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LIFTING UNITS IN SELF-INJECTIVE RINGS AND AN INDEX THEORY FOR RICKART C^*-ALGEBRAS

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In this paper we study the following question: ***If R is a right self-injective ring and I an ideal of R, when can the units of R/I be lifted to units of R?***

We answer this question in terms of $K_0(I)$. For a purely infinite regular right self-injective ring R we obtain an isomorphism between $K_1(R/I)$ and $K_0(I)$ which can be viewed as an analogue of the index map for Fredholm operators.

By giving a purely algebraic description of the connecting map $K_1(A/I) \to K_0(I)$ in the case where A is a Rickart C^*-algebra, we are able to extend the classical index theory to Rickart C^*-algebras in a way which also includes Breuer's theory for W^*-algebras.

0. Preliminary results. Throughout this paper R will denote an associative ring with 1. By a *rng* we mean a ring which does not necessarily have a 1.

We write $M_n(R)$ for the ring of all $n \times n$ matrices over R, and $\mathrm{GL}_n(R)$ for the group of units of $M_n(R)$, though we shall write $U(R)$ rather than $\mathrm{GL}_1(R)$. For $1 \le i,\ j \le n$ let $e_{ij} \in M_n(R)$ be the usual matrix units. Define $E_n(R)$ to be the subgroup of $\mathrm{GL}_n(R)$ generated by all the matrices of the form $1 + re_{ij}$, $r \in R$, $i \neq j$; and $GE_n(R)$ to be the subgroup of $\mathrm{GL}_n(R)$ generated by $E_n(R)$ together with the subgroup $D_n(R)$ of all invertible diagonal matrices. If $GE_n(R) = \mathrm{GL}_n(R)$, then we say that R is a *GE_n-ring*; if R is a GE_n-ring for all $n > 1$ then R is said to be a *GE-ring*.

If R is a GE_n-ring, then $E_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$ and hence $\mathrm{GL}_n(R) = D_n(R)E_n(R)$.

Let $\mathrm{GL}(R)$ denote the direct limit of the directed system

$$U(R) \to \mathrm{GL}_2(R) \to \mathrm{GL}_3(R) \to \cdots$$

where each $a \in \mathrm{GL}_n(R)$ is mapped to

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{GL}_{n+1}(R)$. Then $K_1(R)$ is defined to be $\mathrm{GL}(R)^{\mathrm{ab}}$, that is $\mathrm{GL}(R)$ abelianized.

Note that the canonical map $U(R) \to K_1(R)$ is onto in the case where R is a GE-ring.

Let I be a rng and R a ring containing I as an ideal. Let $P(I)$ denote the class of all finitely generated projective right R-modules A such that $AI = A$. We say that A, $B \in P(I)$ are equivalent if $A \oplus C \approx B \oplus C$ for some $C \in P(I)$. Denote by $[A]$ the equivalence class of $A \in P(I)$. Thus the set $\{[A] \mid A \in P(I)\}$ with the operation $[A] + [B] = [A \oplus B]$ is a cancellative abelian semigroup. We write $G(I)$ for its associated universal abelian group. Then every element of $G(I)$ has the form $[A]-[B]$ for suitable A, $B \in P(I)$ and $[A] - [B] = [A'] - [B']$ if and only if $A \oplus B' \oplus C \approx A' \oplus B \oplus C$ for some $C \in P(I)$. It is not difficult to show that $P(I)$ consists of all R-modules A such that $A \approx e(R^n)$ for some idempotent $n \times n$ matrix e with entries in I. Thus, we see that $G(I)$ depends only on the structure of the rng I and not on the involving ring R. Note that G is a functor from the category of rngs into the category of abelian groups such that preserves direct limits.

For a ring R, $G(R)$ is simply $K_0(R)$. Recall that Bass and Milnor have defined a functor K_0 on the category of rngs; following Milnor [**14**, §4], we consider any ring R containing I as an ideal, let π: $R \to R/I$ be the natural surjection, and form the pullback

$$\begin{array}{ccc} D(R) & \xrightarrow{p_2} & R \\ \downarrow p_1 & & \downarrow \pi \\ R & \xrightarrow{\pi} & R/I. \end{array}$$

Then $K_0(I, R)$ is defined as the kernel of $K_0(p_1)$: $K_0(D(R)) \to K_0(R)$. In [**2**] it is proved that $K_0(I, R)$ depends only on I. Furthermore, there is an exact sequence, cf. [**14**, §4]:

$$K_1(R) \to K_1(R/I) \xrightarrow{\delta} K_0(I, R) \to K_0(R) \to K_0(R/I).$$

Let I be a rng that is an F-algebra, where F is either $\mathbf{Z}$ or a commutative field. Consider $I^1 = I \oplus F$, the unitification of I by F; by applying the above exact sequence we obtain

$$K_0(I, I^1) = \mathrm{Ker}\big(K_0(I^1) \to K_0(F)\big).$$

When we write $K_0(I)$ we will have $K_0(I, I^1)$ in mind.

If I is a ring with unit e, then there is a ring decomposition $I^1 = I \times (1 - e)F$. Therefore $K_0(I^1) = K_0(I) \oplus K_0(F)$ and so $K_0(I, I^1) = K_0(I)$. Hence we see that $K_0(I)$ agrees with the corresponding K_0 of I, where I is viewed as a ring.

