beta'| < 1 \), \( |\gamma'| < 1 \), and \( |\alpha' - 1| > 1 \), \( |\beta' - 1| > 1 \), \( |\gamma' - 1| > 1 \). If \( \eta_\alpha\eta_\beta > 0 \) and \( \eta_\beta\eta_\gamma > 0 \), there cannot exist any \( b \) such that
\[
|b| \leq |\alpha, \beta, \gamma| < b + 1.
\]

**Lemma 2.10 (Lemma 6.3 of [3]).** Let \( \alpha, \beta \in \mathbb{R} \) such that \( |\alpha'| < 1 \), \( |\beta'| < 1 \), \( \beta > \alpha > 1 \), and \( |\beta'| < |\alpha'| \). If \( \eta_\alpha\eta_\beta > 0 \) and \( |\alpha' + 1| \leq 1 \), then \( \beta \geq \alpha + 1 \).

**Lemma 2.11 (Lemma 6.5 of [3]).** Let \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( |\alpha'| < 1 \), \( |\beta'| < 1 \), \( |\gamma'| < 1 \), \( |\alpha' - 1| > 1 \), \( |\beta' - 1| > 1 \), \( |\gamma' - 1| > 1 \), \( |\beta' + 1| \leq 1 \), \( \eta_\alpha\eta_\beta < 0 \), \( \eta_\beta\eta_\gamma > 0 \). If \( |\beta' - \alpha'| > 1 \) and \( |\beta' - \gamma' + 1| > 1 \), then either \( |\alpha' - \beta'| \leq 1 \) or \( |\alpha' - \beta' + \gamma' - 1| \leq 1 \).

**3. The main results.** We are now able to use the lemmas of §2 to prove our main results. We first prove an extension of Lemma 2.11.

**Theorem 3.1.** If \( \alpha, \beta, \gamma \in \mathbb{R} \), \( |\alpha'| < 1 \), \( |\beta'| < 1 \), \( |\gamma'| < 1 \), \( |\alpha' - 1| > 1 \), \( |\beta' - 1| > 1 \), \( |\gamma' - 1| > 1 \), \( |\beta' + 1| \leq 1 \), \( \eta_\alpha\eta_\beta < 0 \), \( \eta_\beta\eta_\gamma > 0 \), and \( |\beta' - \gamma'| > 1 \), then either \( |\alpha' - \beta'| < 1 \) or \( |\alpha' - \gamma'| \leq 1 \), where \( \lambda = \beta - \gamma + 1 \).
Proof. If |λ'| > 1, the result follows from Lemma 2.11. Note that since \( \eta_\alpha \eta_\beta < 0 \), we cannot have |\( \alpha' - \beta' \) = 1, for this would imply that \( \alpha = \beta \pm 1 \) and \( \eta_\alpha = \eta_\beta \). Similarly |\( \alpha' - \gamma' \) \( \neq 1 \). If |\( \lambda' \) = 1, then \( \beta = \gamma = 2 \). Since |\( \beta' - \gamma' \) > 0 and |\( \beta' \) < 1, |\( \gamma' \) < 1, neither of these is possible.

If |\( \lambda' \) < 1, we will consider two cases; however, we first notice that |\( \gamma' + 1 \) > 1 by Lemma 2.8 and \( \eta_\gamma \eta_\lambda = \eta_\gamma (\eta_\beta - \eta_\gamma) = |\eta_\gamma| (|\eta_\beta| - |\eta_\gamma|) < 0 \) by Lemma 2.3. Also \( \kappa(\lambda) < 1 \) by Corollary 2.6.1.

Case 1 (\( \kappa(\alpha) < 1 \)). In this case we see from Lemma 2.5 that |\( \alpha' - \lambda' \) ≤ 1.

