The closure of the derivation $\lambda D : C_0^1(\mathbb{R}) \to C_0(\mathbb{R})$ defined by $(\lambda D)(f) = \lambda f^\prime$, where $\lambda : \mathbb{R} \to \mathbb{R}$ is continuous, generates a $C_0$-group on $C_0(\mathbb{R})$ (corresponding to a flow on $\mathbb{R}$) if and only if $1/\lambda$ is not locally integrable on either side of any zero of $\lambda$ or at $\pm \infty$.

If $S$ is a flow on a locally compact, Hausdorff, space $X$ with fixed point set $X_0^0$, $\delta_S$ is the generator of the induced action on $C_0(X)$, $\lambda : X \setminus X_0^0 \to \mathbb{R}$ is continuous, and bounded on sets of low frequency under $S$, and $t \to \lambda(S_t \omega)^{-1}$ is not locally integrable on either side of any zero or at $\pm \infty$, then the flows along the orbits of $S$ form a flow on $X$ whose generator acts as $\lambda \delta_S$.

1. Introduction. Let $S$ be a flow on a locally compact, Hausdorff, space $X$, and $\delta_S$ be the generator of the associated one-parameter group of $*$-automorphisms of $C_0(X)$, the commutative $C^*$-algebra of continuous complex-valued functions on $X$ which vanish at infinity. Thus
\[
\delta_S f = \lim_{t \to 0} t^{-1} (f \circ S_t - f)
\]
whenever the limit exists (pointwise, and hence uniformly) and defines a function in $C_0(X)$. Let $\mathcal{D}_S^\infty = \cap_{n \geq 1} \mathcal{D}(\delta_S^n)$. Then $\mathcal{D}_S^\infty$ is a dense $*$-subalgebra of $C_0(X)$. If $\delta : \mathcal{D}_S^\infty \to C_0(X)$ is a $*$-derivation, then there is a function $\lambda : X \to \mathbb{R}$ such that
\[
\delta f = \lambda \delta_S f \quad (f \in \mathcal{D}_S^\infty)
\]
[1]. The function $\lambda$ may be chosen arbitrarily on the fixed point set $X_0^0$:
\[
X_0^0 = \{ \omega \in X : S_t \omega = \omega \text{ for all } t \}
= \{ \omega \in X : \delta_S f(\omega) = 0 \text{ for all } f \text{ in } \mathcal{D}_S^\infty \},
\]
and we shall always assume that $\lambda = 0$ on $X_0^0$. However, $\lambda$ is uniquely determined and continuous on $X \setminus X_0^0$, and satisfies a bound of the form
\[
|\lambda(\omega)| \leq c (1 + v(\omega)^n) \quad (\omega \in X \setminus X_0^0)
\]
for some constant $c \geq 0$, and integer $n \geq 0$, where $v(\omega)$ is the frequency of $\omega$, so
\[
v(\omega)^{-1} = \inf\{ t > 0 : S_t \omega = \omega \}
\]
($v(\omega) = 0$ if $\omega$ is aperiodic) (see [4]).
We shall therefore study the \( \ast \)-derivations \( \lambda \delta_S \) defined by
\[
(\lambda \delta_S)f = \begin{cases} 
\lambda \delta_S f & \text{on } X \setminus X^0_S \\
0 & \text{on } X^0_S
\end{cases}
\]
whenever the right-hand side defines a function in \( C_0(X) \). Here \( \lambda: X \setminus X^0_S \to \mathbb{R} \) is a continuous function. The domain \( \mathcal{D}(\lambda \delta_S) \) contains \( \mathcal{D}_S^\infty \) if and only if \( \lambda \) satisfies a bound of the form (*)

Indeed for any \( \omega \) in \( X \setminus X^0_S \), \( \epsilon > 0 \) such that \( 2\epsilon v(\omega) < 1 \) and \( F \) in \( C^\infty[-\epsilon,\epsilon] \), there exists \( f \) in \( \mathcal{D}_S^\infty \) such that \( f(S,\omega) = F(t) \) (\(|t| \leq \epsilon\)), and \( \text{supp} f \subset X \setminus X^0_S \) \[4\]. In particular, \( f \in \mathcal{D}(\lambda \delta_S) \).

The properties of interest are whether there is a flow \( T \) whose generator \( \delta_T \) extends \( \lambda \delta_S \), and if so whether \( T \) is unique and whether \( \mathcal{D}(\lambda \delta_S) \) (or some smaller subalgebra) is a core for \( \delta_T \). Considering both functions which vary transversally and along the orbits of \( S \), it is apparent that \( T \) should be a flow along the orbits of \( S \) whose speed is given at each point by the function \( \lambda \). Thus
\[
T_sS_t \omega = S_{\tau_{\omega}(s,t)} \omega
\]
where \( \tau_\omega \) is a flow on \( \mathbb{R} \) such that
\[
\partial \tau_\omega/\partial t = \lambda_\omega \circ \tau_\omega
\]
where \( \lambda_\omega(s) = \lambda(S_s \omega) \).

The first stage (§2) therefore is to study flows \( T \) on \( \mathbb{R} \) satisfying the differential equation
\[
\partial T/\partial t = \lambda \circ T
\]
where \( \lambda: \mathbb{R} \to \mathbb{R} \) is a continuous function. If \( 1/\lambda \) is not locally integrable on either side of any zero of \( \lambda \) or at \( \pm \infty \), then there is a unique flow \( T \) of this type, each zero of \( \lambda \) is a fixed point of \( T \), and \( C_c^\infty(\mathbb{R}) \) is a core for \( \delta_T \). Otherwise, there may be no flows or there may be many flows.

In §3, it is shown that if each \( \lambda_\omega \) satisfies these conditions of reciprocal non-integrability, then the flows with speeds \( \lambda_\omega \) along the orbits together define a flow on \( X \) whose generator extends \( \lambda \delta_S \).

There is some overlap between §2 of this paper, a paper of de Laubenfels [6], which left several questions incompletely answered, and an unpublished manuscript of the author's [2] which has circulated and been cited quite widely. The results of §3 are more general than those obtained in [3, 7], where it was assumed that \( \lambda \) satisfies a Lipschitz condition
\[
|\lambda(S,\omega) - \lambda(\omega)| \leq |t|\kappa(\nu)
\]
whenever $v(\omega) \leq v$. Such a condition implies the reciprocal non-integrability conditions.

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2. The real line. Sakai [9] has raised the question of characterizing all flows $T$ on $[0,1]$ whose generator extends $\lambda D$, where $\lambda \in C[0,1]$ and $D$ denotes differentiation defined on $C^1[0,1]$. The motivation for this was the fact that, for any flow $T$ on $[0,1]$, there is a homeomorphism $\theta$ of $[0,1]$ such that $\delta_{\theta T \theta^{-1}}$ extends $\lambda D$ for some $\lambda$. Similar remarks apply to flows on $\mathbb{R}$, where $D$ itself is the generator for the flow of translations, and we shall work on the whole line, at least initially.

