

# Pacific Journal of Mathematics

**USING PREDICTION PRINCIPLES TO CONSTRUCT ORDERED  
CONTINUA**

STEPHEN WATSON

## USING PREDICTION PRINCIPLES TO CONSTRUCT ORDERED CONTINUA

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In this paper, we show that the elements of a  $\diamond$ -sequence can be ordered lexicographically to produce an ordered continuum. An application of this idea answers a question of V. Malyhin and others: Is there a compact Hausdorff space in which no two points have equal character? We show that the consistency strength of the existence of such a space lies between that of an inaccessible and a Mahlo cardinal. We show that compactness is essential in this result by constructing, in ZFC, a  $\sigma$ -compact Hausdorff space in which no two points have equal character.

Let us begin with some definitions:

DEFINITION 1.  $\{f_\alpha: \alpha \in E\}$  is a  $\diamond_\kappa(E)$ -sequence (where  $E \subset \kappa - \{0\}$  and  $f_\alpha: \alpha \rightarrow 2$ ) if, for each  $f: \kappa \rightarrow 2$ , there is  $\alpha \in E$  such that  $f_\alpha \subset f$ .

This is not exactly the standard definition (we use the characteristic functions of subsets of  $\kappa$ , we trap only once and do not require that  $E$  be stationary or even cofinal in  $\kappa$ ) but it is equivalent in most cases. The lexicographic ordering is not well defined because their domains are not equal. We need to subtract some  $f_\alpha$ 's which are "not needed". Let us fix this idea.

DEFINITION 2.  $\{f_\alpha: \alpha \in E\}$  is a *minimal*  $\diamond_\kappa(E)$ -sequence if, whenever  $F \subset E$  and  $\{f_\alpha: \alpha \in F\}$  is a  $\diamond_\kappa(F)$ -sequence,  $F$  must equal  $E$ .

This seems like a strong condition but it is not. We *can* subtract the  $f_\alpha$ 's which are not needed.

LEMMA 1. *If  $\{f_\alpha: \alpha \in E\}$  is a  $\diamond_\kappa(E)$ -sequence, then there is  $F \subset E$  such that*

*$\{f_\alpha: \alpha \in F\}$  is a minimal  $\diamond_\kappa(F)$ -sequence.*

*Proof.* The idea is to inductively subtract any  $f_\alpha$  compatible with  $f_\beta$  when  $\beta < \alpha$ . That is,  $\alpha \in F$  iff, for each  $\beta \in \alpha \cap F$ ,  $f_\beta \cup f_\alpha$  is not a function.  $\{f_\alpha: \alpha \in F\}$  is a  $\diamond_\kappa(F)$ -sequence since, for each  $f: \kappa \rightarrow 2$  there is a minimal  $\alpha \in E$  such that  $f \upharpoonright \alpha = f_\alpha$ . By the construction of  $F$  and the

minimality of  $\alpha$ ,  $\alpha \in F$ .  $\{f_\alpha: \alpha \in F\}$  is minimal since, for each  $\alpha \in F$ , we can find  $f: \kappa \rightarrow 2$  such that  $f \supset f_\alpha$ . By the construction of  $F$ , there does not exist  $\beta \in F$  such that  $\beta \neq \alpha$  and  $f \supset f_\beta$ .

Minimality implies that the lexicographic ordering of a minimal  $\diamond_\kappa(E)$ -sequence is well-defined. Meanwhile, the  $\diamond$  prediction property implies compactness.

Some notation is useful:

Let  $\alpha * \beta$  be  $\inf\{\gamma: f_\alpha(\gamma) \neq f_\beta(\gamma)\}$ .

The lexicographic ordering  $\triangleleft$  is defined by  $f_\alpha \triangleleft f_\beta$  iff  $f_\alpha(\alpha * \beta) = 0$ .

