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In this paper, we show that the elements of a $\diamondsuit$-sequence can be ordered lexicographically to produce an ordered continuum. An application of this idea answers a question of V. Malyhin and others: Is there a compact Hausdorff space in which no two points have equal character? We show that the consistency strength of the existence of such a space lies between that of an inaccessible and a Mahlo cardinal. We show that compactness is essential in this result by constructing, in ZFC, a σ-compact Hausdorff space in which no two points have equal character.

Let us begin with some definitions:

DEFINITION 1. $\{f_\alpha\colon \alpha \in E\}$ is a $\diamondsuit_\kappa(E)$-*sequence* (where $E \subset \kappa - \{0\}$ and $f_\alpha\colon \alpha \to 2$) if, for each $f\colon \kappa \to 2$, there is $\alpha \in E$ such that $f_\alpha \subset f$.

This is not exactly the standard definition (we use the characteristic functions of subsets of κ, we trap only once and do not require that E be stationary or even cofinal in κ) but it is equivalent in most cases. The lexicographic ordering is not well defined because their domains are not equal. We need to subtract some $f_\alpha s$ which are "not needed". Let us fix this idea.

DEFINITION 2. $\{f_\alpha\colon \alpha \in E\}$ is a *minimal* $\diamondsuit_\kappa(E)$-sequence if, whenever $F \subset E$ and $\{f_\alpha\colon \alpha \in F\}$ is a $\diamondsuit_\kappa(F)$-sequence, F must equal E.

This seems like a strong condition but it is not. We *can* subtract the f_α's which are not needed.

LEMMA 1. *If* $\{f_\alpha\colon \alpha \in E\}$ *is a* $\diamondsuit_\kappa(E)$-*sequence, then there is* $F \subset E$ *such that*

$$\{f_\alpha\colon \alpha \in F\} \text{ is a minimal } \diamondsuit_\kappa(F)\text{-sequence.}$$

Proof. The idea is to inductively subtract any f_α compatible with f_β when $\beta < \alpha$. That is, $\alpha \in F$ iff, for each $\beta \in \alpha \cap F$, $f_\beta \cup f_\alpha$ is not a function. $\{f_\alpha\colon \alpha \in F\}$ is a $\diamondsuit_\kappa(F)$-sequence since, for each $f\colon \kappa \to 2$ there is a minimal $\alpha \in E$ such that $f \restriction \alpha = f_\alpha$. By the construction of F and the minimality of α, $\alpha \in F$. $\{f_\alpha\colon \alpha \in F\}$ is minimal since, for each $\alpha \in F$, we can find $f\colon \kappa \to 2$ such that $f \supset f_\alpha$. By the construction of F, there does not exist $\beta \in F$ such that $\beta \neq \alpha$ and $f \supset f_\beta$.

Minimality implies that the lexicographic ordering of a minimal $\diamondsuit_\kappa(E)$-sequence is well-defined. Meanwhile, the $\diamondsuit$ prediction property implies compactness.

Some notation is useful:

Let $\alpha * \beta$ be $\inf\{\gamma\colon f_\alpha(\gamma) \neq f_\beta(\gamma)\}$.

The lexicographic ordering $\lhd$ is defined by $f_\alpha \lhd f_\beta$ iff $f_\alpha(\alpha * \beta) = 0$.

LEMMA 2. *If* $\diamondsuit = \{f_\alpha\colon \alpha \in E\}$ *is a minimal* $\diamondsuit_\kappa(E)$*-sequence, then the ordered space* $X(\diamondsuit)$ *induced by the lexicographic ordering is compact.*

Proof. We show that $\lhd$ is Dedekind-complete. That is, we show that for each $W \subset E$ such that $\alpha \in W$, $\beta \in E - W$ implies $f_\alpha \lhd f_\beta$, there is $\eta \in E$ such that, $\alpha \in W$ impies $f_\alpha \unlhd f_\eta$ and such that $\alpha \in E - W$ implies $f_\eta \unlhd f_\alpha$. Define $f\colon \kappa \to 2$ recursively by $f(\gamma) = 1$ iff there is $\alpha \in W$ such that $f_\alpha \restriction \gamma = f \restriction \gamma$ and $f_\alpha(\gamma) = 1$. This f is "between" $\{f_\alpha\colon \alpha \in W\}$ and $\{f_\alpha\colon \alpha \in E - W\}$. Since $\{f_\alpha\colon \alpha \in E\}$ is a $\diamondsuit$-sequence, there is $\eta \in E$ such that $f \supset f_\eta$. Fix $\alpha \neq \eta$.