In fact, one can, by choosing $\theta$ appropriately, arrange that $\theta T \theta^{-1}$ is one of the flows $T^\varepsilon_U$ described in the following example [10, p. 26]. But this fact does not directly help to decide when $\lambda D$ extends to a generator, nor is it helpful in considering flows on general spaces.

Example 2.1. For each open interval $I$ in $\mathbb{R}$, define flows $T_I$ on $I$ as follows:

$$T_{(a, b)}(x, t) = \frac{b(x - a)e^{(b-a)t} + a(b - x)}{b - x + (x - a)e^{(b-a)t}},$$

$$T_{(a, \infty)}(x, t) = a + (x - a)e^t,$$

$$T_{(-\infty, b)}(x, t) = b + (x - b)e^{-t},$$

$$T_R(x, t) = x + t.$$ 

Now let $U$ be an open subset of $\mathbb{R}$, $\mathcal{C}_U$ be the set of all connected components of $U$, and $\varepsilon$ be a function of $\mathcal{C}_U$ into $\{-1,1\}$. Define

$$T^\varepsilon_U(x, t) = \begin{cases} T_I(x, \varepsilon(I)t) & (x \in I \in \mathcal{C}_U), \\ x & (x \in \mathbb{R} \setminus U). \end{cases}$$

Then $T^\varepsilon_U$ is a flow on $\mathbb{R}$, and its generator is the closure of $\lambda^\varepsilon_U D | C_c^\infty(\mathbb{R})$, where

$$\lambda^\varepsilon_U(x) = \begin{cases} \varepsilon((a, b))(x - a)(b - x) & (x \in (a, b) \in \mathcal{C}_U), \\ \varepsilon((a, \infty))(x - a) & (x \in (a, \infty) \in \mathcal{C}_U), \\ \varepsilon((-\infty, b))(b - x) & (x \in (-\infty, b) \in \mathcal{C}_U), \\ \varepsilon(\mathbb{R}) & (\text{if } U = \mathbb{R}), \\ 0 & (x \in \mathbb{R} \setminus U). \end{cases}$$
Let \( \lambda: \mathbb{R} \to \mathbb{R} \) be any continuous function, and put
\[
Z(\lambda) = \{ x \in \mathbb{R} : \lambda(x) = 0 \},
\]
\[
U(\lambda) = \mathbb{R} \setminus Z(\lambda) = \{ x : \lambda(x) \neq 0 \}.
\]
For \( x \) in \( U(\lambda) \), let
\[
\alpha_x = \sup\{ y < x : \lambda(y) = 0 \},
\]
\[
\beta_x = \inf\{ y > x : \lambda(y) = 0 \}
\]
with the convention that the supremum of the empty set is \(-\infty\), and the infimum is \(+\infty\).

Let \( A_i^+(\lambda) \) (respectively, \( A_i^-(\lambda) \)) be the set of all points \( x \) in \( Z(\lambda) \cup \{-\infty\} \) such that for some \( y < x \), \( \lambda \geq 0 \) (respectively, \( \lambda \leq 0 \)) in \( (y, x) \) and \( 1/\lambda \) is integrable over \( (y, x) \). Let \( A_r^+(\lambda) \) (respectively, \( A_r^-(\lambda) \)) be the set of all \( x \) in \( Z(\lambda) \cup \{ -\infty \} \) such that for some \( z > x \), \( \lambda \geq 0 \) (respectively, \( \lambda \leq 0 \)) in \( (x, z) \) and \( 1/\lambda \) is integrable over \( (x, z) \). Let
\[
A_i(\lambda) = A_i^+(\lambda) \cup A_i^-(\lambda),
\]
\[
A_r(\lambda) = A_r^+(\lambda) \cup A_r^-(\lambda),
\]
\[
A(\lambda) = A_i(\lambda) \cup A_r(\lambda).
\]
The first lemma specifies the properties which amount to a flow on \( \mathbb{R} \) having speed \( \lambda \). The proof is elementary and will be omitted.

**Lemma 2.2.** Let \( T \) be a flow on \( \mathbb{R} \), and \( \lambda: \mathbb{R} \to \mathbb{R} \) be continuous. The following are equivalent:

(i) \( T \) is differentiable with respect to \( t \), and \( \frac{\partial T}{\partial t} = \lambda \circ T \),

(ii) \( C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\delta_T) \) and \( \delta_T \) extends \( \lambda D | C_c^\infty(\mathbb{R}) \),

(iii) \( C_c^1(\mathbb{R}) \subset \mathcal{D}(\delta_T) \) and \( \delta_T \) extends \( \lambda D | C_c^1(\mathbb{R}) \),

(iv) If \( x \in U(\lambda) \) and \( T_t x \in (\alpha_x, \beta_x) \), then
\[
\int_{T_t x} \frac{dy}{\lambda(y)} = t;
\]
if \( x \in \text{int} Z(\lambda) \), then \( T_t x = x \).

**Corollary 2.3.** Let \( T \) be a flow with speed \( \lambda \) (so that \( T \) satisfies the conditions of Lemma 2.2) and \( x \in U_\lambda \). The following are equivalent:

(i) \( \{ T_t x : t \in \mathbb{R} \} \subset U(\lambda) \),

(ii) \( \{ T_t x : t \in \mathbb{R} \} = (\alpha_x, \beta_x) \),

(iii) \( \alpha_x \notin A_r(\lambda) \) and \( \beta_x \notin A_i(\lambda) \).

The following result (for \([0, 1]\) rather than \( \mathbb{R} \)) was included in [6], but no proof was given of the core property. The construction of \( T \) appeared earlier in [11].
THEOREM 2.4. Let $\lambda: \mathbb{R} \to \mathbb{R}$ be a continuous function. The following are equivalent:

(i) There is a flow $T$ such that $\delta_T$ is the closure of $\lambda D | C_c^\infty(\mathbb{R})$,

(ii) $A(\lambda) = \emptyset$.

Proof. (i) $\Rightarrow$ (ii). For $y$ in $Z(\lambda)$, $(\delta_T f)(y) = 0$ for all $f$ in $C_c^\infty(\mathbb{R})$, and hence for all $f$ in $\mathcal{D}(\delta_T)$. It follows that $T_y y = y$. Thus for $x$ in $U(\lambda)$, $(T_x x) \subset U(\lambda)$, so, by Corollary 2.3, $\alpha_x \not\in A_x(\lambda)$ and $\beta_x \not\in A_x(\lambda)$. Now if there exists $z$ in $A_x(\lambda)$, then there exists $x$ in $U(\lambda)$ such that $x < z$ and $1/\lambda$ is integrable over $(x, z)$ and therefore over $(x, \beta_x)$. But then $\beta_x \in A_x(\lambda)$, which is a contradiction. Similarly, $A_x(\lambda)$ is empty.