LEMMA 2. *If  $\diamond = \{f_\alpha: \alpha \in E\}$  is a minimal  $\diamond_\kappa(E)$ -sequence, then the ordered space  $X(\diamond)$  induced by the lexicographic ordering is compact.*

*Proof.* We show that  $\triangleleft$  is Dedekind-complete. That is, we show that for each  $W \subset E$  such that  $\alpha \in W$ ,  $\beta \in E - W$  implies  $f_\alpha \triangleleft f_\beta$ , there is  $\eta \in E$  such that,  $\alpha \in W$  implies  $f_\alpha \triangleleft f_\eta$  and such that  $\alpha \in E - W$  implies  $f_\eta \triangleleft f_\alpha$ . Define  $f: \kappa \rightarrow 2$  recursively by  $f(\gamma) = 1$  iff there is  $\alpha \in W$  such that  $f_\alpha \upharpoonright \gamma = f \upharpoonright \gamma$  and  $f_\alpha(\gamma) = 1$ . This  $f$  is “between”  $\{f_\alpha: \alpha \in W\}$  and  $\{f_\alpha: \alpha \in E - W\}$ . Since  $\{f_\alpha: \alpha \in E\}$  is a  $\diamond$ -sequence, there is  $\eta \in E$  such that  $f \supset f_\eta$ . Fix  $\alpha \neq \eta$ .

If  $\alpha \in W$  and  $f_\alpha \triangleright f_\eta$  then  $f_\alpha(\alpha * \eta) = 1$  and  $f_\eta(\alpha * \eta) = 0$ . By the construction of  $f \supset f_\eta$ ,  $f(\alpha * \eta) = 1$  and that is a contradiction.

If  $\alpha \in E - W$  and  $f_\eta \triangleright f_\alpha$  then  $f_\eta(\alpha * \eta) = 1$  and  $f_\alpha(\alpha * \eta) = 0$ . By the construction of  $f \supset f_\eta$ , there is  $\beta \in W$  such that  $f_\beta \upharpoonright \alpha * \eta = f \upharpoonright \alpha * \eta$  and  $f_\beta(\alpha * \eta) = 1$ . Therefore  $f_\beta \upharpoonright \alpha * \eta = f_\alpha \upharpoonright \alpha * \eta$  and  $f_\alpha(\alpha * \eta) < f_\beta(\alpha * \eta)$  so  $f_\alpha \triangleleft f_\beta$  which is a contradiction.

The space  $X(\diamond)$  may not be connected but it is dense-in-itself when  $E$  consists of limit ordinals and that is what we need to make it connected.

LEMMA 3. *If  $\diamond = \{f_\alpha: \alpha \in E\}$  is a minimal  $\diamond_\kappa(E)$ -sequence and  $E$  is a set of limit ordinals, then  $X(\diamond)$  is dense-in-itself. To prove Lemma 3, one must essentially prove that the character of each point  $f_\alpha$  in  $X(\diamond)$  ( $\chi(f_\alpha, X(\diamond))$ ) (see [1]) is determined by its index:*

LEMMA 4. *If  $\diamond = \{f_\alpha: \alpha \in E\}$  is a minimal  $\diamond_\kappa(E)$ -sequence and  $\alpha \in E$  is a limit ordinal, then  $\chi(f_\alpha, X(\diamond)) = \text{cf } \alpha$ .*

*Proof of Lemmas 3 and 4.* It suffices to construct either (1) a  $\triangleleft$ -increasing  $\text{cf}(\alpha)$ -sequence which converges to  $f_\alpha$  and a  $W \in [E]^{\leq \text{cf}(\alpha)}$  such that, for each  $\beta \in E$ , with  $f_\beta \triangleright f_\alpha$ , there is  $\gamma \in W$  such that  $f_\beta \geq f_\gamma \triangleright f_\alpha$ ,

or (2) a  $\triangleleft$ -decreasing  $\text{cf}(\alpha)$ -sequence which converges to  $f_\alpha$  and a  $W \in [E]^{\leq \text{cf}(\alpha)}$  such that, for each  $\beta \in E$  with  $f_\beta \triangleleft f_\alpha$ , there is  $\gamma \in W$  such that  $f_\beta \triangleleft f_\gamma \triangleleft f_\alpha$ . Let  $B$  be a cofinal subset of  $\alpha$  of cardinality  $\text{cf}(\alpha)$ . For each  $\beta \in B$ , construct  $g_\beta: \kappa \rightarrow 2$  by  $g_\beta \upharpoonright \beta = f_\alpha \upharpoonright \beta$ ,  $g_\beta(\beta) \neq f_\alpha(\beta)$ ,  $g_\beta \upharpoonright \kappa - (\beta + 1) \equiv 1 - f_\alpha(\beta)$  and find  $\gamma(\beta) \in E: f_{\gamma(\beta)} \subset g_\beta$ .