Case 2 (\( \kappa(\alpha) ≥ 1 \)). In this case we have |\( \alpha' + 1 \) < 1 by Corollary 2.4.1. Suppose |\( \alpha' - \gamma' \) > 1 and |\( \alpha' - \lambda' \) > 1. Since |\( \beta' - \gamma' \) > 1, we have |\( \lambda' - 1 \) > 1; thus, |\( \lambda' + 1 \) > 1 by Lemma 2.8. If \( \rho = \alpha - \lambda + 1 \) and |\( \rho' \) > 1, then either |\( \gamma' - \rho' \) ≤ 1 or |\( \gamma' - \alpha' \) ≤ 1 by Lemma 2.11. Since \( \gamma - \rho = \beta - \alpha \), we get |\( \beta' - \alpha' \) < 1. If |\( \rho' \) = 1, then \( \alpha = \lambda \) or \( \alpha = \lambda - 2 \) and, as above, neither of these is possible. If |\( \rho' \) < 1, then \( \kappa(\rho) < 1 \) and also \( \eta_\rho \eta_\gamma > 0 \) (Corollary 2.6.1 and Lemma 2.3). Since \( \kappa(\gamma) < 1 \) by Corollary 2.4.1, we get |\( \gamma' - \rho' \) ≤ 1 by Lemma 2.5.

We are now able to show that if \( \theta_4 < \theta_2 + 1 \), then \( \theta_4 + 1 = \theta_2 + \theta_3 \).

Theorem 3.2. If \( \chi < \theta + 1 \), then \( \eta_\theta \eta_\omega < 0 \), |\( \chi' + 1 \) | < 1, \( \kappa(\theta) < 1 \), \( \kappa(\omega) < 1 \), and \( \chi + 1 = \theta + \omega \).

Proof. We first note that |\( \theta' \) < 1, |\( \omega' \) | < 1, |\( \chi' \) | < 1, and |\( \theta' - 1 \) | > 1, |\( \omega' - 1 \) | > 1, |\( \chi' - 1 \) | > 1. Also, if \( \rho_1, \rho_2 \in \{ \theta, \omega, \chi \} \) and \( \rho_1 \neq \rho_2 \), then |\( \rho_1 - \rho_2 \) | > 1 by Lemma 2.1.

Case 1 (\( \eta_\theta \eta_\omega > 0 \)). By Lemma 2.4 we must have \( \eta_\theta \eta_\chi < 0 \). Further, by Lemma 2.10, we must also have |\( \theta' + 1 \) | > 1. By Theorem 3.1, we get |\( \rho' \) | ≤ 1, where \( \rho = \chi - \omega + \theta - 1 \). Now \( 0 < \rho < \theta \); thus, \( \rho = 1 \) and \( \chi = \omega - \theta + 2 \). Since \( \omega - \theta + 1 = \chi - 1 \), we have |\( \omega' - \theta' + 1 \) | > 1; consequently, |\( \chi' \) | = |\( \omega' - \theta' + 2 \) | > 1 by Lemma 2.2. It follows that we must have

Case 2 (\( \eta_\theta \eta_\omega < 0 \)). Here we have \( \eta_\chi \eta_\theta > 0 \) or \( \eta_\chi \eta_\omega > 0 \). In either case, by Lemma 2.10 we get |\( \chi' + 1 \) | < 1. If \( \eta_\chi \eta_\omega > 0 \), then |\( \omega' + 1 \) | > 1 by Lemma 2.8 and \( \kappa(\omega) < 1 \) by Corollary 2.4.1. Also, by Theorem 3.1 |\( \rho' \) | ≤ 1, where \( \rho = \theta - \chi + \omega - 1 \). Since \( -1 < \rho < \theta \), we get \( \rho = 0 \) or 1.
As before, we cannot have \( \rho = 1 \); hence, \( \rho = 0 \) and \( \chi + 1 = \theta + \omega \). Since \( \theta = \chi - \omega + 1 \), we get \( \kappa(\theta) < 1 \) from Corollary 2.6.1. Similarly, if \( \eta_\chi \eta_\theta > 0 \), then \( \kappa(\theta) < 1, \kappa(\omega) < 1 \), and \( \chi + 1 = \theta + \omega \).

By using the remarks at the beginning of §2, we can extend this result to show that if

\[
\theta_{n+3} < \theta_{n+1} + \theta_n
\]
in (1.1), then

\[
\theta_{n+3} + \theta_n = \theta_{n+1} + \theta_{n+2}.
\]

We can also improve two of the results of Theorem 3.2 in

**Lemma 3.3.** If \( \chi < \theta + 1 \), then \( |\theta' + 1| > 1 \) and \( |\omega' + 1| > 1 \).

**Proof.** If \( |\theta' + 1| \leq 1 \), then \( \xi_\theta \leq 0 \) and

\[
-2\xi_\theta \geq \xi_\theta^2 + \eta_\theta^2 > \xi_\omega^2 + \eta_\omega^2 > 2\xi_\omega \quad (|\omega' - 1| > 1).
\]

It follows that \( \xi_\theta + \xi_\omega < 0 \) and, as a consequence, \( |\chi'| = |\theta' + \omega' - 1| > 1 \), which is impossible.