(ii) $\Rightarrow$ (i). For $x$ in $U(\lambda)$, there is a (unique) function $q$ such that $q(x) = 0$ and $q' = 1/\lambda$ in $(\alpha_x, \beta_x)$; $q$ is injective, and, by assumption, $q$ maps $(\alpha_x, \beta_x)$ onto $\mathbb{R}$. Define $T_x x = q^{-1}(t)$. For $y$ in $Z(\lambda)$, define $T_x y = y$. It is easy to verify that $T$ is a flow with speed $\lambda$.

The open set $U(\lambda)$ may be decomposed into a countable union of disjoint open intervals $(a_i, b_i)$. Let $\mathcal{D}(\lambda)$ be the algebra of all functions $f$ in $C_c^1(\mathbb{R})$ which are constant in some neighborhood of each $a_i$ and in some neighborhood of each $b_i$. Since $T$ fixes each $a_i$ and each $b_i$, $\mathcal{D}(\lambda)$ is invariant under the dual action of $T$—the derivative of $f \circ T_t$ is $(\lambda \circ T_t)(f' \circ T_t)/\lambda$ on $U(\lambda)$. Since $\mathcal{D}(\lambda)$ is dense in $C_0(\mathbb{R})$, and contained in $\mathcal{D}(\delta_T)$, it follows that $\mathcal{D}(\lambda)$, and therefore $C_c^1(\mathbb{R})$, is a core for $\delta_T$.

Finally, given $f$ in $C_c^1(\mathbb{R})$ with support in $[-N, N]$, there is a sequence $f_n$ in $C_c^\infty(\mathbb{R})$ with support in $[-N, N]$ such that $\|f - f_n\| \to 0$, $\|f' - f_n'\| \to 0$. Then $\|\delta_T f_n - \delta_T f\| \to 0$. Thus $C_c^\infty(\mathbb{R})$ is a core for $\delta_T$.

If $A(\lambda) \not= \emptyset$, there may or may not be a flow with speed $\lambda$, and any such flow may or may not be unique. Suppose for example that there exists $x$ in $A_x^+(\lambda) \cap A_x^-(\lambda)$. Then any flow with speed $\lambda$ would reach $x$ from neighboring points on either side in a finite length of time, but would have no way of leaving $x$. So there is no flow with speed $\lambda$. On the other hand, if there are sufficiently many zeros of $\lambda$, a flow $T$ may be delayed at the zeros. These delays are measured by $\mu$ where

$$
\mu(I_T(x, t)) = |t| - \int_{I_T(x, t)} \frac{dy}{|\lambda(y)|}
$$

for $x$ in $U(\lambda)$, where $I_T(x, t)$ is the open interval between $x$ and $T_t x$. Since the intervals $I_T(x, t)$ are disjoint from the fixed point space $\mathbb{R}_T^0$, there is no restriction on $\mu$ on $\mathbb{R}_T^0$, and, for standardisation, one may as well assume that $\mu(\mathbb{R}_T^0) = 0$. Thus a (positive) measure $\mu$, defined on the
Borel subsets of \( \mathbb{R} \), will be said to be a \textit{delay measure} for \( T \) if (1) is satisfied and \( \mu(\mathbb{R}^0_t) = 0 \).

Conversely, it is possible to reconstruct \( T \) from \( \mu \) by observing that
\[
T_t x = y \text{ if } x < y \text{ and } \int_x^y \frac{dz}{\lambda(z)} + (\text{sgn } t) \mu(x, y) = t.
\]

This sets up a bijective correspondence between flows with speed \( \lambda \) and a certain class of measures, which have to be identified. A formal statement will be made in Theorem 2.5, for which the following notation and terminology is needed. As suggested above, finiteness of the delays and integrability of \( 1/\lambda \) on one side of a zero of \( \lambda \) has to be balanced on the other side with no change of sign of \( \lambda \).

For a measure \( \mu \) on \( \mathbb{R} \), let \( F^+ (\mu) \) (respectively, \( F^- (\mu) \)) be the set of all \( x \) in \( (-\infty, \infty] \) (respectively, \( [-\infty, \infty) \)) for which \( \mu(y, x) < \infty \) for some \( y < x \) (respectively, \( \mu(x, z) < \infty \) for some \( z > x \)). Then \( \mu \) will be said to be a \textit{fluid measure} for \( \lambda \) if \( \mu \) is non-atomic,
\[
(2) \quad A^+(\lambda) \cap F^+ (\mu) = A^- (\lambda) \cap F^- (\mu),
\]
and \( \mu \) is carried by \( A_i (\lambda) \cap F_i (\mu) \) (= \( A_r (\lambda) \cap F_r (\mu) \)). Note that all these sets are Borel measurable, and that \( A_i (\lambda) \setminus A_r (\lambda) \) etc. are countable and therefore null for measures \( \mu \) which are non-atomic.

\textbf{Theorem 2.5.} \textit{Let} \( \lambda: \mathbb{R} \to \mathbb{R} \) \textit{be a continuous function. For any fluid measure} \( \mu \) \textit{for} \( \lambda \), \textit{there is a unique flow} \( T \) \textit{on} \( \mathbb{R} \) \textit{with speed} \( \lambda \) \textit{for which} \( \mu \) \textit{is a delay measure. Conversely, for any flow} \( T \) \textit{with speed} \( \lambda \), \textit{there is a unique delay measure} \( \mu \) \textit{for} \( T \), \textit{and} \( \mu \) \textit{is a fluid measure for} \( \lambda \).

\textit{Proof.} For simplicity, we shall write \( A^+_i, F_i \) etc. in place of \( A^+_i (\lambda), F_i (\lambda) \) etc., and put
\[
V^+ = \{ x: \lambda(x) \geq 0 \}, \quad V^- = \{ x: \lambda(x) \leq 0 \},
\]
\[
U^+ = \{ x: \lambda(x) > 0 \}, \quad U^- = \{ x: \lambda(x) < 0 \}.
\]

Let \( \mu \) be a fluid measure. Define an equivalence relation on \( \mathbb{R} \) by saying that points \( x \) and \( y \) with \( x < y \) are equivalent if \( \mu(x, y) < \infty \) and \( 1/\lambda \) is integrable over \( (x, y) \). Let \( C_x \) be the equivalence class of \( x \); it is clear that \( C_x \) is some interval in \( \mathbb{R} \). If \( C_x \) consists of the single point \( x \), define \( T_t x = x \). Otherwise, let \( a \) and \( b \) be the endpoints of \( C_x \), so that \( -\infty \leq a < x < b \leq \infty \). To define \( T_t x \), the first stage is to show that \( C_x \) is contained in \( V^+ \) or in \( V^- \). Suppose that there exist \( y^- \) in \( C_x \cap U^- \) and \( y^+ \)
in \( C_x \cap U^+ \), and suppose for the sake of argument that \( y^- < y \). Let \( y = \sup((y^-, y^+) \cap U^+) \), so that \( y^- < y < y^+ \). Then \( (y, y^+) \) is contained in \( V^+ \), and \( y \) is equivalent to \( y^+ \), so \( y \in A^+_r \cap F_r \). By (2), \( y \in A^+_i \) which contradicts the fact that \( y \) is the limit of an increasing sequence in \( U^- \).