Find  $i \in 2: |f_\alpha^{-1}(i) \cap B| = \text{cf}(\alpha)$ . If  $i = 0$ , we construct (2) If  $i = 1$ , we construct (1) Fix  $i = 0$  for this proof.

$\{f_{\gamma(\beta)}: \beta \in B; f_\alpha(\beta) = 0\}$  is the  $\alpha$ -sequence.

Let  $\beta > \beta'$  be fixed where  $f_\alpha(\beta) = f_\alpha(\beta') = 0$ .

By the construction  $f_{\gamma(\beta')}(\beta') = g_{\beta'}(\beta') \neq f_\alpha(\beta') = 0$  and  $f_{\gamma(\beta)}(\beta') = g_\beta(\beta') = f_\alpha(\beta') = 0$  while  $f_{\gamma(\beta')} \upharpoonright \beta' = g_{\beta'} \upharpoonright \beta' = f_\alpha \upharpoonright \beta' = g_\beta \upharpoonright \beta' = f_{\gamma(\beta)} \upharpoonright \beta'$ . This implies  $f_{\gamma(\beta)} \triangleleft f_{\gamma(\beta')}$  and so the  $\alpha$ -sequence is  $\triangleleft$ -decreasing. We show that the  $\alpha$ -sequence converges to  $f_\alpha$ . Otherwise  $\exists \delta \in E \forall \beta \in f_\alpha^{-1}(0) \cap B. f_\alpha \triangleleft f_\delta \triangleleft f_{\gamma(\beta)}$ .  $|f_\alpha^{-1}(0) \cap B| = \alpha \Rightarrow \exists \beta > (\alpha * \delta): f_\alpha(\beta) = 0. f_{\gamma(\beta)} \subset g_\beta$  and  $g_\beta \upharpoonright \beta = f_\alpha \upharpoonright \beta$  and  $f_{\gamma(\beta)}$  incompatible with  $f_\alpha$  implies  $\gamma(\beta) > \beta$ .

$f_{\gamma(\beta)} \upharpoonright (\alpha * \delta) + 1 = g_\beta \upharpoonright (\alpha * \delta) + 1 = f_\alpha \upharpoonright (\alpha * \delta) + 1$  implies  $f_{\gamma(\beta)} \upharpoonright (\alpha * \delta) = f_\alpha \upharpoonright (\alpha * \delta)$ .  $f_{\gamma(\beta)}(\alpha * \delta) = f_\alpha(\alpha * \delta) < f_\delta(\alpha * \delta)$  implies  $f_{\gamma(\beta)} \triangleleft f_\delta$ .

Let  $W = \{\gamma(\beta): f_\alpha(\beta) = 1, \beta \in B\}$ . Suppose  $\beta \in E$  and  $f_\beta \triangleleft f_\alpha$ . We find  $\delta \in W$  such that  $f_\beta \triangleleft f_\delta \triangleleft f_\alpha$ .  $f_\beta(\beta * \alpha) = 0$  and  $f_\alpha(\beta * \alpha) = 1$ . Let  $\delta = \gamma(\beta * \alpha)$ ,  $f_\delta \subset g_{(\beta * \alpha)}$  and  $f_\alpha \upharpoonright \beta * \alpha = g_{(\beta * \alpha)} \upharpoonright \beta * \alpha = f_\alpha \upharpoonright \beta * \alpha$  while  $f_\delta(\beta * \alpha) = g_{(\beta * \alpha)}(\beta * \alpha) \neq f_\alpha(\beta * \alpha) = 1$  so that  $f_\delta \triangleleft f_\alpha$ .  $f_\beta \upharpoonright \beta * \alpha = f_\alpha \upharpoonright \beta * \alpha = f_\delta \upharpoonright \beta * \alpha$  while  $f_\beta(\beta * \alpha) = f_\delta(\beta * \alpha) = 0$  and  $f_\delta \upharpoonright \delta - ((\beta * \alpha) + 1) \equiv 1$  implies  $f_\beta \triangleleft f_\delta$ . Lemma 3 enables us to make  $X(\diamond)$  connected.