Let I be a rng. With each $A \in P(I)$ we can associate its class in $K_0(I)$. In this way we obtain a group homomorphism ϕ: $G(I) \to K_0(I)$. In the case where ϕ is an isomorphism we shall write $G(I) = K_0(I)$. When this occurs there is a very simple form for the elements in $K_0(I, R)$. More precisely, if $A \in P(I)$, then $0 \times A$ is a projective $D(R)$-module, and one easily obtains a group isomorphism

$$K_0(I) = G(I) \to K_0(I, R)$$

in which $[A] \mapsto [0 \times A]$.

In general we do not know whether $K_0(I) = G(I)$ but the following easy result will be enough for our purposes.

PROPOSITION 0.1. *Let I be an ideal of an F-algebra R, where F is either $\mathbf{Z}$ or a commutative field. Suppose there exists a set E of idempotents of I such that for each pair e, $f \in E$ there exists $g \in E$ such that $eRe + fRf \subseteq gRg$, so the subrings $eRe + F \cdot 1$ form a directed system. If the induced map*

$$\operatorname*{dir.lim.}_{e \in E} K_0(eRe + F1) \to K_0(I + F1)$$

is a group isomorphism then $K_0(I) = G(I)$.

Proof. There is an obvious commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \operatorname*{dir.lim.}_{e\in E} K_0(eRe) & \to & \operatorname*{dir.lim.}_{e\in E} K_0(eRe + F1) & \to & K_0(F) & \to & 0 \\
 & & \downarrow \alpha & & \downarrow & & \| & & \\
0 & \to & K_0(I) & \to & K_0(I + F1) & \to & K_0(F) & \to & 0
\end{array}$$

with exact rows, and by hypothesis the middle column is an isomorphism so α is also. On the other hand G preserves direct limits, so we have a map β: $\text{dir.lim.}_{e \in E} G(eRe) \to G(I)$. As $G(eRe) = K_0(eRe)$ for all $e \in E$, it follows that $\beta\alpha^{-1}$: $K_0(I) \to G(I)$ provides an inverse for ϕ. Therefore $K_0(I) = G(I)$. □

We shall need another result. First recall Milnor's definition of the connecting map δ: $K_1(R/I) \to K_0(I, R)$. Consider any element μ of $K_1(R/I)$; it lies in the image of $\mathrm{GL}_n(R/I)$ for some n and so can be represented as the image of a matrix $u \in M_n(R)$ for which there exists $v \in M_n(R)$ such that the elements $i = uv - 1$, $j = vu - 1$ lie in $M_n(I)$. Write

$$M = \{(x, y) \in {}^nR \times {}^nR \mid u(x) - y \in {}^nI\}.$$

In [**14**, Theorem 2.1] it is proved that M is a finitely generated projective $D(R)$-module. Now $\delta(\mu)$ is defined as $[M] - [{}^nD]$ and this gives the connecting map. In this situation we have:

LEMMA 0.2 *As D-modules* ${}^nD \oplus (0 \times i({}^nR)) \approx M \oplus (0 \times j({}^nR))$.

Proof. By using the Morita equivalence between Mod-D and Mod-$M_n(D)$ we see that the claimed isomorphism is equivalent to an $M_n(D)$-module isomorphism

$$M_n(D) \oplus (0, i)M_n(D) \approx {}^nM \oplus (0, j)M_n(D).$$

It is clear that

$${}^nM \approx \{(x, y) \in M_n(D) \times M_n(D) \mid ux - y \in M_n(I)\}$$

This shows that without loss of generality we may assume that $n = 1$. Now any element of M can be expressed in the form

$$(x, y) = (1, u)(x, vy) - (0, i)(y, y)$$

so $M = (1, u)D + (0, i)D$. Now define a D-module homomorphism

$$\alpha \colon D \oplus (0, i)D \to M, \quad ((x, y), (0, i)d) \mapsto (1, u)(x, y) - (0, i)d.$$

Clearly α is onto, and $\operatorname{Ker}\alpha = \{((0, y), (0, iy')) \in D \oplus (0, i)D \mid uy - iy' = 0\}$. But if $uy - iy' = 0$ then from the relation

$$\begin{pmatrix} -j & v \\ -u & 1 \end{pmatrix}\begin{pmatrix} 1 & -v \\ u & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} j \\ u \end{pmatrix}(vy' - y) = \begin{pmatrix} y \\ y' \end{pmatrix}$$

So $\operatorname{Ker}\alpha = ((0, j), (0, iu))D \approx (0, j)D$. Since M is D-projective, α splits and the result follows. □

1. Regular rings. Let R be a ring.

Recall that R is said to be *regular* if for every $x \in R$ there exists $y \in R$ such that $x = xyx$. An element x of R is called *unit-regular in R* if there exists a unit u of R such that $x = xux$. We say that R is *unit-regular* if every element in R is unit-regular.

An ideal I of R has *stable range* 1 if for all a, $b \in I$, if $(1 + a)R + bR = R$ then there exists $c \in I$ such that $(1 + a + bc)R = R$; cf. [**17**], [**18**]. Vasershtein [**17**] proves that I having stable range 1 depends only on the rng structure of I, and not on the ambient ring R. Now one can see that for a ring R the stable range 1 condition is equivalent to saying that for all a, $b \in R$, $aR + bR = R$ implies $a + bc$ is a unit for some $c \in R$, cf. [**18**, Theorem 2.6], [**2**, p. 231].

