EXTENSIONS OF VALUATION AND ABSOLUTE VALUED TOPOLOGIES

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It is known that if \( L \) is a separable, finite dimensional extension of a field \( K \) and if \( \nu \) is a proper valuation (absolute value) on \( K \), then each ring topology on \( L \) whose restriction to \( K \) is the topology \( \mathcal{T}_\nu \) defined on \( K \) by \( \nu \) is the supremum of a finite family of valuation (absolute valued) topologies. We give a characterization of the fields \( K \) and \( L \) and the valuations (absolute values) \( \nu \) on \( K \) for which each ring topology on \( L \) extending \( \mathcal{T}_\nu \) is the supremum of a family of valuation (absolute valued) topologies on \( K \) when \( L \) is an arbitrary finite dimensional extension of \( K \).

Let \( R \) be a ring and let \( \mathcal{T} \) be a ring topology on \( R \), that is, \( \mathcal{T} \) is a topology on \( R \) making \((x, y) \rightarrow x - y \) and \((x, y) \rightarrow xy \) continuous from \( R \times R \) to \( R \). A subset \( A \) of \( R \) is bounded for \( \mathcal{T} \) if given any neighborhood \( U \) of zero, there exists a neighborhood \( V \) of zero such that \( VA \cup AV \subseteq U \). \( \mathcal{T} \) is a locally bounded topology on \( R \) if there exists a fundamental system of neighborhoods of zero for \( \mathcal{T} \) consisting of bounded sets.

Recall that a norm \( N \) on a ring \( R \) is a function from \( R \) to the nonnegative reals satisfying \( N(x) = 0 \) if and only if \( x = 0 \), \( N(x - y) \leq N(x) + N(y) \) and \( N(xy) \leq N(x)N(y) \) for all \( x \) and \( y \) in \( R \). Each norm \( N \) on \( R \) defines a locally bounded topology \( \mathcal{T}_N \) on \( R \) in a natural way. In particular, if \( \cdot \cdot\cdot \mid \cdot \cdot\cdot \mid \) is a proper absolute value on a field \( K \), then there exists a locally bounded topology \( \mathcal{T}_{\cdot \cdot\cdot \mid \cdot \cdot\cdot \mid} \) on \( K \) defined by \( \cdot \cdot\cdot \mid \cdot \cdot\cdot \mid \). We note further that if \( N \) is a nontrivial norm on a field \( K \), that is, \( \mathcal{T}_N \) is nondiscrete, then a subset \( A \) of \( K \) is bounded in norm if and only if \( A \) is a \( \mathcal{T}_N \)-bounded subset of \( K \).

Each proper valuation \( \nu \) on a field \( K \) defines a locally bounded topology \( \mathcal{T}_\nu \) on \( K \) as well. If each of \( \nu \) and \( w \) is a valuation or an absolute value on \( K \), then \( \nu \) and \( w \) are independent if \( \mathcal{T}_\nu \neq \mathcal{T}_w \).

In [11], Rigo and Warner proved that if \( L \) is a separable, finite dimensional extension of a field \( K \) and if \( \nu \) is a proper valuation (absolute value) on \( K \), then each ring topology on \( L \) inducing \( \mathcal{T}_\nu \) on \( K \) is the supremum of a finite family of valuation (absolute valued) topologies on \( L \) (Theorem 2). In this paper we characterize the fields \( K \) and \( L \) and valuations (absolute values) \( \nu \) on \( K \) for which each ring topology on \( L \)
extending $\mathcal{T}_v$ is the supremum of a finite family of valuation (absolute valued) topologies on $L$ when $L$ is an arbitrary finite dimensional extension of $K$.

**Theorem 1.** Let $K$ be a field, $v$ a proper valuation (absolute value) on $K$, $\hat{K}$ the completion of $K$ for $\mathcal{T}_v$, $L$ a purely inseparable, finite dimensional extension of $K$, $w$ the unique extension of $v$ to $L$ and $\hat{L}$ the completion of $L$ for $\mathcal{T}_w$. The following are equivalent.

1°. $[\hat{L} : \hat{K}] = [L : K]$.

2°. $\mathcal{T}_w$ is the only ring topology on $L$ whose restriction to $K$ is $\mathcal{T}_v$.

3°. $\mathcal{T}_w$ is the only locally bounded topology on $L$ whose restriction to $K$ is $\mathcal{T}_v$.