Now suppose for the sake of argument that \( C_x \) is contained in \( V^+ \) (the other case is similar). If \( a = x \), then \( x \not\in A^+_i \cap F_i = A^+_r \cap F_r \), so \( b = x \). Thus we need only consider the case \( a < x < b \). Define

\[
\varphi(x') = \begin{cases} 
- \int_{x'}^x \frac{dy}{\lambda(y)} - \mu(x', x) & (a \leq x' \leq x), \\
\int_x^{x'} \frac{dy}{\lambda(y)} + \mu(x, x') & (x \leq x' \leq b).
\end{cases}
\]

By definition of the equivalence relation, and (2),

\[
a \not\in A_i \cap F_i \supset A^+_r \cap F_r.
\]

Since \((a, x) \subset V^+\), it follows that either \( \mu(a, x) = \infty \) or \( \int_a^x \lambda(y)^{-1} \, dy = \infty \), so \( \varphi(a) = -\infty \). Similarly, \( \varphi(b) = \infty \). In particular, neither \( a \) nor \( b \) is equivalent to \( x \), so \( C_x = (a, b) \).

Since \( \mu \) is non-atomic, \( \varphi \) is continuous, and \( \varphi \) is clearly strictly increasing. Thus for each \( t \) in \( R \), there is a unique point \( T_t x \) in \((a, b)\) such that \( \varphi(T_t x) = t \), and \( t \mapsto T_t x \) is a homeomorphism of \( R \) onto \((a, b) = C_x \).

It is clear that \( T_0 x = x \) and (1) holds.

If \( T \) is defined on \( R \times R \) in this way, then for \( s, t \geq 0 \) and with the above notation and assumptions, using (1) with \( x \) replaced by \( T_t x \),

\[
\varphi(T_{s+t} x) = s + t = \int_{T_x x}^{T_{s+t} x} \frac{dy}{\lambda(y)} + \mu(T_t x, T_{s+t} x) + t
\]

\[
\varphi(T_{s+t} x) = \varphi(T_t x) + \varphi(T_s x) = \varphi(T_{s+t} x),
\]

so \( T_{s+t} x = T_s T_t x \). Dealing similarly with other cases, it follows that \( T \) satisfies the group property. Since \( T_t \) is an order-preserving homeomorphism of each \( C_x \), it is a homeomorphism of \( R \). It is clear from the construction that \( t \mapsto T_t x \) is continuous, so \( T \) is a continuous flow on \( R \). (For flows on \( R \), it is elementary to establish joint continuity from separate continuity, but flows on general spaces have the same property (see [5, Lemma 2.4]) for example).

For \( x \) in \( A_i \cap F_i \), \( C_x \) is non-trivial, so \( x \) is not fixed by \( T \). Thus \( A_i \cap F_i \) is disjoint from \( R^0_T \) (actually \( R^0_T = R \setminus (A_i \cap F_i) \)). Since \( \mu \) is carried by \( A_i \cap F_i \), \( \mu \) is a delay measure. Since \( \mu(U) = 0 \), it follows from (1) and the construction that Lemma 2.2(iv) is satisfied, so that \( T \) has speed \( \lambda \).
Let \( S \) be any flow with speed \( \lambda \) for which \( \mu \) is a delay measure. For \( x \) in \( U^+ \), \( S_t x \) increases with \( t \) for small \( t \) by Lemma 2.2(iv), and hence for all \( t \) (since \( t \mapsto S_t x \) is either strictly monotone or constant by the group property). Now \( S_t x \) is determined by (1). Similarly \( S_t x \) is uniquely determined for \( x \) in \( U^- \). Any interior point of \( Z \) is fixed under \( S \). Thus \( S_t x \) is uniquely determined for all \( x \) in a dense subset of \( \mathbb{R} \), so by continuity \( S \) is unique.

Now let \( T \) be a flow with speed \( \lambda \), let \( x \) be a point in \( \mathbb{R} \setminus \mathbb{R}^0_T \) and \( C \) be the trajectory of \( x \). Now \( t \mapsto T_t x \) is injective, hence strictly monotone, and suppose for the sake of argument that it is increasing, so \( C \) is contained in \( V^+ \) by Lemma 2.2(i). If for some \( \varepsilon > 0 \) and \( s_1 < s_2 \), \( \lambda(T_t x) < \varepsilon \) whenever \( s_1 < t < s_2 \), then by Lemma 2.2(iv),

\[
T_{s_2} x - T_{s_1} x < \varepsilon(s_2 - s_1).
\]

For \( t_1 < t_2 \), \( \{ y \in (T_{t_1} x, T_{t_2} x): \lambda(y) < \varepsilon \} \) is a countable union of disjoint intervals of the type \( (T_{s_1} x, T_{s_2} x) \), so it follows that its Lebesgue measure is less than \( \varepsilon(t_2 - t_1) \). Hence \( Z \cap (T_{t_1} x, T_{t_2} x) \) is (Lebesgue) null.

If \( \lambda(T_t x) > 0 \) whenever \( s_1 < t < s_2 \), then by Lemma 2.2(iv),

\[
s_2' - s_1' = \int_{T_{s_1} x}^{T_{s_2} x} \frac{dy}{\lambda(y)}.
\]

Now \( U^+ \cap (T_{t_1} x, T_{t_2} x) \) is a countable union of disjoint intervals of the form \( (T_{s_1} x, T_{s_2} x) \) and, taking the sum over these intervals and using the nullity of \( Z \cap (T_{t_1} x, T_{t_2} x) \) gives

\[
t_2 - t_1 \geq \int_{U \cap (T_{t_1} x, T_{t_2} x)} \frac{dy}{\lambda(y)} = \int_{T_{t_1} x}^{T_{t_2} x} \frac{dy}{\lambda(y)}.
\]

Define a function \( F_C \) on \( C \) by

\[
F_C(T_t x) = \begin{cases} t + \int_{T_t x}^x \frac{dy}{\lambda(y)} & (t \leq 0), \\ t - \int_x^{T_t x} \frac{dy}{\lambda(y)} & (t > 0). \end{cases}
\]

Then \( F_C \) is continuous and (4) shows that \( F_C \) is increasing. So \( F_C \) determines a (positive) non-atomic Lebesgue-Stieltjes measure \( \mu_C \) on \( C \), and \( \mu_C \) may be regarded as a measure on \( \mathbb{R} \). Furthermore \( \mu_C \) is independent of the choice of \( x \) in \( C \), since replacing \( x \) by \( T_t x \) alters \( F_C \) only by a constant. For \( t > 0 \) it is immediate that

\[
\int_x^{T_t x} \frac{dy}{\lambda(y)} + \mu_C(x, T_t x) = t.
\]
Also (3) shows that any compact subinterval of the open set \( C \cap U^+ \), and hence \( C \cap U^+ \) itself, is \( \mu_C \)-null, so \( \mu_C \) is carried by \( C \cap Z \).