A theorem of Fuchs and Kaplansky [7, Proposition 4.12] asserts that the unit-regular rings are precisely those regular rings with stable range 1. We shall use Evans' theorem [7, Proposition 4.13]: if the endomorphism ring of a right R-module M has stable range 1, then M can be cancelled from direct sums of right R-modules, that is, $M \oplus N \approx M \oplus N'$ for some right R-modules N and N' implies $N \approx N'$. By [18, Theorem 2.4, Theorem 3.9] the stable range condition carry over to corners and it is Morita-invariant. Hence, if R has stable range 1 then all finitely-generated projective right R-modules cancel from direct sums.

Now we shall give a description for $K_0(I)$ in the case where I is an ideal of a regular ring R. In an earlier version of this paper we had obtained such a description in the case where R is unit-regular, and then Goodearl provided us with the general case.

First we need a more-or-less known lemma.

LEMMA 1.1. *Let R be a regular ring and let e, $f \in R$ be idempotents. Then*

(i) *If $eR \subseteq fR$, then there exists an idempotent g in R with $gR = fR$ and $ge = eg = e$.*

(ii) *Let I be an ideal of R. If e, $f \in I$ then there exist idempotents g, $h \in I$ such that $eRf \subseteq hRh$ and $eRe + fRf \subseteq gRg$. Moreover if $a \in I$ then there exists an idempotent $k \in I$ such that $a \in kRk$.*

Proof. (i) Define $g = (1 + ef(1 - e)f(1 - ef(1 - e))$.

(ii) Let h be an idempotent such that $eR + fR = hR$. Clearly $h \in I$ and, by (i), we can choose h such that $fh = hf = f$. Then $eRf \subseteq hRh$.

Let c and d be idempotents in I such that $eR + fR = cR$ and $Re + Rf = Rd$. Then $eRe + fRf \subseteq (eR + fR) \cap (Re + Rf) = cRd$. It follows from the above that $cRd \subseteq gRg$, for some idempotent g in I.

If $a \in I$, then by regularity there exists $x \in R$ such that $a = axa$, so $e = ax$ and $f = xa$ are idempotents in I and $a \in eRf$. Now the result follows from the above. □

PROPOSITION 1.2 (*Goodearl*). *If I is an ideal of a regular ring R then $G(I) = K_0(I)$.*

Proof. Let E be the set of all idempotents in I. By Lemma 1.1 (ii), $I + \mathbf{Z} \cdot 1$ is the directed union of the subrings $eRe + \mathbf{Z} \cdot 1$. Therefore $\text{dir.lim}_{e \in E} K_0(eRe + \mathbf{Z} \cdot 1) = K_0(I + Z \cdot 1)$. By Proposition 0.1 the result follows. □

Now we shall obtain a tidier expression for the connecting map $\delta : K_1(R/I) \to K_0(I)$ in the case where R is a regular ring.

If a is an $n \times n$ matrix over R we write $\operatorname{Ker} a$ for the set of elements $x \in {}^nR$ such that $a(x) = 0$. We define Coker a to be any complement of $a({}^nR)$ in ${}^nR$, so Coker a is determined up to isomorphism.

PROPOSITION 1.3 (*with Goodearl*). *Let R be a regular ring and I an ideal of R. Then the connecting map*

$$\delta : K_1(R/I) \to K_0(I)$$

satisfies $\delta(\bar{a}) = [\operatorname{Coker} a] - [\operatorname{Ker} a]$, *where a is any matrix over R representing $\bar{a} \in K_1(R/I)$.*

Proof. Suppose $a \in M_n(R)$. By regularity there exists an $n \times n$ matrix b over R such that $a = aba$. Since a is a unit modulo I, we have

$$ab - 1 = i \in M_n(I)$$

$$ba - 1 = j \in M_n(I).$$

Now $j({}^nR) = \operatorname{Ker} a$ and $i({}^nR) \oplus a({}^nR) = {}^nR$. With the same notation as in Lemma 0.2 we have $\delta(\bar{a}) = [M] - [{}^nD] = [0 \times \operatorname{Coker} a] - [0 \times \operatorname{Ker} a] \in K_0(I, R)$. Hence $\delta(\bar{a}) = [\operatorname{Coker} a] - [\operatorname{Ker} a] \in K_0(I)$. □

We now use the preceding propositions to obtain some results on lifting units.

LEMMA 1.4. *If R is a regular ring and I is an ideal of R then the following are equivalent*

(i) *For each idempotent e in I the corner ring eRe is unit-regular.*

(ii) *$I + Z$ is a unit-regular subring of R, where Z is the centre of R.*

(iii) *I has stable range* 1.

Proof. (i) ⇒ (ii) By Lemma 1.1, $I + Z$ is the directed union of the subrings $eRe + Z$, where e is an idempotent in I. Now $eRe + Z \approx eRe \times (1 - e)Z$ is the direct product of two unit-regular rings, so $eRe + Z$ is unit-regular. Since unit regularity is preserved by taking direct limits we see that $I + Z$ is unit-regular.

(ii) $\Rightarrow$ (iii) By hypothesis $I + Z$ is unit-regular and so has stable range 1. It follows from [**18**, Theorem 3.6 (g)] that I has stable range 1.