**Proof.** We first consider the case when $[L : K] = p$ where $p$ is the characteristic of $K$.

Suppose $[\hat{L} : \hat{K}] = p$. Then there exists $a$ in $L \setminus \hat{K}$. Hence $L = K(a)$ and the minimal polynomial of $a$ over $K$ is irreducible in $\hat{K}[X]$. Thus by [11, Corollary 2 of Theorem 1], $\mathcal{T}_w$ is the only ring topology on $L$ whose restriction to $K$ is $\mathcal{T}_v$.

Clearly 2° implies 3°. So it suffices to prove that if $\mathcal{T}_w$ is the only locally bounded topology on $L$ whose restriction to $K$ is $\mathcal{T}_v$, then $[\hat{L} : \hat{K}] = p$. If $[\hat{L} : \hat{K}] = 1$, let $a \in L \setminus K$ and let $f(X)$ be the minimal polynomial of $a$ over $K$. Then $f(X) = (X - a)^p$ and $X - a \in \hat{K}[X]$. Hence by [11, Theorem 1], there are $p$ ring topologies $\mathcal{T}_1, \ldots, \mathcal{T}_p$ on $L$ inducing $\mathcal{T}_v$ on $K$ and the completion $\hat{L}_i$ of $L$ for $\mathcal{T}_i$ is a finite dimensional $\hat{K}$-algebra for each $i \in [1, p]$. If $v$ is a valuation on $K$, let $\hat{v}$ be its extension to $\hat{K}$, let $G$ be the order group of $\hat{v}$ and let $\{x_1, \ldots, x_n\}$ be a basis for $\hat{L}_i$ over $\hat{K}$ where $x_1 = 1$. Then $\{V_{\alpha} : \alpha \in G\}$ is a fundamental system of neighborhoods of zero for a Hausdorff topology on $\hat{L}_i$ compatible with the vector space structure of $\hat{L}_i$ where for each $\alpha \in G$,

$$V_\alpha = \left\{ \sum_{j=1}^n a_j x_j : a_j \in \hat{K}, \inf\{ \hat{v}(a_j) : 1 \leq j \leq n \} \geq \alpha \right\}.$$ 

Hence by [8, Theorem 7], $\{V_\alpha : \alpha \in G\}$ is a fundamental system of neighborhoods of zero for the completion $\hat{\mathcal{T}}_i$ of $L$ for $\mathcal{T}_i$. It follows that the restriction of $\hat{\mathcal{T}}_i$ to $\hat{K}$ is the topology defined on $\hat{K}$ by $\hat{v}$. Thus as $L \subset \hat{K}$, $\hat{\mathcal{T}}_i|_L$ is a locally bounded topology on $L$, that is, each $\mathcal{T}_i$ is a locally bounded topology on $L$, a contradiction. If $v$ is an absolute value on $K$, then each $\mathcal{T}_i$ is normable and hence locally bounded. Indeed, by [2, Theorem 2, p. 27; 3, Proposition 10, p. 69 and Theorem 1, p. 70], there exist a vector space norm $N$ on $\hat{L}_i$ and a positive number $c$ such that
\[ N(xy) \leq cN(x)N(y) \] for all \( x \) and \( y \) in \( \hat{L}_i \). Therefore the function \( \| \cdot \| \) defined on \( \hat{L}_i \) by, \( \| x \| = cN(x) \), is an algebra norm on \( \hat{L}_i \) defining the topology on \( \hat{L}_i \). So \([\hat{L} : \hat{K}] = p\) by [11, Corollary 1 of Theorem 1].

Now let \( L \) be an arbitrary, purely inseparable, finite dimensional extension of \( K \).

Suppose that \([\hat{L} : \hat{K}] = [L : K]\) and let \( \mathcal{T} \) be a ring topology on \( L \) whose restriction to \( K \) is \( \mathcal{T}_v \). Let \( K_1 \) be a maximal subfield of \( L \) containing \( K \) such that \( \mathcal{T}|_{K_1} \) is defined by a valuation (absolute value) \( v_1 \) extending \( v \) to \( K_1 \). If \( K_1 \neq L \), let \( a \in L \setminus K_1 \) be such that \([K_1(a) : K_1] = p\). Denote \( K_1(a) \) by \( K_2 \), let \( v_2 \) be the unique extension of \( v \) to \( K_2 \), let \( \hat{K}_1 \) be the completion of \( K_1 \) for \( \mathcal{T}|_{K_1} \) and let \( \hat{K}_2 \) be the completion of \( K_2 \) for \( \mathcal{T}_{v_2} \). If \( a \notin \hat{K}_1 \), then by the previous argument, \( \mathcal{T}|_{K_2} = \mathcal{T}_{v_2} \), contradicting the maximality of \( K_1 \). If \( a \in \hat{K}_1 \), then \([\hat{K}_2 : \hat{K}_1] = 1\). So \([\hat{L} : \hat{K}] = [\hat{L} : \hat{K}_2][\hat{K}_1 : \hat{K}] \leq [L : K_2][K_1 : K] < [L : K] \), a contradiction. Hence \( K_1 = L \).