If \( |\omega' + 1| \leq 1 \), then \( \xi_\omega \leq 0 \) and

\[
|\eta_\omega| \leq \sqrt{1 - (1 - |\xi_\omega|)^2}.
\]

Since

\[
2|\xi_\omega| = \xi_\omega^2 + 1 - (1 - |\xi_\omega|)^2,
\]

we get

\[
2|\xi_\omega| \geq 1 - (1 - |\xi_\omega|)^2 > |\eta_\omega|(1 - (1 - |\xi_\omega|)^2).
\]

Also, since

\[
\left( |\eta_\theta|\sqrt{2|\xi_\omega|} - \sqrt{1 - (1 - |\xi_\omega|)^2} \right)^2 \geq 0,
\]

we see, using (3.2), that

\[
(1 - \eta_\theta^2)\xi_\omega^2 \leq \left( \sqrt{2|\xi_\omega|} - |\eta_\theta|\sqrt{1 - (1 - |\xi_\omega|)^2} \right)^2
\]

and

\[
|\xi_\omega|\sqrt{1 - \eta_\theta^2} + |\eta_\theta|\sqrt{1 - (1 - |\xi_\omega|)^2} \leq \sqrt{2|\xi_\omega|}
\]

by (3.3). Now \( \xi_\theta < \sqrt{1 - \eta_\theta^2} \); hence from (3.1) we get

\[
|\xi_\theta||\xi_\omega| + |\eta_\omega||\eta_\theta| - |\xi_\omega| < \sqrt{2|\xi_\omega|} - |\xi_\omega| \leq 1/2.
\]
Since $\xi_\omega \leq 0$ and $\eta_\omega \eta_\theta < 0$, we find that

$$-2\xi_\omega + 2\xi_\omega \xi_\theta + 2\eta_\omega \eta_\theta > -1.$$ 

But

$$|\chi'|^2 - |\omega'|^2 = |\theta' + \omega' - 1|^2 - |\omega'|^2$$

$$= |\theta' - 1|^2 - 2\xi_\omega + 2\xi_\omega \xi_\theta + 2\eta_\theta \eta_\omega;$$

thus, since $|\theta' - 1| > 1$, we have $|\chi'| > |\omega'|$ when $|\omega' + 1| \leq 1$ and this is impossible.  

We will also need to make use of the following result and its corollaries.

**Theorem 3.4.** If $\chi < \theta + 1$ and there exists some $\rho \in \mathcal{R}$ such that $\rho \notin \{\theta, \omega, \chi\}$, $|\rho'| < 1$, $|\rho' - 1| > 1$, then $|\rho' - \psi'| < 1$ for some $\psi \in \{\theta, \omega, \chi\}$.  

**Proof.** Suppose that there exists some $\rho \in \mathcal{R}$ such that $\rho \notin \{\theta, \omega, \chi\}$, $|\rho'| < 1$, $|\rho' - 1| > 1$, and $|\rho' - \psi'| \geq 1$ for each $\psi \in \{\theta, \omega, \chi\}$. We first note that if $|\rho' - \psi'| = 1$, then $\rho = \psi + 1$. If $\rho = \psi - 1$, then $0 < \rho < \theta$, which contradicts the definition of $\theta$. If $\rho = \psi + 1$, then $|\rho' - 1| = |\psi'| < 1$, which is also impossible. Thus, $|\rho' - \psi'| > 1$ for all $\psi \in \{\theta, \omega, \chi\}$. Since $\eta_\theta \eta_\omega < 0$, $|\theta' + 1| > 1$, $|\omega' + 1| > 1$, we must have $|\rho' + 1| < 1$ (Lemma 2.8). Put $\alpha$ equal to that one of $\theta$ or $\omega$ such that $\eta_\theta \eta_\rho < 0$ and let $\beta$ be the other one. We have $\alpha + \beta = \theta + \omega = \chi + 1$. Further, $|\rho' - \alpha'| > 1$ and $|\alpha' + 1| > 1$; thus, by Theorem 3.1, we get $|\beta' - \lambda'| \leq 1$, where $\lambda = \rho - \alpha + 1$. Since $\beta - \lambda = \beta - \rho + \alpha - 1 = \chi - \rho$, this is impossible.  