Similarly for a non-trivial trajectory \( C \) contained in \( V^- \), one may construct a non-atomic measure \( \mu_C \), carried by \( C \cap Z \), such that

\[
\int_x^{T,x} \frac{dy}{\lambda(y)} - \mu_C(x, T_t x) = t \quad (t < 0).
\]

There are only countably many non-trivial trajectories \( C \); let \( \mu \) be the sum of all the corresponding measures \( \mu_C \). It is clear that \( \mu(\mathbb{R}_T^0) = 0 \), and (5) and (6) show that (1) also holds, so \( \mu \) is a delay measure for \( T \).

Suppose \( x \) is a point in \( Z \) with non-trivial trajectory \( C \). Assuming that \( C \) is contained in \( V^+ \), (5) gives

\[
\int_{T_1 x}^{T_2 x} \frac{dy}{\lambda(y)} + \mu_C(T_1 x, T_1 x) = 2,
\]

so \( x \in A_I^+ \cap F_i \cap A_r^+ \cap F_r \).

Now consider a point in \( \mathbb{R}_T^0 \cap A_i^+ \). For all sufficiently large \( x' < x \), \((x', x)\) is contained in \( V^+ \) and \( 1/\lambda \) is integrable over \((x', x)\). Let \( x'' \) be any point of \( U^+ \cap (x', x) \). The trajectory \( C \) of \( x'' \) is contained in \((-\infty, x)\), so

\[
\mu(x', x) \geq \mu_C(x'', x) \geq \lim_{t \to \infty} \mu_C(x'', T_t x'')
\]

\[
= \lim_{t \to \infty} \left\{ t - \int_{x''}^{T_t x''} \frac{dy}{\lambda(y)} \right\} = \infty,
\]

using (5) in the penultimate step. Thus \( x \notin F_i \).

These and similar arguments show that

\[
(A_I \cap F_i) \cup (A_r \cap F_r)
\]

\[
\subset Z \setminus \mathbb{R}_T^0 \subset [(A_i^+ \cap A_i^+) \cup (A_i^- \cap A_i^-)] \cap F_i \cap F_r.
\]

Thus \( \mu \) is a fluid measure.

Finally, let \( \mu' \) be any delay measure for \( T \). Then (1) shows that \( \mu' \) is uniquely determined on any open subinterval of a non-trivial trajectory, and is \( \sigma \)-finite on the trajectory. Hence \( \mu' \) is uniquely determined on each non-trivial trajectory. Since \( \mu' \) is carried by the union of the countable set of non-trivial trajectories, it follows that \( \mu' \) is unique. This completes the proof of Theorem 2.5.

From Theorem 2.5, it is a routine matter of measure theory to determine those \( \lambda \) for which there is a (unique) flow with speed \( \lambda \).
**Corollary 2.6.** There is at least one flow on $\mathbb{R}$ with speed $\lambda$ if and only if $(x, y) \cap Z(\lambda)$ is uncountable whenever $-\infty \leq x < y \leq \infty$ and either $x \in (A^+_i(\lambda) \setminus A^-_i(\lambda)) \cup (A^-_r(\lambda) \setminus A^-_i(\lambda))$ or $y \in (A^+_i(\lambda) \setminus A^-_r(\lambda)) \cup (A^-_i(\lambda) \setminus A^-_r(\lambda))$. The flow is unique if and only if $A^+_i(\lambda) = A^-_r(\lambda)$, $A^-_j(\lambda) = A^-_r(\lambda)$ and $A(\lambda)$ is countable. If there are two distinct flows with speed $\lambda$, then there are uncountably many.

If $\lambda D|_{C_c^\infty(\mathbb{R})}$ generates a $C_0$-semigroup $\tau$, then the derivation law implies that $\tau_t$ is an endomorphism of $C_0(\mathbb{R})$. Since all $C_0$-groups of $*$-automorphisms arise from flows, Theorem 2.5 covers all cases when $\lambda D|_{C_c^\infty(\mathbb{R})}$ generates a $C_0$-group. A $C_0$-semigroup of endomorphisms corresponds to a half-flow $T$ on $\mathbb{R} = \mathbb{R} \cup \{ \pm \infty \}$ which fixes $\pm \infty$, that is, a continuous mapping $T : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that

$$T_0 x = x, \quad T_s T_t = T_{s+t}, \quad T_\infty = \infty, \quad T_t(\infty) = -\infty.$$  

The analogue of Theorem 2.4 follows.

**Proposition 2.7.** Let $\lambda : \mathbb{R} \to \mathbb{R}$ be continuous. The following are equivalent:

(i) $\lambda D|_{C_c^\infty(\mathbb{R})}$ generates a $C_0$-semigroup on $C_0(\mathbb{R})$,
(ii) $A^+_r(\lambda) = A^-_i(\lambda) = \emptyset$.

The $C_0$-semigroup in Proposition 2.7 arises from a half-flow on $\mathbb{R}$ (as opposed to $\overline{\mathbb{R}}$) if and only if $-\infty \notin A^+_r(\lambda)$ and $\infty \notin A^+_i(\lambda)$, that is, $1/\lambda$ is not integrable at $\pm \infty$.

All the results of this section have analogues for $\mathbb{T} (= \mathbb{R}/\mathbb{Z})$ and $[0, 1]$, provided that $A^+_i(\lambda)$ etc. are interpreted correctly. For $\mathbb{T}$, regard $\lambda : \mathbb{T} \to \mathbb{R}$ as a periodic function on $\mathbb{R}$ and let $A^+_i(\lambda)$ consist of those $x$ in $Z(\lambda)$ such that for some $y < x$, $\lambda \geq 0$ in $(y, x)$ and $1/\lambda$ is integrable over $(y, x)$, etc. The statements of Theorems 2.4 and 2.5 and Proposition 2.7 are almost unchanged. For $[0, 1]$, let $A^+_i(\lambda)$ consist of those $x \neq 0$ in $Z(\lambda)$ such that, for some $0 < y < x$, $\lambda \geq 0$ in $(y, x)$ and $1/\lambda$ is integrable over $(y, x)$; let $A^+_r(\lambda)$ consist of those $x \neq 1$ in $Z(\lambda)$ such that for some $x < z < 1$, $\lambda \leq 0$ in $(x, z)$ and $1/\lambda$ is integrable over $(x, z)$, etc. The statements of Theorem 2.4 become:

(i) There is a flow $T$ on $[0, 1]$ such that $\delta_T = \lambda D$,
(ii) $A(\lambda) = \emptyset$; $\lambda(0) = \lambda(1) = 0$.