If $\alpha \in W$ and $f_\alpha \rhd f_\eta$ then $f_\alpha(\alpha * \eta) = 1$ and $f_\eta(\alpha * \eta) = 0$. By the construction of $f \supset f_\eta$, $f(\alpha * \eta) = 1$ and that is a contradiction.

If $\alpha \in E - W$ and $f_\eta \rhd f_\alpha$ then $f_\eta(\alpha * \eta) = 1$ and $f_\alpha(\alpha * \eta) = 0$. By the construction of $f \supset f_\eta$, there is $\beta \in W$ such that $f_\beta \restriction \alpha * \eta = f \restriction \alpha * \eta$ and $f_\beta(\alpha * \eta) = 1$. Therefore $f_\beta \restriction \alpha * \eta = f_\alpha \restriction \alpha * \eta$ and $f_\alpha(\alpha * \eta) < f_\beta(\alpha * \eta)$ so $f_\alpha \lhd f_\beta$ which is a contradiction.

The space $X(\diamondsuit)$ may not be connected but it *is* dense-in-itself when E consists of limit ordinals and that is what we need to make it connected.

LEMMA 3. *If* $\diamondsuit = \{f_\alpha\colon \alpha \in E\}$ *is a minimal* $\diamondsuit_\kappa(E)$*-sequence and E is a set of limit ordinals, then* $X(\diamondsuit)$ *is dense-in-itself. To prove Lemma* 3, *one must essentially prove that the character of each point* f_α *in* $X(\diamondsuit)$ $(\chi(f_\alpha, X(\diamondsuit)))$ (*see* [**1**]) *is determined by its index*:

LEMMA 4. *If* $\diamondsuit = \{f_\alpha\colon \alpha \in E\}$ *is a minimal* $\diamondsuit_\kappa(E)$*-sequence and* $\alpha \in E$ *is a limit ordinal, then* $\chi(f_\alpha, X(\diamondsuit)) = \operatorname{cf} \alpha$.

Proof of Lemmas 3 *and* 4. It suffices to construct either (1) a $\lhd$-increasing $\operatorname{cf}(\alpha)$-sequence which converges to f_α and a $W \in [E]^{\leq \operatorname{cf}(\alpha)}$ such that, for each $\beta \in E$, with $f_\beta \rhd f_\alpha$, there is $\gamma \in W$ such that $f_\beta \unrhd f_\gamma \rhd f_\alpha$, or (2) a $\lhd$-decreasing cf(α)-sequence which converges to f_α and a $W \in [E]^{\leq \mathrm{cf}(\alpha)}$ such that, for each $\beta \in E$ with $f_\beta \lhd f_\alpha$, there is $\gamma \in W$ such that $f_\beta \unlhd f_\gamma \lhd f_\alpha$. Let B be a cofinal subset of α of cardinality cf(α). For each $\beta \in B$, construct g_β: $\kappa \to 2$ by $g_\beta \upharpoonright \beta = f_\alpha \upharpoonright \beta$, $g_\beta(\beta) \neq f_\alpha(\beta)$, $g_\beta \upharpoonright \kappa - (\beta + 1) \equiv 1 - f_\alpha(\beta)$ and find $\gamma(\beta) \in E$: $f_{\gamma(\beta)} \subset g_\beta$.

Find $i \in 2$: $|f_\alpha^{-1}(i) \cap B| = \mathrm{cf}(\alpha)$. If $i = 0$, we construct (2) If $i = 1$, we construct (1) Fix $i = 0$ for this proof.

$\{ f_{\gamma(\beta)}$: $\beta \in B$; $f_\alpha(\beta) = 0\}$ is the α-sequence.

Let $\beta > \beta'$ be fixed where $f_\alpha(\beta) = f_\alpha(\beta') = 0$.

By the construction $f_{\gamma(\beta')}(\beta') = g_{\beta'}(\beta') \neq f_\alpha(\beta') = 0$ and $f_{\gamma(\beta)}(\beta') = g_\beta(\beta') = f_\alpha(\beta') = 0$ while $f_{\gamma(\beta')} \upharpoonright \beta' = g_{\beta'} \upharpoonright \beta' = f_\alpha \upharpoonright \beta' = g_\beta \upharpoonright \beta' = f_{\gamma(\beta)} \upharpoonright \beta'$. This implies $f_{\gamma(\beta)} \lhd f_{\gamma(\beta')}$ and so the α-sequence is $\lhd$-decreasing. We show that the α-sequence converges to f_α. Otherwise $\exists \delta \in E \forall \beta \in f_\alpha^{-1}(0) \cap B$. $f_\alpha \lhd f_\delta \lhd f_{\gamma(\beta)}$. $|f_\alpha^{-1}(0) \cap B| = \alpha \Rightarrow \exists \beta > (\alpha * \delta)$: $f_\alpha(\beta) = 0$. $f_{\gamma(\beta)} \subset g_\beta$ and $g_\beta \upharpoonright \beta = f_\alpha \upharpoonright \beta$ and $f_{\gamma(\beta)}$ incompatible with f_α implies $\gamma(\beta) > \beta$.