(iii) $\Rightarrow$ (i) Every corner of a rng with stable range 1 also has stable range 1 cf. [**18**, Theorem 3.9]. $\square$

It follows from [**17**, Theorem 4] that the sum of two ideals with stable range 1 has stable range 1. Hence there is a unique largest ideal R_0 of R having stable range 1, namely, the sum of all ideals of R with stable range 1.

If an R-module A is isomorphic to a direct summand of an R-module B then we write $A \lesssim B$. Two idempotents e and f of R are said to be *isomorphic* if the modules $A = eR$, $B = fR$ are isomorphic. The notations $e \leq f$ and $e \lesssim f$ mean $eR \subseteq fR$ and $eR \lesssim fR$ respectively.

LEMMA 1.5. *If R is a regular ring then R_0 coincides with the ideal I generated by all idempotents of R whose corner is unit-regular.*

Proof. By Lemma 1.1 any ideal of R is the directed union of its corners, so by Lemma 1.4 (i) $\Leftrightarrow$ (iii), we see that $R_0 \subseteq I$.

Conversely, if e is an idempotent in I then $e = \Sigma x_i e_i y_i$, where x_i, $y_i \in R$ and the e_i's are idempotents with $e_i R e_i$ unit-regular. From the R-linear map $\oplus e_i R \to R$, $\Sigma e_i r_i \mapsto \Sigma x_i e_i r_i$ we see that $eR \lesssim \oplus e_i R$. It follows from [**12**, Corollary 10(ii)] that the endomorphism ring of the R-module $\oplus e_i R$ has stable range 1 and since eRe is a corner of this endomorphism ring it also has stable range 1. $\square$

If I is an ideal of R write $\bar{x}$ for $x + I \in R/I$ and denote by π the natural projection $R \to R/I$.

PROPOSITION 1.6. *Let R be a regular ring and I an ideal of R with stable range* 1, *then the map*

$$\alpha\colon U(R/I) \to K_0(I), \qquad \bar{a} \mapsto [\operatorname{Coker} a] - [\operatorname{Ker} a],$$

is a group homomorphism. Moreover

$$\operatorname{Ker} \alpha = \pi(U(R)) = \{\bar{a} \in U(R/I)\colon a \text{ is unit-regular}\}.$$

Proof. By Proposition 1.3 we see that α is the composition of the maps $U(R/I) \to K_1(R/I)$ and $\delta\colon K_1(R/I) \to K_0(I)$ and so it is a group homomorphism.

If Z is the centre of R then $K_0(I)$ is a subgroup of $K_0(I + Z)$. Notice that $\bar{a}$ lies in $\operatorname{Ker} \alpha$ if and only if $[\operatorname{Coker} a] = [\operatorname{Ker} a]$ in $K_0(I + Z)$.

Since I and so $I + Z$ has stable range 1, we have that $\bar{a} \in \operatorname{Ker} \alpha$ if and only if $\operatorname{Coker} a \approx \operatorname{Ker} a$ and this occurs if and only if a is unit-regular, cf. [7, Proof of Theorem 4.1].

Conversely, let $\bar{a} \in \operatorname{Ker} \alpha$, If a is a representative in R for $\bar{a}$, then $a = aua$ for some unit u in R. Now since $\bar{a} \in U(R/I)$, $(\bar{a} - \bar{u}^{-1})\bar{u} = 0$ so $\bar{a} = \bar{u}^{-1}$ and $\bar{a}$ belongs to $\pi(U/R))$. □

Now we consider regular right self-injective rings. The reader is referred to [7] for background. We mention, however, that every regular right self-injective ring can be uniquely expressed as a direct product of a unit-regular ring and a purely infinite regular ring (recall that an idempotent e of a ring R is said to be *purely infinite* if $(eR) \approx (eR)^2$, so R is a purely infinite regular right self-injective ring if 1 is a purely infinite idempotent in R).

LEMMA 1.7. *If R is a purely infinite regular right self-injective ring and I is an ideal of R, then $\pi(U(R)) = U(R/I)'$.*

Proof. By [13, Corollary 2.8] $U(R)$ is a perfect group. Hence $\pi(U(R)) \subseteq U(R/I)'$.

Conversely, take u in the commutator group $U(R/I)'$. Since $R \approx R^2$ there exist matrices $X \in R^2$ and $Y \in {}^2R$ such that $XY = 1$ and $YX = I_2$. Then $\overline{Y}u\overline{X}$ is a 2×2 invertible matrix. By [13, Theorem 2.2] $\overline{Y}u\overline{X} \in E_2(R/I)$, hence there exists $Z \in \mathrm{GL}_2(R)$ such that $\overline{Z} = \overline{Y}u\overline{X}$. Therefore $v = XZY$ is a unit of R with $\bar{v} = u$. The result follows. □

If e is an idempotent of a regular right self-injective ring, then we denote by $\mathrm{cc}(e)$ its *central cover*, that is, the minimum central idempotent such that $\mathrm{cc}(e)e = e$.