**Corollary 3.4.1.** If $\chi < \theta + 1$ and there exists $\rho \in \mathcal{R}$ such that $\rho \in \{\theta, \omega, \chi\}$, $|\rho'| < 1$, and $|\rho| < \theta + 1$, then $\rho = 0$.  

**Proof.** Since $|-\rho'| = |\rho'|$, we may assume with no loss of generality that if $\rho \neq 0$, then $\rho > 0$. Since $|\rho'| < 1$, we must have $\theta < \rho < \theta + 1$. Thus, by Lemma 2.1, $|\rho' - \psi'| \geq 1$ for all $\psi \in \{\theta, \omega, \chi\}$, which is impossible by the theorem.  

**Corollary 3.4.2.** If $\chi < \theta + 1$, there does not exist any $\rho \in \mathcal{R}$ such that $|\rho'| < 1$ and $\chi < \rho < \chi + 1$. 


Proof. Suppose such a $\rho$ does exist. If $|\rho' - 1| < 1$, then, since $|\rho - 1| < \theta + 1$, we can only have $\rho - 1 \in \{\theta, \omega, \chi\}$ by the previous result. Since $\rho \neq \chi + 1$, $|\theta' + 1| > 1$, $|\omega' + 1| > 1$, we must have $|\rho' - 1| > 1$ and, as a consequence, $|\rho' - \psi'| < 1$ for some $\psi \in \{\theta, \omega, \chi\}$. Since $0 < \rho - \chi < \chi + 1 - \chi \leq \omega$, we find by the previous corollary that $\rho - \psi = \theta$. If $\psi = \omega$ or $\chi$, then $\rho \geq \chi + 1$; thus, $\psi = \theta$ and $\rho = 2\theta$. If $\rho = 2\theta$, then $|\omega'| < |\theta'| < 1/2$ and $|\omega' - \theta'| < 1$, which is impossible. 

Let $\rho = \theta_5$, the minimum adjacent to $\chi = \theta_4$. We can now show the following unconditional result concerning $\rho$.

**Theorem 3.5.** $\rho \geq 1 + \omega$ or $\theta_{n+5} \geq \theta_{n+3} + \theta_n$ in (1.1).

**Proof.** Suppose $\rho < 1 + \omega$ and let $R^* = (1/\theta) R$. If $\theta^* = \omega/\theta$, $\omega^* = \chi/\theta$, $\chi^* = \rho/\theta$, then $\theta^*$ is the minimum adjacent to 1 in $R^*$, $\omega^*$ is the minimum adjacent to $\theta^*$, and $\chi^*$ is the minimum adjacent to $\omega^*$. Since $\rho < 1 + \omega$, we have $\chi^* < (1 + \omega)/\theta < \omega/\theta + 1 = \theta^* + 1$. By Theorem 3.2, we have $\theta^* + \omega^* = \chi^* + 1$ and

$$\omega + \chi = \rho + \theta.$$  

If $\chi \geq \theta + 1$, then $\rho \geq \omega + 1$. If $\chi < \theta + 1$, then $\rho \geq \chi + 1 > \omega + 1$ by Corollary 3.4.2. 

In fact, we actually get cases in which $\rho = 1 + \omega$. For example, consider $D = 239$, $\delta^3 = D$, $R_1 = \langle 1, \delta, \delta^2 \rangle$. In $R = R_{312}$, we get

$$\theta = (6 + 17\delta + 7\delta^2)/247,$$

$$\omega = (74 + 45\delta + 4\delta^2)/247,$$

$$\chi = (253 + 17\delta + 7\delta^2)/247 = \theta + 1,$$

$$\rho = (321 + 45\delta + 4\delta^2)/247 = \omega + 1.$$  

Note also that if $R = R_{313}$ here, we have $\theta = (191 - 3\delta + 7\delta^2)/332$, $\omega = (217 + 47\delta + \delta^2)/332$, $\chi = (76 + 44\delta + 8\delta^2)/332$. In this case $\chi < \theta + 1$ and $\chi = \theta + \omega - 1$. Also, $\rho = (408 + 44\delta + 8\delta^2)/332 = \chi + 1$.