Theorem 2.5 is valid, but only for functions satisfying $\lambda(0) = \lambda(1) = 0$. The conditions of Proposition 2.7 are:

(i) $\lambda D$ generates a $C_0$-semigroup on $C[0, 1]$,
(ii) $A^-_r(\lambda) = A^-_i(\lambda) = \emptyset$; $\lambda(0) \geq 0, \lambda(1) \leq 0$. 
This answers a question raised in [6]. In particular, Theorem 4 of [6] remains valid if the assumption that the derivation is well-behaved is dropped, provided that the assertion that \( p(0) = p(1) = 0 \) is replaced by the conditions \( p(0) \geq 0, \ p(1) \leq 0 \). Some of the claims made in [6] about the example on p. 77 are incorrect, and the true position is set out below. (In comparing this paper with [6], the reader should bear in mind that there is a difference in sign conventions in defining generators.)

**Example 2.8 [6, p. 77].** Consider \( \lambda: [0,1] \to \mathbb{R} \) defined by \( \lambda(x) = -2x^{1/2} \). Then

\[
A_r^-(\lambda) = \{0\}, \quad A_i^-(\lambda) = A_i^+(\lambda) = \emptyset.
\]

Thus condition (ii) is satisfied, and \( \lambda D \) is the generator of the half-flow \( T^- \), where

\[
T_t^- x = (\max(x^{1/2} - t, 0))^2.
\]

On the other hand, \( -\lambda \) does not satisfy (ii) because \( -\lambda(1) < 0 \) and \( 0 \in A_r^+(-\lambda) \). The half-flow \( T^+ \) defined by

\[
T_t^+ x = (\min(x^{1/2} + t, 1))^2
\]

satisfies

\[
\delta_{T_t} f(x) = -\lambda(x)f'(x)
\]

for \( 0 < x < 1 \), but behaves differently at both endpoints.

3. **General spaces.** Let \( S \) be a flow on a locally compact Hausdorff space \( X \), with fixed point set \( X^0_S \), and let \( \lambda: X \setminus X^0_S \to \mathbb{R} \) be a continuous function. The problem now is to determine conditions under which there is a flow with "speed \( \lambda \) relative to \( S \)" and how such flows behave at the points of \( X^0_S \). The first result interprets the relative speed in two different, but equivalent, ways.

**Proposition 3.1.** Let \( T \) be a flow on \( X \), and \( \lambda: X \setminus X^0_S \to \mathbb{R} \) be a continuous function. The following are equivalent:

(i) For \( \omega \in \text{int} X^0_S \), \( T_t\omega = \omega \); for \( \omega \in X \setminus X^0_S \), there is a function \( \tau_\omega: \mathbb{R} \to \mathbb{R} \) such that \( T_t\omega = S_{\tau_\omega(t)}\omega \) \((t \in \mathbb{R})\) and \( \tau_\omega(0) = \lambda(\omega) \),

(ii) If \( f \in \mathcal{D}(\delta_S) \) and \( g \in C_0(X) \) are such that

\[
g = \begin{cases} 
\lambda \delta_S f & \text{on } X \setminus X^0_S \\
0 & \text{on } X^0_S
\end{cases}
\]

then \( f \in \mathcal{D}(\delta_T) \) and \( \delta_T f = g \).
Proof. (i) ⇒ (ii). This is a standard argument, but the details are included for completeness. For $\omega$ in $X \setminus X_S^0$,

$$
\lim_{t \to 0} \frac{f(T_t\omega) - f(\omega)}{t} = \frac{d}{dt} \left. \left( f(S_{\tau_t(\omega)}\omega) \right) \right|_{t=0} = \tau'_\omega(0) \delta_S f(\omega) = g(\omega).
$$

Replacing $\omega$ by $T_s\omega$, it follows that

$$
\frac{f(T_t\omega) - f(\omega)}{t} - g(\omega) \leq \frac{1}{|t|} \int_0^t \left| g(T_s\omega) - g(\omega) \right| ds \leq \sup_{|s| \leq |t|} \| g \circ T_s - g \|.
$$

By continuity, this estimate remains valid for $\omega$ in $X \setminus X_S^0$, while it is trivially valid for $\omega$ in $\text{int} X_S^0$. Thus

$$
\left\| t^{-1}(f \circ T_t - f) - g \right\| \leq \sup_{|s| \leq |t|} \| g \circ T_s - g \| \to 0 \quad \text{as } t \to 0.
$$

Thus $f \in \mathcal{D}(\delta_T)$, and $\delta_T f = g$.

(ii) ⇒ (i). Firstly, consider $\omega$ in $X \setminus X_S^0$. The argument used in [3] to show that $\{ T_t\omega \} \subset \{ S_s\omega \}$ is still valid, so there is a function $\tau_\omega$ such that $T_t = S_{\tau_t(\omega)}\omega$. Furthermore, $\tau_\omega$ is uniquely determined modulo the $S$-period of $\omega$, and one may (uniquely) arrange that $\tau_\omega$ is continuous and $\tau_\omega(0) = 0$.

It was shown in [4, Theorem 2.1] that there exists $f$ in $\mathcal{D}(\delta_S)$ such that $f(S_s\omega) = s$ for all small $|s|$, and $\text{supp} f \subset X \setminus X_S^0$. It follows from (ii) that $f \in \mathcal{D}(\delta_T)$ and

$$
\lambda(\omega) = (\delta_T f)(\omega) = \lim_{t \to 0} \frac{\tau_\omega(t)}{t} = \tau'_\omega(0).
$$

Next, for any function $h$ in $C_0(X)$ with $\text{supp} h$ contained in $\text{int} X_S^0$, it follows from (ii) that $h \in \mathcal{D}(\delta_T)$ and $\delta_T h = 0$. The local nature of $\delta_T$ ensures that each point of $\text{int} X_S^0$ is fixed by $T$.

Remark. The class $\mathcal{D}$ of functions $f$ which satisfy condition (ii) of Proposition 3.1 is a *-subalgebra of $\mathcal{D}(\delta_S)$, but it may not separate the points of $X_S^0$. Furthermore the flow $T$ may not fix every point of $X_S^0$ (so that $T$ may not be a “fluctuation” of $S$ in the sense of [2]). For example, let $X = \mathbb{R}^2$, $S_t(x, y) = (x + ty, y)$, $T_t(x, y) = (x + t, y)$. Here $X_S^0 = \mathbb{R} \times \{ 0 \}$ and $\lambda(x, y) = 1/y$ ($y \neq 0$), while $\mathcal{D}$ fails to separate any points of $X_S^0$. 


A sufficient condition that \( T \) fixes each point of \( X^0_S \) is condition (i) in Theorem 3.2 below (see [3] and the proof of Theorem 3.2). Sufficient conditions that \( \mathcal{D} \) is a core for \( \delta_T \) (in particular, \( \mathcal{D} \) separates the points of \( X \), and \( T \) fixes \( X^0_S \)) were given in [3, 7, 8].