PROPOSITION 1.8. *Let R be a purely infinite regular right self-injective ring and I an ideal. If A, $B \in P(I)$, then*

(i) $[A] = [B] \in K_0(I)$ *if and only if there exists a purely infinite idempotent e in I such that $A \oplus eR \approx B \oplus eR$.*

(ii) (*with Goodearl*) $K_0(I) = 0$ *if and only if every idempotent in I is sub-isomorphic to a purely infinite idempotent in I.*

(iii) $[A] = [B] \in K_0(I)$ *if and only there exists a purely infinite idempotent e in I such that $A \oplus \mathrm{cc}(e)R \approx B \oplus \mathrm{cc}(e)R$.*

(iv) $[A] = [B] \in K_0(I)$ *if and only if there exists a purely infinite idempotent e in I such that $(1 - \mathrm{cc}(e))A \approx (1 - \mathrm{cc}(e))B$.*

Proof. (i) By Proposition 1.2, $K_0(I) = G(I)$. Thus $[A] = [B]$ if and only if $A \oplus C \approx B \oplus C$ for some $C \in P(I)$. It follows from [7, Theorem 10.32] that C can be written as $C_1 \oplus C_2$, where C_2 is purely infinite and the endomorphism ring of C_1 has stable range 1. But then C_1 cancels from direct sums and we have $A \oplus C_2 \approx B \oplus C_2$. Since $R \approx R^2$, C_2 is cyclic and so $C_2 \approx eR$, for some purely infinite idempotent e in I.

(ii) Since R is purely infinite, we see that every finitely generated right R-module is cyclic.

Suppose $K_0(I) = 0$ and let e be an idempotent in I. By (i) there exists a purely infinite idempotent f in I such that $eR \oplus fR \approx fR$. Thus $eR \lesssim fR$ as desired.

Conversely, let e be an idempotent in I. By hypothesis $eR \lesssim fR$ for some purely infinite idempotent f in I. Then, since $fR \lesssim eR \oplus fR \lesssim (fR)^2$, by [7, Theorem 10.14] we have $eR \oplus fR \approx fR$. So $[eR] = 0$ in $K_0(I)$.

(iii) Suppose $[A] = [B] \in K_0(I)$. By (i), $A \oplus eR \approx B \oplus eR$ for some purely infinite idempotent e in I and a fortiori $A \oplus \mathrm{cc}(e)R \approx B \oplus \mathrm{cc}(e)R$.

Conversely, if $A \oplus \mathrm{cc}(e)R \approx B \oplus \mathrm{cc}(e)R$ then we have $(1 - \mathrm{cc}(e))A \approx (1 - \mathrm{cc}(e))B$. Hence it suffices to prove that $[\mathrm{cc}(e)A] = [\mathrm{cc}(e)B] = [0]$. By cutting down to $\mathrm{cc}(e)R$ we may assume e faithful and we need only verify $[A] = [0]$. Thus we are reduced to the case A directly finite. By the general comparability axiom there exists a central idempotent h such that $heR \lesssim hA$ and $(1 - h)A \lesssim (1 - h)eR$. Since hA is directly finite and heR purely infinite we deduce that $heR = 0$. But e is faithful so $h = 0$. Then $A \lesssim eR$ and the result follows from the proof of (ii).

(iv) The relation $A \oplus \mathrm{cc}(e)R \approx B \oplus \mathrm{cc}(e)R$ is equivalent to $(1 - \mathrm{cc}(e))A \approx (1 - \mathrm{cc}(e))B$ and $\mathrm{cc}(e)A \oplus \mathrm{cc}(e)R \approx \mathrm{cc}(e)B \oplus \mathrm{cc}(e)R$. Since $\mathrm{cc}(e)$ is purely infinite the latter relation always holds. So the result follows from (iii). □

THEOREM 1.9. *Let R be a purely infinite, regular, right self-injective ring and let I be an ideal of R. Then*

(i) *The map*

$$\alpha : U(R/I) \to K_0(I), \quad \alpha(\bar{a}) = [\mathrm{Coker}\, a] - [\mathrm{Ker}\, a]$$

is a group homomorphism which induces an isomorphism

$$K_1(R/I) = U(R/I)^{\mathrm{ab}} \xrightarrow{\approx} K_0(I).$$

(ii) *A unit $\bar{a} \in U(R/I)$ can be lifted to a unit in R if and only if* $[\mathrm{Coker}\, a] = [\mathrm{Ker}\, a] \in K_0(I)$.

Proof. (i) Let $f\colon U(R/I) \to K_1(R/I)$ be the natural map. It follows from [**13**, Theorem 1.2 (iii) and Theorem 2.2] that f is onto and $\operatorname{Ker} f = U(R/I)'$. So $K_1(R/I) = U(R/I)^{\mathrm{ab}}$. By [**13**, Theorem 2.7 (ii)] $K_1(R) = 0$ and it follows from [7, Proposition 15.6] that $K_0(R) = 0$. Thus (i) follows from Proposition 1.3.

(ii) This is an immediate consequence of (i) and Lemma 1.7. □

LEMMA 1.10. *Let R be a regular right self-injective ring and I an ideal of R. If e is an idempotent of I, then the following are equivalent*

(i) *$e \lesssim f$ for some purely infinite idempotent f in I.*

(ii) *$e \le f$ for some purely infinite idempotent f in I.*

Proof. Clearly (ii) $\Rightarrow$ (i). Conversely, by [7, Theorem 10.32] there exists a central idempotent h in R such that heR is purely infinite and $(1-h)eR$ is directly finite. So without loss of generality we may assume that eR is directly finite. We have $eR \approx e'R \subseteq fR$ for some idempotent e'. Since eRe has stable range 1, $(1-e)R \approx (1-e')R$, so there exists a unit u in R such that $e = u^{-1}e'u$. The idempotent $u^{-1}fu$ is a purely infinite idempotent in I and $e \le u^{-1}fu$. □

COROLLARY 1.11. *Let R be a regular right self-injective ring. Let e_1 be the central idempotent in R such that e_1R is purely infinite and $(1-e_1)R$ is directly finite. Then the following are equivalent*

(i) *Every unit in R/I can be lifted to a unit in R.*

(ii) *For every idempotent $e \in e_1 I$ there exists a purely infinite idempotent $f \in I$ such that $e \le f$.*

(iii) $K_0(e_1I) = 0$.

