ON PEŁ CZYŃSKI’S PROPERTIES (V) AND (V*)

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It is shown that a Banach lattice $X$ has Pelczynski’s property (V*) if and only if $X$ contains no subspace isomorphic to $c_0$. This result is used to show that there is a Banach space $E$ that has Pelczynski’s property (V*) but such that its dual $E^*$ fails Pelczynski’s property (V), thus answering in the negative a question of Pelczynski.

In his fundamental paper [7], Pelczynski introduced two properties of Banach spaces, namely property (V) and property (V*). For a Banach space $X$ we say that $X$ has property (V*) if any subset $K \subset X$ such that $\lim_n \sup_{x \in K} x_n^*(x) = 0$ for every weakly unconditionally Cauchy series (w.u.c.) $\sum_{n=1}^{\infty} x_n^*$ in $X^*$, then $K$ is relatively weakly compact. We say that $X$ has property (V) if any subset $K \subset X^*$ such that $\lim_n \sup_{x^* \in K} x_n^*(x^*) = 0$ for every weakly unconditionally Cauchy series (w.u.c.) $\sum_{n=1}^{\infty} x_n$ in $X$ then $K$ is relatively weakly compact. In [7] Pelczynski noted that it follows directly from the definition that if $X^*$ has property (V) then $X$ has property (V*), and he asked [7, Remark 3, p. 646] if the converse is true. As we shall soon show Example 5 below will provide a negative answer to Pelczynski’s question.

In this paper we will concentrate on property (V*) and we shall refer the reader to [4] and [7] for more on property (V). Among classical Banach spaces that have property (V*), $L^1$-spaces are the most notable ones. In [7] Pelczynski showed that if a Banach space has property (V*), then it must be weakly sequentially complete. He also noted that for a closed subspace $X$ of a space with unconditional basis, the space $X$ has property (V*) if and only if $X$ contains no subspace isomorphic to $c_0$. This prompted the following natural question:

**Problem 1.** Let $(\Omega, \Sigma, \lambda)$ be a probability space, and let $X$ be a closed subspace of a Banach space with unconditional basis. Does the Banach space $L^1(\lambda, X)$ of Bochner integrable $X$-valued functions have property (V*) whenever $X$ has (V*)?

In this paper we shall give an affirmative answer to this question, in fact we shall prove a more general result, namely if $X$ is a separable subspace of an order continuous Banach lattice, then $L^1(\lambda, X)$ has
property \((V^*)\) if and only if \(X\) has \((V^*)\). It will also be shown that for a Banach lattice \(X\), the space \(X\) has property \((V^*)\) if and only if \(X\) contains no subspace isomorphic to \(c_0\).

First, let us fix some notations and terminology. We say that a series \(\sum_{n=1}^{\infty} x_n\) in a Banach space \(X\) is weakly unconditionally Cauchy (w.u.c.) if for every \(x^* \in X^*\), the series \(\sum_{n=1}^{\infty} x^*(x_n)\) is unconditionally convergent or equivalently if

\[
\sup \left\{ \left\| \sum_{i \in \sigma} x_i \right\| : \sigma \text{ finite subset of } \mathbb{N} \right\} < \infty.
\]

If \((\Omega, \Sigma, \lambda)\) is a probability space and \(X\) is a Banach space, then \(L^1(\lambda, X)\) will stand for the Banach space of all (classes of) Bochner integrable \(X\)-valued functions defined on \(\Omega\). For a compact Hausdorff space \(T\), we shall denote by \(M(T, X)\) the space of all countably additive \(X\)-valued measures defined on the \(\sigma\)-field of Borel subsets of \(T\), and that are of bounded variation. The space \(M(T, X)\) is a Banach space under the variation norm.

Recall that a Banach space \(X\) has the separable complementation property if every separable subspace \(E\) of \(X\) is contained in a separable complemented subspace \(F\) of \(X\). In this paper we shall need the fact that any order continuous Banach lattice has the separable complementation property [6, p. 9].

Any other notation or terminology used and not defined can be found in [5] or [6].

1. **The main result.** The next theorem gives a characterization of those separable subspaces of an order continuous Banach lattice that have Pelczynski's property \((V^*)\).

**Theorem 2.** Let \(X\) be a separable subspace of an order continuous Banach lattice \(Y\). Then \(X\) has property \((V^*)\) if and only if \(X\) contains no subspace isomorphic to \(c_0\).

**Proof.** Of course if \(X\) has \((V^*)\), then \(X\) is weakly sequentially complete [7], hence one direction is obvious.