**LEMMA 5.** *If  $(X, <)$  is a compact ordered space which is dense-in-itself then letting an equivalence relation  $\sim$  on  $X$  be defined by  $x \sim y$  if there is no  $z \in X$  such that  $x < z < y$ ,  $(X/\sim, <)$  is a continuum.*

The basic theorem can be proved now.

**THEOREM 1.** *If there is a cardinal  $\kappa$  and a set of regular infinite cardinals  $E \subset \kappa$  such that  $\diamond_\kappa(E)$  holds, then there is a ordered continuum with no two points of equal character.*

*Proof.* If  $\diamond$  is a  $\diamond_\kappa(E)$ -sequence, then Lemma 1 implies that we may assume  $\diamond$  is minimal. Lemmas 2 and 3 implies that  $X(\diamond)$  is compact and dense-in-itself. Lemma 4 implies that the character of a point is the regular infinite cardinal by which it is indexed. Lemma 5 produces an ordered continuum  $X(\diamond)/\sim$  in which the character of an equivalence

class is the maximum of the characters of its elements (since equivalence classes are finite), and the theorem is proved.

A partial converse can be proved.

**THEOREM 2.** *If there is a compact Hausdorff space  $X$  with no two points of equal character, then this is a cardinal  $\kappa$  and a set of infinite cardinals  $E \subset \kappa$  such that there is a  $\diamond_\kappa(E)$ -sequence.*

*Proof of Theorem 2.* Let  $\kappa = |X|$ . Let  $\{P_\alpha: \alpha \in \kappa\}$  enumerate  $[X]^2$ . Define  $F \subset \kappa^2$  and  $\{U_f: f \in F\}$  a family of open sets such that

1.  $|\cup\{U_{f \upharpoonright \beta} : \beta < \text{dom } f\}| > 1$  iff  $f \in F$
2.  $f \in F \Rightarrow \cup\{U_{f \upharpoonright \beta} : \beta < \text{dom } f\} = \emptyset$
3.  $f, f' \in F, |f' - f| = 1$  and  $f' \supset f$  implies  $\bar{U}_{f'} \subset U_f$
4.  $f, f', f'' \in F, |f' - f| = 1 = |f'' - f|$  and  $f' \neq f''$  and  $f' \supset f, f'' \supset f$  implies  $U_f \cap U_{f'} = \emptyset$
5.  $f \in F \Rightarrow |U_f \cap P_{\text{dom } f}| \leq 1$
6.  $U_\emptyset$  is an open set containing no isolated point and no point of character  $\kappa$ .

For any  $g: \kappa \rightarrow 2$ , there is a unique  $x_g \in \cap\{U_f: f \in F, f \subset g\}$ . If the character of  $x_g$  is  $\alpha_g < \kappa$  then let  $f_{\alpha_g}: \alpha_g \rightarrow \alpha$  be defined by  $f_{\alpha_g} \subset g$ . This is well-defined since no two points have equal character. Let  $E = \{\alpha_g: g \in \kappa^2\}$ .

This proof is a simple modification of P.291 of [1].

**COROLLARY TO THEOREM 1.** ( $V = L + \exists$  Mahlo cardinal.) *There is an ordered continuum with no two points of equal character.*

*Proof.* The definition of a Mahlo cardinal is a regular cardinal  $\kappa$  which has a stationary subset  $E$  of regular infinite cardinals. Under  $V = L$ , whenever  $\kappa$  is a regular cardinal and  $E$  is a stationary subset of  $\kappa$  then  $\diamond_\kappa(E)$  holds (see p. 181 of [2]).