Proof. R decomposes into the direct product of the rings $R_1 = e_1R$ and $R_2 = (1-e_1)R$. Since R_2 is unit-regular it is clear that a unit in a factor ring of R_2 can be lifted to a unit in R_2. Thus without loss of generality we may assume that R is purely infinite, that is, $e_1 = 1$.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from Proposition 1.8 (ii) and Corollary 1.10. It is clear from Theorem 1.9 (ii) that (i) $\Leftrightarrow$ (iii). □

COROLLARY 1.12. *If R is a regular right self-injective ring of Type* III *and I is an ideal of R, then every unit in R/I can be lifted to a unit in R.*

Proof. Since R is Type III every idempotent is purely infinite. The result follows from Corollary 1.11. □

Now it is a simple matter to extend Corollary 1.11 to arbitrary right self-injective rings. For this we first need a lemma.

For any ring R denote by $J = J(R)$ its Jacobson radical. We shall use the fact that an element of R is a unit if and only if so is modulo J. Recall that if R is right self-injective then R/J is regular and right self-injective. Moreover every idempotent in R/J can be lifted to an idempotent in R.

We denote by R_∞ the right ideal generated by all purely infinite idempotents in R.

LEMMA 1.13. *If R is right self-injective, then R_∞ is an ideal of R.*

Proof. If e is a purely infinite idempotent in R then it suffices to prove that $xe \in R_\infty$ for all x in R. In the case x is a unit we have that xex^{-1} is a purely infinite idempotent, hence $xex^{-1} \in R_\infty$ and so $xe \in R_\infty$. Now write $R/J = R_1 \times R_2$ where R_1 is purely infinite and R_2 is unit-regular. Let S_1 and S_2 be the ideals of R such that $S_1/J = R_1$ and $S_2/J = R_2$. Since $R = S_1 S_2$ it suffices to consider separately the cases $x \in S_1$ and $x \in S_2$.

Suppose first $x \in S_1$. Since R_1 is purely infinite $R_1 \approx M_2(R_1)$ and hence every element of R_1 is a sum of an even number of units in R_1. But then, every element of R_1 is a sum of units in $R_1 \times R_2$ and so every element of S_1 is a sum of units in R. Now it is clear that $xe \in R_\infty$.

Assume now $x \in S_2$. Since R_2 is unit-regular we can find an idempotent f and a unit u in R such that $xu - f \in J$. So $x - fu^{-1}$ is a sum of two units. On the other hand $fRe \subseteq J$ so also $fu^{-1}e$ is a sum of two units. Therefore $xe = (x - fu^{-1})e + fu^{-1}e \in R_\infty$. □

THEOREM 1.14 *If R is a right self-injective ring and I is an ideal of R, then the following are equivalent.*

(i) *Every unit in R/I can be lifted to a unit in R.*

(ii) *If e is an idempotent in I which is contained in a purely infinite idempotent in R, then there exists a purely infinite idempotent in I containing e.*

(iii) $K_0(IR_\infty) = 0$.

Proof. Write $\bar{R} = R/J$ and denote images in $\bar{R}$ by overbars. Note that $R/(I + J)$ is a factor ring of the regular ring R/J. So $J(R/(I + J)) = 0$. Therefore $J(R/I) = (I + J)/I$. Now we have the following commutative diagram

$$\begin{array}{ccc} R & \rightarrow & R/I \\ \downarrow & & \downarrow \\ \bar{R} & \rightarrow & (R/I)/J(R/I) \approx \bar{R}/\bar{I} \end{array}$$

where the rows and columns are the natural projections. Now it is easily seen that $U(R) \to U(R/I)$ is onto if and only if $U(\overline{R}) \to U(\overline{R}/\overline{I})$ so is.

If $e_1\overline{R}$ is the purely infinite part of $\overline{R}$, then $\overline{IR_\infty} = e_1\overline{I}$. Thus $K_0(e_1\overline{I}) \approx K_0(IR_\infty)$ (for this notice that the kernel of the natural projection $IR_\infty \to e_1\overline{I}$ is contained in J). Now it follows from Corollary 1.11, applied to the pair $(\overline{R}, \overline{I})$ that (i) $\Leftrightarrow$ (iii). The result will follow by using Corollary 1.11 and noting that (ii) holds for the pair (R, I) if and only if it holds for $(\overline{R}, \overline{I})$.

Suppose first that $(\overline{R}, \overline{I})$ satisfies (ii). Let e be an idempotent in I such that $e \leq f$ for some purely infinite idempotent f in R. Then $\bar{e} \leq \bar{f}$ and so there exists a purely infinite idempotent g in R such that $\bar{e} \leq \bar{g}$ and $\bar{g}$ belonging to $\overline{I}$. In fact $g \in I + J$ and thus $g \in I$.