Conversely, assume \(X\) contains no subspace isomorphic to \(c_0\), hence \(X\) is weakly sequentially complete, and let \(K \subset X\) such that

\[
\limsup_{n} \sup_{x \in K} x_n^*(x) = 0
\]
for every w.u.c. series $\sum_{n=1}^{\infty} x_n^*$ in $X^*$. Let $(x_n)_{n \geq 1}$ be a sequence in $K$. By
[6, p. 9] let $X_0$ be a band with weak order unit in $Y$ such that $X \subset X_0$. By
[6, p. 25] there exists a probability space $(\Omega, \Sigma, \nu)$ such that $X_0$ is an order
ideal of $L^1(\nu)$ and such that
\[ L_\infty(\nu) \subset X_0 \subset L^1(\nu), \]
and if $f \in L_\infty(\nu)$, then
\[ \frac{1}{2} \| f \|_1 \leq \| f \|_{X_0} \leq \| f \|. \]
Since $K$ satisfies (') and $L^1(\nu)$ has property (V*), one can find a
subsequence $(x_{n_k})_{k \geq 1}$ of $(x_n)_{n \geq 1}$ such that $(x_{n_k})_{k \geq 1}$ is weakly convergent in $L^1(\nu)$. We claim that $(x_{n_k})_{k \geq 1}$ is in fact weak Cauchy in $X$. For
this, let $g \in X^*$, by the Hahn-Banach theorem, let $h \in X_0^*$ such that
$h = g$ on $X$ and $\| h \|_{X_0^*} = \| g \|_{X^*}$. Since $h \in X_0^*$, we know [6, p. 25] that we
can consider $h$ as a measurable mapping on $\Omega$ and
\[ \| h \|_{X_0^*} = \sup \left\{ \int hf : dv : f \in X_0, \| f \|_{X_0} \leq 1 \right\} < \infty. \]
Without loss of generality we may assume that $h \geq 0$. For each $n \geq 1$ let
$\Omega_n = \{ \omega \in \Omega \mid n - 1 < h(\omega) \leq n \}$. Then it follows from the duality be-
tween $X_0$ and $X_0^*$ that the series $\sum_{n=1}^{\infty} h \cdot 1_{\Omega_n}$ converges weak* to $h$,
morover the series $\sum_{n=1}^{\infty} h \cdot 1_{\Omega_n}$ is a w.u.c. series in $X_0^*$. To see that
$\sum_{n=1}^{\infty} h 1_{\Omega_n}$ is a w.u.c. series, note that if $\sigma$ is a finite subset of $N$ and
$Z = \bigcup_{n \in \sigma} \Omega_n$, then
\[ \left\| \sum_{n \in \sigma} h X_{\Omega_n} \right\| = \sup \left\{ \sum_{n \in \sigma} \int_{\Omega_n} hf \, dv : f \geq 0, \| f \|_{X_0} \leq 1 \right\} \]
\[ = \sup \left\{ \int_{Z} hf \, dv : f \geq 0, \| f \|_{X_0} \leq 1 \right\} \]
\[ \leq \| h \|_{X_0^*}. \]
Hence $\sup\{ \| \sum_{n \in \sigma} h 1_{\Omega_n} \| : \sigma \text{ finite subset of } IN \} < \infty$; and the series
$\sum_{n=1}^{\infty} h 1_{\Omega_n}$ is a w.u.c. series in $X_0^*$. Therefore $\sum_{n=1}^{\infty} h 1_{\Omega_n}$ when restricted to
$X$ is a w.u.c. series in $X^*$. Since $K$ satisfies ('), it follows that the series
$\sum_{n=1}^{\infty} h 1_{\Omega_n}$ converges unconditionally uniformly on $K$. If not, one can find
$\delta > 0$ $p_1 < p_2 < \cdots < p_n < \cdots$ such that for every $n \geq 1$
\[ \sup_{x \in K} \left( \sum_{j=p_n+1}^{p_{n+1}} \langle h 1_{\Omega_n}, x \rangle \right) > \delta. \]
For each $n \geq 1$, let $y^*_n = \sum_{j=p_n+1}^{p_{n+1}} h1_{\Omega_j}$, the series $\sum_{n=1}^{\infty} y^*_n$ is also w.u.c. but 
$$\lim_{n \to \infty} \sup_{x \in K} y^*_n(x) \neq 0,$$
thus contradicting ('). This implies that for $\varepsilon > 0$, there exists $m > 0$ such that for all $n \geq 1$
$$\left| \sum_{j=m+1}^{\infty} \int_{\Omega_j} hx_n \, dv \right| < \varepsilon.$$ 

Let $e^* = \sum_{j=1}^{m} h1_{\Omega_j}$, then $e^* \in L^\infty(\nu)$. Since the sequence $\{ e^*(x_n) \}_{k \geq 1}$ is Cauchy, it follows that there exists $N > 0$ such that for $p, q > N$
$$\left| e^*(x_{n_p} - x_{n_q}) \right| < \varepsilon,$$
this of course implies that for $p, q > N$
$$\left| g(x_{n_p} - x_{n_q}) \right| < 3\varepsilon.$$
This shows that $K$ is weakly precompact, and hence $K$ is relatively weakly compact since $X$ is weakly sequentially complete.