$f_{\gamma(\beta)} \upharpoonright (\alpha * \delta) + 1 = g_\beta \upharpoonright (\alpha * \delta) + 1 = f_\alpha \upharpoonright (\alpha * \delta) + 1$ implies $f_{\gamma(\beta)} \upharpoonright (\alpha * \delta) = f_\delta \upharpoonright (\alpha * \delta)$. $f_{\gamma(\beta)}(\alpha * \delta) = f_\alpha(\alpha * \delta) < f_\delta(\alpha * \delta)$ implies $f_{\gamma(\beta)} \lhd f_\delta$.

Let $W = \{\gamma(\beta)$: $f_\alpha(\beta) = 1$, $\beta \in B\}$. Suppose $\beta \in E$ and $f_\beta \lhd f_\alpha$. We find $\delta \cap W$ such that $f_\beta \unlhd f_\delta \lhd f_\alpha$. $f_\beta(\beta * \alpha) = 0$ and $f_\alpha(\beta * \alpha) = 1$. Let $\delta = \gamma(\beta * \alpha)$, $f_\delta \subset g_{(\beta * \alpha)}$ and $f_\alpha \upharpoonright \beta * \alpha = g_{(\beta * \alpha)} \upharpoonright \beta * \alpha = f_\alpha \upharpoonright \beta * \alpha$ while $f_\delta(\beta * \alpha) = g_{(\beta * \alpha)}(\beta * \alpha) \neq f_\alpha(\beta * \alpha) = 1$ so that $f_\delta \lhd f_\alpha$. $f_\beta \upharpoonright \beta * \alpha = f_\alpha \upharpoonright \beta * \alpha = f_\delta \upharpoonright \beta * \alpha$ while $f_\beta(\beta * \alpha) = f_\delta(\beta * \alpha) = 0$ and $f_\delta \upharpoonright \delta - ((\beta * \alpha) + 1) \equiv 1$ implies $f_\beta \unlhd f_\delta$. Lemma 3 enables us to make $X(\diamondsuit)$ connected.

LEMMA 5. *If* $(X, <)$ *is a compact ordered space which is dense-in-itself then letting an equivalence relation* $\sim$ *on X be defined by* $x \sim y$ *if there is no* $z \in X$ *such that* $x < z < y$, $(X/\sim, <)$ *is a continuum.*

The basic theorem can be proved now.

THEOREM 1. *If there is a cardinal* κ *and a set of regular infinite cardinals* $E \subset \kappa$ *such that* $\diamondsuit_\kappa(E)$ *holds, then there is a ordered continuum with no two points of equal character.*

Proof. If $\diamondsuit$ is a $\diamondsuit_\kappa(E)$-sequence, then Lemma 1 implies that we may assume $\diamondsuit$ is minimal. Lemmas 2 and 3 implies that $X(\diamondsuit)$ is compact and dense-in-itself. Lemma 4 implies that the character of a point is the regular infinite cardinal by which it is indexed. Lemma 5 produces an ordered continuum $X(\diamondsuit)/\sim$ in which the character of an equivalence class is the maximum of the characters of its elements (since equivalence classes are finite), and the theorem is proved.

A partial converse can be proved.

THEOREM 2. *If there is a compact Hausdorff space X with no two points of equal character, then this is a cardinal* κ *and a set of infinite cardinals* $E \subset \kappa$ *such that there is a* $\diamondsuit_\kappa(E)$*-sequence.*

Proof of Theorem 2. Let $\kappa = |X|$. Let $\{P_\alpha\colon \alpha \in \kappa\}$ enumerate $[X]^2$. Define $F \subset {}^\kappa 2$ and $\{U_f\colon f \in F\}$ a family of open sets such that

1. $|\bigcup\{U_{f\restriction\beta}\ \beta < \operatorname{dom} f\}| > 1$ iff $f \in F$
2. $f \in F \Rightarrow \bigcup\{U_{f\restriction\beta}\colon \beta < \operatorname{dom} f\} = \varnothing$
3. $f, f' \in F$, $|f' - f| = 1$ and $f' \supset f$ implies $\overline{U}_{f'} \subset U_f$
4. $f, f', f'' \in F$, $|f' - f| = 1 = |f'' - f|$ and $f' \neq f''$ and $f' \supset f$, $f'' \supset g$ implies $U_f \cap U_{f'} = \varnothing$
5. $f \in F \Rightarrow |U_f \cap P_{\operatorname{dom} f}| \leq 1$
6. $U_\varnothing$ is an open set containing no isolated point and no point of character κ.