Now we have $\overline{ge} = \bar{e}$ so $ge - e = j \in J$. From this we easily obtain $g(1 + j)e = (1 + j)e$. But then $g_1 = (1 + j)^{-1}g(1 + j)$ is a purely infinite idempotent in I such that $e = g_1 e \leq g_1$.

Conversely, let $\bar{e}$ be an idempotent in $\overline{I}$ such that $\bar{e} \leq \bar{f}$ for some purely infinite idempotent $\bar{f}$ in $\overline{R}$. Clearly we may assume f is a purely infinite idempotent in R and e is an idempotent in I. Then $fe - e = j \in J$. As in the preceding paragraph we obtain $e \leq f_1 = (1 + j)^{-1}f(1 + j)$. Clearly f_1 is purely infinite and so, by hypothesis, there exists a purely infinite idempotent g in I with $e \leq g$. Therefore $g \in I$ is a purely infinite idempotent such that $\bar{e} \leq \bar{g}$. □

COROLLARY 1.15. *If R is a prime, regular, right self-injective ring, and I is an ideal of R, then*

(i) *If $I = R_0$, then a unit $\bar{a} \in R/I$ can be lifted to a unit in R if and only if a is unit regular or equivalently* $\operatorname{Ker} a \approx \operatorname{Coker} a$.

(ii) *If $I \neq R_0$, then every unit in R/I can be lifted to a unit in R.*

Proof. (i) It follows from Proposition 1.6.

(ii) If $I \neq R_0$ then, by Lemma 1.5, there exists an idempotent e in I such that eRe is not unit-regular, but R being prime, regular, right self-injective this implies that e is purely infinite. By Theorem 1.14 we must prove that every idempotent f in I is contained in a purely infinite idempotent in I. Without loss of generality we way assume that f is directly finite. Since R satisfies the comparability axiom we have either $e \lesssim f$ or $f \lesssim e$. Since $e \neq 0$ we must have $f \lesssim e$, as desired. □

EXAMPLE. Let $R = \operatorname{End}_k(V)$ where V is an infinite-dimensional K-vector space. In this case $R_0 = \{x \in R \mid \dim_K x(V) < \infty\}$. If we associate with each $[eR] \in K_0(R_0)$ the K-dimension of $e(V)$, we obtain an isomorphism $K_0(R_0) \xrightarrow{\approx} \mathbf{Z}$. By Theorem 1.9 $U(R/R_0)^{\mathrm{ab}} \approx \mathbf{Z}$, furthermore a unit $\bar{a}$ in R/R_0 can be lifted to a unit in R if and only if $\dim_K \operatorname{Coker} a = \dim_K \operatorname{Ker} a$. □

2. Computation of $K_0(I)$. Let R be a purely infinite regular right self-injective ring and let I be an ideal of R. Our goal now is to realize $K_0(I)$ as a group of continuous functions. This has been motivated by Olsen's work in W^*-algebras [**15**].

The starting point in Olsen's proof is Wils' characterization of the closed ideals of W^*-algebras. Although in the regular case such a characterization is not our disposal, we can obtain our results by extending some computations due to Goodearl and Boyle.

If M is a right R-module and $n \geq 0$ is an integer we shall write nM for M^n.

LEMMA 2.1 *Let R be a regular ring. Let A and B be nonsingular injective right R-modules such that the endomorphism ring* $\operatorname{End}_R A$ *is Type* II *and $pA \approx qB$ for some positive integers p, q. Let r be a positive integer.*

(i) *If $r \leq p$ then there exists a right R-module D such that $D \subseteq B$ and $qD \approx rA$.*

(ii) *Assume A is directly finite. Let C be a finitely generated projective right R-module such that $A, B \subseteq C$ and $rA \lesssim qC$. If $r \geq p$ then there exists a right R-module D such that $B \subseteq D \subseteq C$ and $qD \approx rA$.*

Proof. (i) We have $rA \approx B_1 \subseteq qB$ for some B_1. Since $\operatorname{End}_R B_1$ is Type II (see [7, Theorem 10.10]) by [7, Proposition 10.28] $B_1 \approx qB_2$ for some B_2. So $qB_2 \lesssim qB$ and by [7, Theorem 10.34] there exists a right R-module D such that $B_2 \approx D \subseteq B$.

(ii) As in (i) there exists $A_1 \subseteq C$ such that $rA \approx qA_1$. Now consider the submodule of C, $B + A_1$, which is finitely generated and so projective. Then $B + A_1 \lesssim B \oplus A_1$ and by [7, Corollary 9.20] $B + A_1$ is a directly finite nonsingular injective right R-module. Thus $\operatorname{End}_R(B + A_1)$ is unit regular.

On the other hand $qB \approx pA \subseteq rA \approx qA_1$, so $B \approx B_1 \subseteq A_1$ for some B_1. Then by [7, Corollary 4.4] there are decompositions $B + A_1 = B \oplus B' = B_1 \oplus B'$ and thus $D = B \oplus (A_1 \cap B')$ is the desired R-module.