**Proposition 3.** Let $X$ have the separable complementation property. Then $X$ has property $(V^*)$ if and only if every separable subspace of $X$ has property $(V^*)$.

**Proof.** Since property $(V^*)$ is easily seen to be stable by subspaces one implication is immediate.

Conversely, assume that every separable subspace of $X$ has $(V^*)$ and let $K \subset X$ such that $\lim_{n} \sup_{x \in K} x^*(x) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} x^*_n$ in $X^*$. Let $\{x_n\}_{n \geq 1}$ be a sequence in $K$. Since $X$ has the separable complementation property there exists a separable complemented subspace $Z$ of $X$ such that $\{x_n\}_{n \geq 1} \subset Z$. Since $\lim_{n} \sup_{m} x^*_n(x_m) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} x^*_n$ in $X^*$ and since $Z$ is complemented in $X$, it follows that $\lim_{n} \sup_{m} z^*_n(x_m) = 0$ for every w.u.c. series $\sum_{n=1}^{\infty} z^*_n$ in $Z^*$. By hypothesis the space $Z$ has property $(V^*)$, hence it follows that there exists a subsequence $\{x_{n_k}\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$ which is weakly convergent in $Z$ and therefore is weakly convergent in $X$. This completes the proof and shows that $X$ has property $(V^*)$.

**Theorem 4.** If $X$ is a Banach lattice, then $X$ has property $(V^*)$ if and only if $X$ contains no subspace isomorphic to $c_0$.

**Proof.** If $X$ is a Banach lattice that contains no subspace isomorphic to $c_0$, then $X$ has an order continuous norm. By Theorem 2 every separable subspace of $X$ has property $(V^*)$, since $X$ has the separable
complementation property, it follows from Proposition 3 that $X$ has property (V*).

We are now in a position to answer Pelczynski's question [7, Remark 3, p. 646].

**Example 5.** A Banach space $E$ such that $E$ has property (V*) but $E^*$ fails property (V).

**Proof.** To answer Pelczynski's question one needs to take a weakly sequentially complete Banach lattice $E$ such that $E^{**}$ is not weakly sequentially complete. This space will have the property (V*) by Theorem 4 but its dual $E^*$ does not have property (V) [7]. An example of such a Banach lattice can be provided by the space constructed by M. Talagrand in [9]. Indeed the space $E$ exhibited in [9] is weakly sequentially complete but is such that the space $M([0,1], E)$ contains a subspace isomorphic to $c_0$. This in particular shows that the space $M([0,1], E)$ cannot be weakly sequentially complete, therefore it follows from [8] that $E^{**}$ cannot be weakly sequentially complete.

The next theorem gives a positive answer to Problem 1 stated at the beginning of this paper.

**Theorem 6.** Let $X$ be a separable subspace of an order continuous Banach lattice $Y$. If $(\Omega, \Sigma, \lambda)$ is a probability space, then $L^1(\lambda, X)$ has property (V*) if and only if $X$ has property (V*).

**Proof.** If $L^1(\lambda, X)$ has (V*), then $X$ has (V*) since it is easily checked that property (V*) is stable by subspace.

Conversely, let $X$ be a separable subspace of an order continuous Banach lattice $Y$. If $X$ has property (V*), then $X$ contains no subspace isomorphic to $c_0$. Of course $L^1(\lambda, X)$ is a subspace of $L^1(\lambda, Y)$ which is an order continuous Banach lattice [3]. The proof now follows from Theorem 2 and from a result of [8] (see also [1]) which guarantees that $L^1(\lambda, X)$ contains no subspace isomorphic to $c_0$.

2. Notes and remarks.

**Remark A.** Theorem 4 fails for arbitrary Banach spaces. Indeed not every weakly sequentially complete Banach space has property (V*) the first Delbaen-Bourgain space [2] DBI is an example of a weakly sequentially complete Banach space that fails (V*). Indeed, the space DBI has
the Schur property (weakly compact sets are compact), its dual is isomorphic to an $L^1$-space, but DBI fails $(V^*)$ due to the following easy proposition.

**Proposition 7** For a non-reflexive Banach space $X$, if $X^*$ is weakly sequentially complete, then $X$ fails $(V^*)$.

**Remark B.** In [7] Pelczynski noted that if a Banach space $X$ has property $(V)$ then $X^*$ has property $(V^*)$, and he asked [7, Remark 3, p. 646] if the converse is true. Here the first Delbaen-Bourgain space DBI provides a counter example to Pelczynski's question, for DBI fails property $(V)$ since it has the Schur property, but its dual has property $(V^*)$ since it is isomorphic to an $L^1$-space.

**References**


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