Assume \( 3^\circ \) holds. Let \([L : K] = p^n \) and let \( a_1, a_2, \ldots, a_n \in L \) be such that \( L = K(a_1, \ldots, a_n) \), \([K(a_1) : K] = p\) and for all \( i \in [1, n - 1] \), \([K(a_1, \ldots, a_i+1) : K(a_1, \ldots, a_i)] = p\). Denote \( K \) by \( K_0 \). For each \( i \in [1, n] \) let \( K_i = K(a_1, \ldots, a_i) \), let \( v_i \) be the unique extension of \( v \) to \( K_i \) and let \( \hat{K}_i \) be the completion of \( K_i \) for \( \mathcal{T}_{v_i} \). If \( p^n > [\hat{L} : \hat{K}] \), then as \([\hat{L} : \hat{K}] = \prod_{i=0}^{n-1} [\hat{K}_{i+1} : \hat{K}_i] \), there exists an \( i \) such that \([\hat{K}_{i+1} : \hat{K}_i] = 1\). So by the previous argument there exists a locally bounded topology \( \mathcal{T}' \) on \( K_{i+1} \) whose restriction to \( K_i \) is \( \mathcal{T}_{v_i} \) but \( \mathcal{T}' \neq \mathcal{T}_{v_{i+1}} \). By [12, Satz 1.6], \( \mathcal{T}' \) extends to a locally bounded topology \( \mathcal{T} \) on \( L \). Clearly \( \mathcal{T}|_K = \mathcal{T}_v \) but \( \mathcal{T} \neq \mathcal{T}_w \), a contradiction. So \([\hat{L} : \hat{K}] = p^n = [L : K]\).

**Theorem 2.** Let \( L \) be a finite dimensional extension of a field \( K \), let \( D \) be the separable closure of \( K \) in \( L \), let \( v \) be a proper valuation (absolute value) on \( K \) and let \( \{v_i; 1 \leq i \leq m\} \) be a complete family of pairwise independent valuations (absolute values) on \( D \) extending \( v \). For each \( i \in [1, m] \), let \( w_i \) be the unique extension of \( v_i \) to \( L \), let \( \hat{L}_i \) denote the completion of \( L \) for \( \mathcal{T}_{w_i} \) and let \( \hat{D}_i \) denote the completion of \( D \) for \( \mathcal{T}_{v_i} \). The following are equivalent.

1°. Each ring topology on \( L \) whose restriction to \( K \) is \( \mathcal{T}_v \) is the supremum of a finite family of valuation (absolute valued) topologies on \( L \).

2°. Each locally bounded topology on \( L \) whose restriction to \( K \) is \( \mathcal{T}_v \) is the supremum of a finite family of valuation (absolute valued) topologies on \( L \).

3°. There are \( 2^m - 1 \) locally bounded topologies on \( L \) inducing \( \mathcal{T}_v \) on \( K \), namely the topologies \( \text{sup}_{i \in M} \mathcal{T}_{w_i} \) where \( M \) runs through all nonempty subsets of \([1, m]\).
4°. \([\hat{L}_i : \hat{D}_i] = [L : D]\) for all \(i \in [1, m]\).

**Proof.** Clearly 1° implies 2° and 3° implies 2°. We first show that 2° implies 3°. Suppose that \(T\) is a locally bounded topology on \(L\) and \(T = \sup_{1 \leq i \leq n} T_{u_i}\) where each \(u_i\) is a proper valuation (absolute value) on \(L\) and \(T_{u_i} \neq T_{u_j}\) for \(i \neq j\). Then \(T_v = T|_K = \sup_{1 \leq i \leq n} T_{u_i}|_K\). As the completion of \(K\) for \(T_v\) is a field, the Approximation Theorem [7, Theorem 3.4, p. 292] yields that each \(u_i|_K\) is equivalent to \(v\). Hence for each \(i \in [1, n]\), there exists \(j(i) \in [1, m]\) such that \(T_{u_i} = T_{w_{j(i)}}\).