For any $g\colon \kappa \to 2$, there is a unique $x_g \in \bigcap\{U_f\colon f \in F,\ f \subset g\}$. If the character of x_g is $\alpha_g < \kappa$ then let $f_{\alpha_g}\colon \alpha_g \to \alpha$ be defined by $f_{\alpha_g} \subset g$. This is well-defined since no two points have equal character. Let $E = \{\alpha_g\colon g \in {}^\kappa 2\}$.

This proof is a simple modification of P.291 of [**1**].

COROLLARY TO THEOREM 1. ($V = L + \exists$ *Mahlo cardinal.*) *There is an ordered continuum with no two points of equal character.*

Proof. The definition of a Mahlo cardinal is a regular cardinal κ which has a stationary subset E of regular infinite cardinals. Under $V = L$, whenever κ is a regular cardinal and E is a stationary subset of κ then $\diamondsuit_\kappa(E)$ holds (see p. 181 of [**2**]).

COROLLARY TO THEOREM 2. ($\nexists$ *weakly inaccessible cardinal.*) *Any compact Hausdorff space has two points of equal character.*

Proof. Let κ be the least cardinal such that there is a set of infinite cardinals $E \subset \kappa$ and a $\diamondsuit_\kappa(E)$-sequence $\{f_\alpha\colon \alpha \in E\}$. If κ is a successor, then κ is not minimal. If κ is a singular limit, then κ is also not minimal but an argument is needed.

Let C be a closed unbounded set of order-type $\mathrm{cf}(\kappa)$.

Let D be the set of limit points of C.

Any cardinal in D has cofinality less than $\mathrm{cf}(\kappa)$; thus $D \cap E \subset \mathrm{cf}(\kappa)$.

Let $F = D - \mathrm{cf}(\kappa)$; thus $E \cap F = \varnothing$. If $\alpha \in F$, whenever possible, let α^- be the greatest element of $F \cup \{0\}$ which is smaller than α. For each $\alpha \in F$, by minimality, there is g_α: $(\{\gamma \in \kappa\colon \alpha^- \leq \gamma < \alpha\}) \to 2$ which does not contain any $f_\gamma \upharpoonright \gamma - \alpha$ whenever $\alpha^- \leq \gamma < \alpha$.

Let $h\colon \kappa \to 2$ contain each g_α. h does not contain any f_γ whenever $\gamma \in E$.

We demonstrate that compactness is an essential condition in these results by proving

THEOREM 3. (ZFC) *There is a σ-compact Hausdorff space X in which no two points have equal character.*

We need a set-theoretic lemma.

LEMMA 6. *There is an infinite cardinal κ such that $\aleph_\kappa = \kappa$.*

Proof. Induction on $\alpha \leq \omega_1$. Let $\kappa_0 = \omega$. Let $\kappa_{\alpha+1} = \aleph_{\kappa_\alpha}$. Let $\kappa_\alpha = \sup\{\kappa_\beta\colon \beta < \alpha\}$ when α is a limit. Let $\kappa = \kappa_{\omega_1}$.

We need to construct a tree

LEMMA 7. *There is an infinitely branching tree $(T, <)$ of height ω such that (letting $t^* = \{t' \in T\colon t' > t$ and level $t' =$ level $t + 1\}$; the immediate successors of t) $t, t' \in T$ and $t \neq t'$ implies $|t^*| \neq |t'^*|$.*

Proof. Construct the tree on κ of Lemma 6 by induction on level. If level n has been constructed, let T_n be the set of nodes at height less than n, let S_n be the set of nodes at height n, let A be the set of cardinals less than κ.

We carry an induction hypothesis that $|A - \{|t^*|\colon t \in T_n\}| = \kappa$ and find an injection $\pi\colon S_n \to A - \{|t^*|\colon t \in T_n\}$.

Define $t \in S_n$ to have $\pi(t)$-many immediate successors.

Proof of Theorem 3. Topologize the tree of Lemma 7 by letting a neighborhood of $t \in T$ be defined in each $F \in [t^*]^{<\omega}$ by $U_f(t) = \{s \in T\colon$ (1) $s > t$ and (2) $u \in F$ implies $s \not\geq u\}$. T_n is a compact subset of T and so T is σ-compact.

The corollary to Theorem 2 was independently obtained by Peg Daniels. The author thanks the referee for many useful comments on the proof of Theorem 2 and the corollary to Theorem 2.

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