**Theorem 3.2.** Let \( \lambda: X \setminus X^0_S \) be a continuous function, and suppose that

(i) For any compact set \( K \subset X \), there exists \( \varepsilon > 0 \) such that \( \lambda \) is bounded on \( \{ \omega \in K \setminus X^0_S: \nu(\omega) < \varepsilon \} \),

(ii) If \( \lambda(\omega) = 0 \) for some \( \omega \) in \( X \setminus X^0_S \), then \( t \mapsto \lambda(S_t \omega)^{-1} \) is not integrable over \((0, a)\) or over \((-a, 0)\) for any \( a > 0 \),

(iii) For any \( \omega \) in \( X \setminus X^0_S \), \( t \mapsto \lambda(S_t \omega)^{-1} \) is not integrable over \((0, \infty)\) or over \((-\infty, 0)\).

Then there is a unique flow \( T \) on \( X \) with speed \( \lambda \) relative to \( S \) (so that the conditions of Proposition 3.1 are valid).

**Proof.** For \( \omega \) in \( X \setminus X^0_S \), let \( \lambda_\omega(t) = \lambda(S_t \omega) \). It follows from assumptions (ii) and (iii) and Theorem 2.5 that there is a unique flow \( \theta_\omega \) on \( \mathbb{R} \) with speed \( \lambda_\omega \). This flow is characterised by the properties:

- \( x \) is a fixed point of \( \theta_\omega \iff \lambda(S_x \omega) = 0 \),

\[
\int_x^{\theta_\omega(x,t)} \frac{ds}{\lambda(S_s \omega)} = t \quad \text{if } \lambda(S_x \omega) \neq 0.
\]

The uniqueness of the flows, together with the relation

\[
\lambda_{S_t \omega}(x) = \lambda_\omega(x + t),
\]

ensures that the flows \( \theta_\omega \) are coherent in the sense that

\[
\theta_{S_t \omega}(x, s) + t = \theta_\omega(x + t, s).
\]

Let \( \tau_\omega(t) = \theta_\omega(0, t) \) and

\[
T_i \omega = \begin{cases} 
S_{\tau_\omega(t)} \omega & (\omega \in X \setminus X^0_S), \\
\omega & (\omega \in X^0_S).
\end{cases}
\]

Then \( T \) satisfies the group property \( T_s T_i = T_{i+s} \).

In order to show that \( T \) is a flow, it remains to show that \( (\omega, t) \mapsto T_i \omega \) is jointly continuous. Let \( (\omega_\alpha) \) and \( (t_\alpha) \) be nets such that \( \omega_\alpha \to \omega \), \( t_\alpha \to t \). By passing to subnets and replacing \( \lambda \) by \(-\lambda\), it suffices to assume that \( t_\alpha \geq 0 \) and to consider six cases:

1. \( \omega_\alpha \in X^0_S \);
2. \( \omega_\alpha \in X \setminus X^0_S \), \( \lambda(\omega_\alpha) = 0 \);
3. $\omega_\alpha \in X \setminus X^0_S$, $\omega \in X \setminus X^0_S$, $\lambda(\omega_\alpha) > 0$, $\lambda(\omega) > 0$, $\tau_{\omega_\alpha}(t_\alpha) \to \tau$, where $0 \leq \tau \leq \infty$;
4. $\omega_\alpha \in X \setminus X^0_S$, $\omega \in X \setminus X^0_S$, $\lambda(\omega_\alpha) > 0$, $\lambda(\omega) = 0$;
5. $\omega_\alpha \in X \setminus X^0_S$, $\omega \in X^0_S$, $\lambda(\omega_\alpha) > 0$, $\nu(\omega_\alpha) > \nu$, where $\nu > 0$;
6. $\omega_\alpha \in X \setminus X^0_S$, $\omega \in X_S^0$, $\lambda(\omega_\alpha) > 0$, $\nu(\omega_\alpha) \to 0$.

Cases 1 and 2. Since $X^0_S$ is closed and $\lambda$ is continuous, either $\omega \in X^0_S$ or $\lambda(\omega) = 0$. Thus

$$T_{t_\alpha} \omega_\alpha = \omega_\alpha \to \omega = T_t \omega.$$  

Case 3. Firstly, suppose that $\tau > \tau_\omega(t)$. Then, by construction of $\tau_\omega$, there exists $\theta$ such that $\tau_\omega(t) < \theta < \tau$, $\lambda(S_s \omega) > 0$ for $0 \leq s \leq \theta$. Since $S$ is jointly continuous, $\lambda(S_s \omega_\alpha)^{-1} \to \lambda(S_s \omega)^{-1}$ as $\alpha \to \infty$ uniformly for $0 \leq s \leq \theta$, and therefore

$$\int_0^\theta \frac{ds}{\lambda(S_s \omega)} \to \int_0^\theta \frac{ds}{\lambda(S_s \omega)}.$$  

But for large $\alpha$, $\tau_\omega(t) < \theta < \tau_{\omega_\alpha}(t_\alpha)$, so

$$t_\alpha > \int_0^\theta \frac{ds}{\lambda(S_s \omega)} \to \int_0^\theta \frac{ds}{\lambda(S_s \omega)} > t.$$  

This is a contradiction, so it follows that $\tau \leq \tau_\omega(t)$. For all sufficiently small $\theta' > \tau$, $\lambda(S_s \omega) > 0$ for $0 \leq s \leq \theta'$, and the same argument as above shows that

$$\int_0^\theta \frac{ds}{\lambda(S_s \omega)} \to \int_0^{\theta'} \frac{ds}{\lambda(S_s \omega)}.$$  

Hence $\theta' \geq \tau_\omega(t)$. Since $\theta' > \tau$ is arbitrarily small, it follows that $\tau \geq \tau_\omega(t)$. Thus $\tau = \tau_\omega(t)$ and

$$T_{t_\alpha} \omega_\alpha = S_{\tau_\omega(t_\alpha)} \omega_\alpha \to S_\tau \omega = S_{\tau_\omega(t)} \omega = T_t \omega.$$  

Case 4. By assumption (ii), for any $\eta > 0$, $\int_0^\eta |\lambda(S_s \omega)|^{-1} ds = \infty$, and therefore

$$\lim_{\epsilon \to 0^+} \int_0^\eta \frac{ds}{|\lambda(S_s \omega)| + \epsilon} = \infty.$$  

Since $(|\lambda(S_s \omega)| + \epsilon)^{-1} \to (|\lambda(S_s \omega)| + \epsilon)^{-1}$ uniformly on $(0, \eta)$, it follows that

$$\lim_{\epsilon \to 0^+} \lim_{\alpha \to \infty} \int_0^\eta \frac{ds}{|\lambda(S_s \omega_\alpha)| + \epsilon} = \infty.$$  

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It follows that
\[
\lim_{\alpha \to \infty} \int_0^\eta \frac{ds}{|\lambda(S_\alpha \omega)|} = \infty
\]
and therefore \( \tau_\omega(t_\alpha) < \eta \) for large \( \alpha \). Thus \( \tau_\omega(t_\alpha) \to 0 \), so
\[
T_{t_\alpha} \omega = S_{\tau_\omega(t_\alpha)} \omega \to \omega = T_\omega.
\]