We next show that 3° implies 4°. Let \(T\) be a locally bounded topology on \(L\) whose restriction to \(D\) is \(T_v\) and let \(M\) be a nonempty subset of \([1, m]\) such that \(T = \sup_{\nu \in M} T_{w_{\nu}}\). Note that for \(\nu, \mu \in M, \nu \neq \mu, T_{w_{\nu}|_D} \neq T_{w_{\mu}|_D}\). Then \(T_{w_{\nu}} = T|_D = \sup_{\nu \in M} T_{w_{\nu}|_D}\) and so the Approximation Theorem implies that the cardinality of \(M\) is one. Thus \(M = \{i\}\) by the definition of \(w_i\), that is, \(T = T_{w_i}\). As \(L\) is a purely inseparable extension of \(D\), 4° follows from Theorem 1.

Finally suppose that 4° holds. Let \(T\) be a ring topology on \(L\) whose restriction to \(K\) is \(T_v\). By Theorems 2 and 4 of [11], there exist a nonempty subset \(M\) of \([1, m]\) and ring topologies \(T_i\) on \(L\) for each \(i \in M\) such that \(T_i|_D = T_{w_{\nu}}\) and \(T = \sup_{\nu \in M} T_{w_{\nu}}\). Hence \(T = T_{w_i}\) for all \(i \in M\) by Theorem 1 and so 1° holds.

**Corollary.** Let \(K\) be the field \(F(X)\) of rational functions over the field \(F\), let \(L\) be a finite dimensional extension of \(K\) and let \(v\) be a proper valuation or absolute value on \(K\), improper on \(F\). Define \(D, L, i, \text{ and } \hat{D}_i\) as in Theorem 2. Then \([\hat{L}_i : \hat{D}_i] = [L : D]\) for all \(i\) and each ring topology on \(L\) inducing \(T_v\) on \(K\) is a locally bounded topology.

**Proof.** First note that if \(v\) is a proper valuation on \(K\) improper on \(F\), then \(v\) is equivalent to a real-valued valuation [1, Example 4, p. 106]. It suffices to establish 2° of Theorem 2. Let \(T\) be a locally bounded topology on \(L\) whose restriction to \(K\) is \(T_v\). Then there exists a nonzero topological nilpotent for \(T_v\) and hence for \(T\). So by [5, Theorem 6.1], there exists a norm \(N\) on \(L\) such that \(T = T_N\). As \(F\) is a bounded subset of \(K\) for \(T_v\) and as \(T_N|_K = T_v\), \(F\) is bounded in norm (for \(N\)). Consequently, \(F\) is a \(T\)-bounded subset of \(L\) as well. Thus by [6, Theorem 4] and the argument used to establish Theorem 3 of [4], \(T\) is the supremum of a finite family of valuation topologies on \(L\).

In [9], Nagata gave an example of fields \(L\) and \(K\), each of prime characteristic \(p\), and a discrete valuation \(v\) on \(K\) such that \(\hat{K} = L\) is a
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purely inseparable extension of $K$ of degree $p$ over $K$ (p. 56). Thus $\hat{L} = \hat{K}$ and so conditions $1^\circ$–$4^\circ$ of Theorem 2 need not hold in general.

**Theorem 3.** Let $K$ be a field and let $\mathcal{T}_0$ denote $\sup_{1 \leq i \leq m} \mathcal{T}_{v_i}$ where each $v_i$ is a proper valuation or absolute value on $K$ and for $i \neq j$, $\mathcal{T}_{v_i} \neq \mathcal{T}_{v_j}$.

Let $L$ be a finite dimensional extension of $K$ and let $D$ be the separable closure of $K$ in $L$. For each $i \in [1, m]$, let $\{v_{ij}; 1 \leq j \leq M(i)\}$ be a complete family of pairwise independent valuations or absolute values extending $v_i$ to $D$. For each $i \in [1, m]$, $j \in [1, M(i)]$, let $w_{ij}$ denote the unique extension of $v_{ij}$ to $L$, let $\hat{L}_{ij}$ denote the completion of $L$ for $\mathcal{T}_{w_{ij}}$, and let $\hat{D}_{ij}$ denote the completion of $D$ for $\mathcal{T}_{v_{ij}}$. The following are equivalent.

$1^\circ$. Each ring topology on $L$ whose restriction to $K$ is $\mathcal{T}_0$ is the supremum of a finite family $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of topologies on $L$ where for each $i$, $\mathcal{T}_i$ is defined by a proper valuation or absolute value on $L$.

$2^\circ$. Each locally bounded topology on $L$ whose restriction to $K$ is $\mathcal{T}_0$ is the supremum of a finite family $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of topologies on $L$ where for each $i$, $\mathcal{T}_i$ is defined by a proper valuation or absolute value on $L$.

$3^\circ$. There are $\prod_{i=1}^n (2^{M(i)} - 1)$ locally bounded topologies on $L$ inducing $\mathcal{T}_0$ on $K$, namely the topologies $\sup_{1 \leq i \leq m} (\sup_{j \in S(i)} \mathcal{T}_{w_{ij}})$ where $S(i)$ runs through all nonempty subsets of $[1, M(i)]$.