If we let $R_1 = \langle 1, \mu, \nu \rangle$, where $\{1, \mu, \nu\}$ is a basis of the algebraic integers of $\mathcal{R}$, then $R_1$ is a reduced lattice and there exists an integer $p$ such that $R_{p+1} = R_1$. In this case $\varepsilon_0 (> 1)$, the fundamental unit of $\mathcal{R}$, is given by the formula

$$\varepsilon_0 = \theta_{p+1} = \prod_{i=1}^{p} \theta_g^{(i)}.$$
The value \( p \) is called the period of Voronoi's continued fraction algorithm for finding \( \varepsilon_0 \). By using the reasoning similar to that of Pen and Skubenko [2], we can prove

**Theorem 3.6.** If \( p \) is the period of Voronoi's continued fraction algorithm for finding \( \varepsilon_0 \), then \( \varepsilon_0 > \tau^{p/2} \), where \( \tau = (1 + \sqrt{5})/2 \).

**Proof.** If \( \mathcal{R} = \mathcal{R}_i \), then \( \rho \geq \omega + 1 \) and

\[
\theta_g^{(i)}\theta_g^{(i+1)}\theta_g^{(i+2)}\theta_g^{(i+3)} \geq 1 + \theta_g^{(i)}\theta_g^{(i+1)}.
\]

Since \( \mathcal{R}_{p+1} = \mathcal{R}_1, \mathcal{R}_{p+2} = \mathcal{R}_2, \mathcal{R}_{p+3} = \mathcal{R}_3 \), we get \( \theta_g^{(p+1)} = \theta_g^{(1)}, \theta_g^{(p+2)} = \theta_g^{(2)}, \theta_g^{(p+3)} = \theta_g^{(3)} \); thus, we get

\[
\varepsilon_0^4 = \left( \prod_{i=1}^{p} \theta_g^{(i)} \right)^4 = \prod_{i=1}^{p} \theta_g^{(i)}\theta_g^{(i+1)}\theta_g^{(i+2)}\theta_g^{(i+3)} \\
\geq \prod_{i=1}^{p} \left( 1 + \theta_g^{(i)}\theta_g^{(i+1)} \right) \\
\geq \prod_{i=1}^{p} \left( 1 + \left( \prod_{i=1}^{p} \theta_g^{(i)}\theta_g^{(i+1)} \right)^{1/p} \right)^p \\
= \left( 1 + \varepsilon_0^{2/p} \right)^p.
\]

If we put \( \eta = \varepsilon_0^{2/p} > 1 \), then \( \eta^2 \geq \eta + 1 \). It follows that \( \varepsilon_0^{2/p} > \tau \). \( \Box \)

Thus, if \( R \) is the regulator of \( \mathcal{R} \), we have \( R > p(\log \tau)/2 \).

4. **Further results.** In this section we will obtain some results on the spacing of the first few minima of \( \mathcal{R} \). We first require the following technical lemma.

**Lemma 4.1.** If \( \chi < \theta + 1 \), then

(i) \( |\theta'|, |\omega'| > 1/2 \);
(ii) \( |2\omega' + \chi'| > |\omega'|, |2\theta' + \chi'| > |\theta'|, |2\theta' + \omega'| > |\theta'| \);
(iii) \( |\theta' + \chi'| > |\chi'| \);
(iv) \( |2\chi' + \theta'| > |\chi'| \).