**Case 5.** For each \( \alpha \),
\[
\tau_\omega(t_\alpha) = m_\alpha \nu(\omega_\alpha)^{-1} + \theta_\alpha
\]
where \( m_\alpha \) is an integer, \( 0 \leq \theta_\alpha < \nu(\omega_\alpha)^{-1} \leq \nu^{-1} \). Passing to a subnet, one may assume that \( \theta_\alpha \to 0 \). Then
\[
T_{t_\alpha} \omega = S_{\tau_\omega(t_\alpha)} \omega = S_{\theta_\alpha} \omega \to S_\theta \omega = \omega.
\]

**Case 6.** Let \( K \) be any compact neighbourhood of \( \omega \), and let
\[
\tau_\alpha = \inf\{ t > 0 : S_t \omega_\alpha \notin K \}.
\]
Suppose that \( \tau_\alpha \to \tau < \infty \). Then \( S_{\tau_\alpha} \omega_\alpha \to S_\tau \omega = \omega \), so \( \omega \in X \setminus K \). This is a contradiction. It follows (on passing to subnets) that \( \tau_\alpha \to \infty \).

By assumption (i), there is a constant \( c \) such that \( |\lambda(S_\alpha \omega_\alpha)| \leq c \) whenever \( 0 < s < \tau_\alpha \), so that, for any \( \eta > 0 \),
\[
\int_0^\eta \frac{ds}{|\lambda(S_\alpha \omega_\alpha)|} \geq \frac{\eta}{c}
\]
for all sufficiently large \( \alpha \). In particular, \( \tau_\omega(t_\alpha) \leq c t_\alpha \). Passing to a subnet, one may assume that \( \tau_\omega(t_\alpha) \to \tau < \infty \). Then
\[
T_{t_\alpha} \omega_\alpha \to S_\tau \omega = \omega = T_\omega.
\]

It is clear that \( T \) satisfies condition (i) of Proposition 3.1, and it remains only to establish uniqueness. If \( \tilde{T} \) is any flow with relative speed \( \lambda \), then for \( \omega \) in \( X \setminus X_\tau^0 \), there is a unique continuous function \( \tilde{\tau}_\omega : \mathbb{R} \to \mathbb{R} \) such that \( \tilde{\tau}_\omega(0) = 0 \) and \( T_s \omega = S_{\tilde{\tau}_\omega(s)} \omega \). Furthermore \( \tau_\omega'(0) = \lambda(\omega) \). The uniqueness ensures that
\[
\tilde{\tau}_\omega(s + t) = \tilde{\tau}_\omega(s) + \tilde{\tau}_{S_{\tilde{\tau}_\omega}(s)} \omega(t)
\]
and therefore there is a flow \( \tilde{\theta}_\omega \) on \( \mathbb{R} \) given by
\[
\tilde{\theta}_\omega(x, t) = \tilde{\tau}_{S_{\tilde{\theta}_\omega}(x)}(t) + x.
\]
Now $\tilde{\theta}_\omega$ has speed $\lambda_\omega$, and it follows from the uniqueness of flows with speed $\lambda_\omega$ that $\tilde{\theta}_\omega = \theta_\omega$. In particular

$$\tilde{\tau}_\omega(t) = \tilde{\theta}_\omega(0,t) = \theta_\omega(0,t) = \tau_\omega(t),$$

so $\tilde{T}_t \omega = T_t \omega$ ($\omega \in X \setminus X^0_S$).

For $\omega \in \text{int } X^0_S$, $\tilde{T}_t \omega = \omega = T_t \omega$. Thus $\tilde{T}_t$ and $T_t$ coincide on a dense subset of $X$, and therefore $\tilde{T} = T$.

**Remark.** Under the assumptions of Theorem 3.2, the algebra $\mathcal{D}$ considered in the remark following Proposition 3.1 equals $\mathcal{D}(\delta_S) \cap \mathcal{D}(\delta_T)$, but it is still unclear whether it is automatically a core for $\delta_T$. Let

$$\mathcal{D}_0 = \{ f \in \mathcal{D} : f(S, \omega) \in \mathcal{D}(\lambda_\omega) \text{ for all } \omega \in X \setminus X^0_S, \text{ } f \text{ has compact support} \},$$

where $\mathcal{D}(\lambda_\omega)$ is as defined in the proof of Theorem 2.4. Then $\mathcal{D}_0$ is a $T$-invariant *-subalgebra of $\mathcal{D}$, but it is not clear that $\mathcal{D}_0$ separates the points of $X$. If so, then $\mathcal{D}$ is a core for $\delta_T$.

**Example 3.3.** In Theorem 3.2, it is not possible to replace (ii) and (iii) by the weaker assumption

(iii)' For each $\omega$ in $X \setminus X^0_S$, there is a unique flow on $\mathbb{R}$ with speed $\lambda_\omega$ (where $\lambda_\omega(t) = \lambda(S_t \omega)$),

even if (i) is replaced by the stronger assumption that $\lambda$ is bounded. For example, let

$$X = \mathbb{R} \times [0, 1], \quad S_t(x, y) = (x + t, y)$$

$$\lambda(x, y) = \begin{cases} |x|^{1/2} & (|x| \leq 1, y \neq 0), \\ \frac{1}{1 + (1/y + 1)(1 - |x|)^1/|x|^{1/2}} & (|x| \leq 1, y = 0), \\ 1 & (|x| \geq 1). \end{cases}$$

Then

$$\int_0^2 \frac{dx}{\lambda(x, 0)} = 3 = \int_0^1 \frac{dx}{\lambda(x, y)} \quad (y \neq 0).$$

Since $Z(\lambda_{(0,y)}) = A_t^+(\lambda_{(0,y)}) = A_t^-(\lambda_{(0,y)}) = \emptyset$ and $A_t^-(\lambda_{(0,y)}) = A_r^-(\lambda_{(0,y)}) = \emptyset$, there is a unique measure $\mu$ satisfying the conditions of Theorem 2.5 for $\lambda = \lambda_{(0,y)}$, namely $\mu = 0$. The corresponding flow $\theta_y$ on $\mathbb{R}$ satisfies

$$\theta_y(s, t) = s + \tau_y(t) \quad \text{where } \int_0^{\tau_y(t)} \frac{dx}{\lambda(x, y)} = t.$$
If \( T \) is any flow on \( \mathbb{R} \) satisfying the conditions of Proposition 3.1, then \( T \) induces flows \( \tilde{\theta}_y \) on \( \mathbb{R} \) such that

\[
T_t(x, y) = (\tilde{\theta}_y(x, t), y),
\]

and \( \tilde{\theta}_y \) has speed \( \lambda_{(0,y)} \). Hence \( \tilde{\theta}_y = \theta_y \), so

\[
T_3(0, y) = (\tau_y(3), y) = (1, y) \quad (y \neq 0)
\]

\[
T_3(0, 0) = (\tau_0(3), 0) = (2, 0).
\]

This contradicts the continuity of \( T \).

**REFERENCES**


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