$4^\circ$. $[\hat{L}_{ij} : D_{ij}] = [L : D]$ for all $i \in [1, m]$, $j \in [1, M(i)]$.

**Proof.** Clearly $1^\circ$ implies $2^\circ$ and $3^\circ$ implies $2^\circ$. We first prove that $2^\circ$ implies $3^\circ$. Let $\mathcal{T}$ be a locally bounded topology on $L$ inducing $\mathcal{T}_0$ on $K$. Then $\mathcal{T} = \sup_{1 \leq i \leq n} \mathcal{T}_{u_i}$ where each $u_i$ is a proper valuation or absolute value on $L$ and $\sup_{1 \leq j \leq m} \mathcal{T}_{v_j} = \sup_{1 \leq i \leq n} \mathcal{T}_{u_i}|_K$. Suppose that there exists an $i$, $1 \leq i \leq n$, such that for all $j$, $1 \leq j \leq m$, $\mathcal{T}_{u_i}|_K \neq \mathcal{T}_{v_j}$. Without loss of generality assume that $v_1, \ldots, v_r$ are valuations on $K$, $v_{r+1}, \ldots, v_m$ are absolute values on $K$ and $i = 1$. If $u_1$ is an absolute value on $L$, let $a \in K$ be such that $u_1(a) > 1$, $v_j(a) > 0$ for $j \in [1, r]$ and $v_j(a) < 1$ for $j \in [r + 1, m]$. (The existence of $a$ is guaranteed by [7, Theorem 3.4, p. 292].) Then $\{a^t; t = 1, 2, \ldots\}$ is a bounded set for $\sup_{1 \leq j \leq m} \mathcal{T}_{v_j}$ but not for $\mathcal{T}_{u_1}|_K$, a contradiction. (Indeed, if $\{a^t; t = 1, 2, \ldots\}$ is bounded for $\sup_{1 \leq j \leq m} \mathcal{T}_{v_j}$, then there exists a nonzero element $x$ in $K$ such that $x\{a^t; t = 1, 2, \ldots\} \not\subseteq \{y \in K; u_1(y) \leq 1\}$. But $u_1(xz^t) \to \infty$ as $t \to \infty$, a contradiction.) If $u_1$ is a valuation on $L$, let $G$ be the order group of $u_1|_K$ and for each $\alpha \in G$, let $a_\alpha \in K$ be such that $v_j(a_\alpha) > 0$ for $j = 1, 2, \ldots, r$, $v_j(a_\alpha) < 1$ for $j = r + 1, \ldots, m$ and $u_1(a_\alpha) = \alpha$. Then $\{a_\alpha; \alpha \in G\}$ is a bounded set for $\sup_{1 \leq j \leq m} \mathcal{T}_{v_j}$ but not for $\mathcal{T}_{u_1}|_K$, a contradiction. Thus for each $i \in [1, n]$ there exists $j(i) \in [1, m]$
and \( t(i) \in [1, M(j(i))] \) such that \( \mathcal{T}_{u_i} = \mathcal{T}_{w(j(i))} \). Furthermore a similar argument establishes that for each \( j \in [1, m] \), there exists an \( i \in [1, n] \) such that \( \mathcal{T}_{u_i}|_K = \mathcal{T}_{v_i} \).

Assume \( 3^\circ \) holds. Suppose that there exist \( i \in [1, m] \) and \( j \in [1, M(i)] \) with \( [\hat{L}_{ij}, \hat{D}_{ij}] < [L : D] \). By Theorem 2 there exists a locally bounded topology \( \mathcal{T} \) on \( L \) whose restriction to \( K \) is \( \mathcal{T}_{v_i} \) but \( \mathcal{T} \) is not the supremum of a finite family of valuation or absolute valued topologies on \( L \). Let \( \mathcal{T}' = \sup(\mathcal{T}, \sup_{i+1, i} \mathcal{T}_{w_i}) \). Then \( \mathcal{T}'|_K = \mathcal{T}_0 \) but \( \mathcal{T}' \) is not the supremum of a finite family of topologies on \( L \) of the appropriate type. Indeed, if \( \mathcal{T}' \) is such a supremum, then as \( ^{1,\circ} \), Theorem 4.4 of [10] yields that \( \mathcal{T} \) is as well. Thus \( 4^\circ \) holds.

Finally the proof that \( 4^\circ \) implies \( 1^\circ \) is the same as that used in Theorem 2.

**References**


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