**Proof.** (i) The method of proof of (i) is given in the proof of Corollary 3.4.2.

(ii) Since \( |\omega'| > |\chi'| \), we have

\[ |2\omega' + \chi'| \geq 2|\omega'| - |\chi'| > |\omega'|. \]

Similarly, \( |2\theta' + \chi'| > |\theta'| \) and \( |2\theta' + \omega'| > |\theta'| \).
(iii) We note that
\begin{equation}
2\xi _x\xi _\theta + 2\eta _x\eta _\theta = |x'| + 1 - |x' + 1 - \theta'|^2 + |\theta - 1|^2 - 1.
\end{equation}
Since \(\omega = x + 1 - \theta\), we get
\[|\theta' + x'|^2 = |\theta'|^2 + |x'|^2 + |x' + 1|^2 - |\omega'|^2 + |\theta' - 1|^2 - 1.\]
Since \(|\theta'| > |\omega'|\) and \(|\theta' - 1| > 1\), we have
\[|\theta' + x'| > |x'|.\]

(iv) From (4.1) we get
\[2x' + \theta' - Ix' = |2x' + \theta'|^2 - |x'|^2 - |x' + 1|^2 - |\omega'|^2 + 2|x'|^2 + 2|\theta' - 1|^2 - 2.\]
Since
\[|x'|^2 + |x' + 1|^2 = \xi _x + (\xi _x + 1)^2 = \frac{1}{2}(4\xi _x + 4\xi _x + 1) + \frac{1}{2} \geq \frac{1}{2},\]
we get
\[|2x' + \theta'| - |x'| > 0.\]

We are now able to find possible candidates for further minima when \(x < \theta + 1\).

**Lemma 4.2.** If \(x < \theta + 1\), \(x + 1 < \rho < x + 2\), and \(|\rho'| < 1\), then \(\rho \in \{x + \theta, x + \omega, 2x\}\).

**Proof.** Since \(x < \rho - 1 < \chi + 1\), we cannot have \(|\rho' - 1| \leq 1\), by Corollary 3.4.2. Since \(|\rho' - 1| > 1\), by Theorem 3.4, we must have some \(\psi \in \{\theta, \omega, x\}\) such that \(|\rho' - x'| < 1\). If \(\psi = \theta\), then
\[\omega = x + 1 - \theta < \rho - \psi < x + 2 - \theta = \omega + 1 < \chi + 1;\]
hence, \(\rho - \theta = x\) by Corollary 3.4.1 and 3.4.2. If \(\psi = \omega\), then \(\theta < \rho - x < \theta + 1\). By Corollary 3.4.1, we can only have \(\rho = 2\omega\), which is impossible by Lemma 4.1, or \(\rho = \omega + x\). If \(\psi = x\), then \(1 < \rho - \psi < 1 + \theta\) and \(\rho - x \in \{\theta, \omega, x\}\). \(\square\)

**Corollary 4.2.1.** If \(\rho\) satisfies the conditions of the lemma and \(\rho\) is also a minimum of \(\mathcal{R}\), then \(\rho = x + \omega\).

**Proof.** If \(\rho = 2x\) or \(\rho = \theta + x\), then \(|\rho'| > |x'|\), which is not possible. \(\square\)
**Lemma 4.3.** If \( \chi < \theta + 1, \chi + 2 < \rho < \chi + 3, \) and \( |\rho'| < 1, \) then
\[
\rho \in \{ \theta + \chi, \omega + \chi, 2\chi, \chi + \theta + 1, \chi + \omega + 1, \chi + 2\theta, \\
\chi + 2\omega, 2\chi + 1, 2\chi + \theta, 2\chi + \omega, 3\chi \}.
\]

*Proof.* Since \( \chi + 1 < \rho - 1 < \chi + 2, \) we see by Lemma 4.2 that if \( |\rho' - 1| < 1, \) then \( \rho = \chi + \theta + 1, \chi + \omega + 1, 2\chi + 1. \) If \( |\rho' - 1| \geq 1, \) then \( |\rho' - \psi'| < 1 \) for some \( \psi \in \{ \theta, \omega, \chi \}. \) If \( \psi = \theta, \) then
\[
\chi < \omega + 1 < \chi + 2 - \theta < \rho - \psi < \chi + 3 - \theta = \omega + 2 < \chi + 2.
\]
Thus, \( \rho - \theta \in \{ \chi + 1, \chi + \theta, \chi + \omega, 2\chi \}. \) (Note that \( \theta + \omega + \chi = 2\chi + 1. \)) If \( \psi = \omega, \) then \( \chi < \rho - \chi < \chi + 2 \) and \( \rho - \omega \in \{ \chi + 1, \chi + \theta, \chi + \omega, 2\chi \}. \) If \( \psi = \chi, \) then \( 2 < \rho < \chi + 2 \) and \( \rho - \chi \in \{ \theta, \chi, \omega, \chi + 1, \chi + \theta, \chi + \omega, 2\chi \}. \)