**COROLLARY TO THEOREM 2.** ( $\aleph$  weakly inaccessible cardinal.) *Any compact Hausdorff space has two points of equal character.*

*Proof.* Let  $\kappa$  be the least cardinal such that there is a set of infinite cardinals  $E \subset \kappa$  and a  $\diamond_\kappa(E)$ -sequence  $\{f_\alpha: \alpha \in E\}$ . If  $\kappa$  is a successor, then  $\kappa$  is not minimal. If  $\kappa$  is a singular limit, then  $\kappa$  is also not minimal but an argument is needed.

Let  $C$  be a closed unbounded set of order-type  $\text{cf}(\kappa)$ .

Let  $D$  be the set of limit points of  $C$ .

Any cardinal in  $D$  has cofinality less than  $\text{cf}(\kappa)$ ; thus  $D \cap E \subset \text{cf}(\kappa)$ .

Let  $F = D - \text{cf}(\kappa)$ ; thus  $E \cap F = \emptyset$ . If  $\alpha \in F$ , whenever possible, let  $\alpha^-$  be the greatest element of  $F \cup \{0\}$  which is smaller than  $\alpha$ . For each  $\alpha \in F$ , by minimality, there is  $g_\alpha: (\{\gamma \in \kappa: \alpha^- \leq \gamma < \alpha\}) \rightarrow 2$  which does not contain any  $f_\gamma \upharpoonright \gamma - \alpha$  whenever  $\alpha^- \leq \gamma < \alpha$ .

Let  $h: \kappa \rightarrow 2$  contain each  $g_\alpha$ .  $h$  does not contain any  $f_\gamma$  whenever  $\gamma \in E$ .

We demonstrate that compactness is an essential condition in these results by proving

**THEOREM 3. (ZFC)** *There is a  $\sigma$ -compact Hausdorff space  $X$  in which no two points have equal character.*

We need a set-theoretic lemma.

**LEMMA 6.** *There is an infinite cardinal  $\kappa$  such that  $\aleph_\kappa = \kappa$ .*

*Proof.* Induction on  $\alpha \leq \omega_1$ . Let  $\kappa_0 = \omega$ . Let  $\kappa_{\alpha+1} = \aleph_{\kappa_\alpha}$ . Let  $\kappa_\alpha = \sup\{\kappa_\beta: \beta < \alpha\}$  when  $\alpha$  is a limit. Let  $\kappa = \kappa_{\omega_1}$ .

We need to construct a tree

**LEMMA 7.** *There is an infinitely branching tree  $(T, <)$  of height  $\omega$  such that (letting  $t^* = \{t' \in T: t' > t \text{ and level } t' = \text{level } t + 1\}$ ; the immediate successors of  $t$ )  $t, t' \in T$  and  $t \neq t'$  implies  $|t^*| \neq |t'^*|$ .*

*Proof.* Construct the tree on  $\kappa$  of Lemma 6 by induction on level. If level  $n$  has been constructed, let  $T_n$  be the set of nodes at height less than  $n$ , let  $S_n$  be the set of nodes at height  $n$ , let  $A$  be the set of cardinals less than  $\kappa$ .

We carry an induction hypothesis that  $|A - \{|t^*|: t \in T_n\}| = \kappa$  and find an injection  $\pi: S_n \rightarrow A - \{|t^*|: t \in T_n\}$ .

Define  $t \in S_n$  to have  $\pi(t)$ -many immediate successors.

*Proof of Theorem 3.* Topologize the tree of Lemma 7 by letting a neighborhood of  $t \in T$  be defined in each  $F \in [t^*]^{<\omega}$  by  $U_f(t) = \{s \in T: (1) s > t \text{ and } (2) u \in F \text{ implies } s \not\geq u\}$ .  $T_n$  is a compact subset of  $T$  and so  $T$  is  $\sigma$ -compact.

The corollary to Theorem 2 was independently obtained by Peg Daniels. The author thanks the referee for many useful comments on the proof of Theorem 2 and the corollary to Theorem 2.

#### REFERENCES

- [1] R. Engelking, *General Topology*, PWN, Polish Scientific Publishers, Warsaw, 1979.
- [2] K. Kunen, *Set Theory*, North Holland, Amsterdam, 1983.

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