**Corollary 4.3.1.** If \( \rho \) satisfies the conditions of the lemma and \( \rho \) is a minimum of \( \mathcal{R}, \) then
\[
\rho \in \{ \omega + \chi, \omega + \chi + 1, 2\chi + 1, 2\chi + \omega \}.
\]

*Proof.* We have \( 2|\chi'|, 3|\chi'| > |\chi'|; \) the other possibilities are ruled out by Lemma 4.1.

**Theorem 4.4.** If \( \chi < \theta + 1, \) there does not exist a set of minima \( \{ \mu_1, \mu_2, \mu_3, \mu_4 \} \) of \( \mathcal{R} \) such that \( \chi + 1 \leq \mu_1 < \mu_2 < \mu_3 < \mu_4 < \chi + 3. \)

*Proof.* Put \( \mathcal{R}^* = (1/\mu_1)\mathcal{R}, \theta^* = \mu_2/\mu_1, \omega^* = \mu_3/\mu_1, \chi^* = \mu_4/\mu_1. \) Since \( \chi^* < (\chi + 3)/(\chi + 1) < 1 + \theta^*, \) we must have
\[
(4.2) \quad \mu_4 + \mu_1 = \mu_2 + \mu_4 \quad \text{(Theorem 3.2),}
\]
and \( \mu_1, \mu_2, \mu_3, \mu_4 \in \{ \chi + 1, \chi + \omega, \chi + \omega + 1, 2\chi + 1, 2\chi + \omega \} \) by Corollaries 4.2.1 and 4.3.1. If \( \mu_1 \neq \chi + 1, \) then (4.2) cannot hold. If \( \mu_1 = \chi + 1 \) and \( \mu_2 \neq \chi + \omega + 1, \) then (4.2) again cannot hold. Thus, we must have \( \omega_1 = \chi + 1 \) and \( \mu_2 = \chi + \omega + 1. \) It follows that \( \mu_2 - \mu_1 = \omega - 1 \) and we can only have \( \mu_3 = 2\chi + 1, \mu_4 = 2\chi + \omega \).

Since \( \chi + 1 \) is a minimum, we have \( |\chi' + 1| < |\chi'|, \) and therefore \( \xi_\chi < -1/2. \) Since \( \xi_\omega < 1/2, \) we get \( 2\xi_\chi + \xi_\omega < -1/2 \) and \( |2\chi' + \omega' + 1| < |\omega' + \chi'|. \) Thus, if \( \mu_5 \) is the minimum adjacent to \( \mu_4 = 2\chi + \omega, \) then \( \mu_5 \leq 2\chi + \omega + 1. \) Since \( \rho^* = \mu_5/\mu_1, \) the minimum adjacent to \( \chi^* \) in \( \mathcal{R}^*, \) must satisfy \( \rho^* \geq \chi^* + 1, \) we get \( \mu_5 \geq \mu_4 + \mu_1 = 3\chi + \omega + 1 > 2\chi + \omega + 1, \) a contradiction. \( \square \)
Corollary 4.4.1. If $\theta_1 = 1$ in (2.1), then $\theta_8 > 4$.

Proof. If $\theta_4 \geq \theta_1 + 1$, put \( R^* = (1/\theta_4) R \), $\theta^* = \theta_5/\theta_4$, $\omega^* = \theta_6/\theta_4$, $\chi^* = \theta_7/\theta_4$, $\rho^* = \theta_8/\theta_4$. By Theorem 3.5, we have $\rho^* \geq \omega^* + 1$; hence, $\theta_8 = \theta_4 \rho^* \geq (\theta_1 + 1)(\omega^* + 1) > 4$. If $\theta_4 < \theta_1 + 1$, then $\theta_8 > \theta_5 + 3 > 4$ by the theorem. \( \square \)

It follows from Corollary 4.4.1 that in $R_i$ we have

$$\prod_{j=0}^{6} \theta_{8}^{(i+j)} > 4;$$

hence, from (2.1), we get

$$\theta_n > 4^{(n-1)/7}.$$

References


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