Finally note that (i) follows for any ring R. □

LEMMA 2.2. *Let R be a regular right self-injective ring. Let A be a principal right ideal of R such that* $\operatorname{End}_R A$ *is Type* II_f. *Let $\{p_n, q_n\}_{n \in \mathbf{N}}$ be a set of positive integers such that $p_n A \lesssim q_n R$ for every n. Then there exist* *principal right ideals of* R; $B_1, B_2, \dots$ *such that* $q_nB_n \approx p_nA$ *for every* n *and* $B_n \subseteq B_m$ *whenever* $p_n/q_n \leq p_m/q_m$.

Proof. We are going to construct the right ideals B_n by induction on n. Since $p_1A \lesssim q_1R$ and $\mathrm{End}_R A$ is Type II there exists a principal right ideal A_1 such that $p_1A \approx p_1q_1A_1 \lesssim q_1R$ cf. [7, Proposition 10.28]. Then by [7, Theorem 10.34] $p_1A_1 \approx B_1 \subseteq R$ for some right ideal B_1.

Now suppose we have constructed $B_1, \dots, B_n$. Set $\lambda_n = p_n/q_n$ for each n. Assume for simplicity that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Now there are three possibilities: (1) $\lambda_{n+1} \leq \lambda_n$, (2) $\lambda_1 \leq \lambda_{n+1}$ and (3) $\lambda_i \geq \lambda_{n+1} \geq \lambda_{i+1}$ for some $i \in \{1, \dots, n-1\}$.

(1) By the induction hypothesis we have $q_nB_n \approx p_nA$, so $q_{n+1}q_nB_n \approx q_{n+1}p_nA$ and then, by applying Lemma 2.1(i), there exists a principal right ideal B_{n+1} with $B_{n+1} \subseteq B_n$ and $q_{n+1}B_{n+1} \approx p_{n+1}A$.

(2) Let A_1 be a submodule of A such that $A \approx q_{n+1}A_1$. Now $q_1B_1 \approx p_1A \approx p_1q_{n+1}A_1$. On the other hand $p_{n+1}q_{n+1}A_1 \lesssim q_{n+1}R$ implies $p_{n+1}A_1 \lesssim R$. By Lemma 2.1 (ii) there exists B_{n+1} with $B_1 \subset B_{n+1}$ and $q_1B_{n+1} \approx p_{n+1}q_1A_1$, thus $q_{n+1}B_{n+1} \approx p_{n+1}A$.

(3) As in the case (1), there exists a submodule of B_i, say B, such that $q_{n+1}B \approx p_{n+1}A$. From the relation $\lambda_{n+1} \geq \lambda_{i+1}$ we obtain $p_{i+1}q_{n+1}B_{i+1} \lesssim p_{n+1}q_{i+1}B_{i+1} \approx p_{i+1}p_{n+1}A \approx p_{i+1}q_{n+1}B$, so there exists B_{i+1}^* with $B_{i+1} \approx B_{i+1}^* \subseteq B$. Then by [7, Corollary 4.4] there are decompositions $B + B_{i+1} = B_{i+1} \oplus B^* = B_{i+1}^* \oplus B^*$.

Now write B_{n+1} for the module $B_{i+1} \oplus (B \cap B^*)$. Then $B_i \supseteq B_{n+1} \supseteq B_{i+1}$ and $q_{n+1}B_{n+1} \approx p_{n+1}A$. $\square$

Let R be a regular right self-injective ring. If e is a directly finite idempotent of R, then eRe is unit-regular cf [7; Corollary 1.23, Theorem 9.17]. By Lemma 1.5 we see that R_0 coincides with the ideal of R generated by all directly finite idempotents of R.

LEMMA 2.3. *Let* R *be a regular right self-injective ring and* I *an ideal of* R *contained in* R_0. *If* J *is an ideal of* R *contained in* I, *then the natural homomorphism* $K_0(J) \to K_0(I)$, *induced by the inclusion* $J \subseteq I$, *is injective*.

Proof. By Proposition 1.2 every element in $K_0(J)$ can be written in the form $[A] - [B]$ for some finitely generated projective right R-modules in $P(I)$. If $[A] = [B]$ in $K_0(I)$, then there exists a finitely projective right R-module $C \in P(I)$ with $A \oplus C \approx B \oplus C$. Since every idempotent in I

is directly finite, by [7, Corollary 9.20] C is directly finite and then by [7, Corollary 9.18] $A \approx B$. So $[A] = [B]$ in $K_0(J)$. $\square$

From now on we shall identity $K_0(J)$ with its image in $K_0(I)$.

Let $B(R)$ be the set of all central idempotents of R. If $\{e_i\}_{i \in I}$ is a family of elements in R we denote by $\bigvee_{i \in I} e_i$ and by $\bigwedge_{i \in I} e_i$ its supremun and its infimum respectively. If R is regular and right self-injective then by [7, Proposition 9.9] $B(R)$ is a complete Boolean algebra.

Let $X = BS(R)$ be the Boolean spectrum of R, that is, X is the set of all maximal ideals of $B(R)$. Recall that the closed sets in X are of the form $V(S) = \{M \in BS(R) \mid S \subseteq M\}$, where $S \subseteq B(R)$. Recall that with this topology, X is an Stonian space, that is, X is a compact Hausdorff space such that the closure of every open set is open. If $Y \subseteq X$ then we denote the closure of Y in X by $\overline{Y}$.

We shall need the following simple lemma.

Lemma 2.4. *Suppose* $\{e_i\}_{i \in I}$ *is a family of elements in* $B(R)$. *If* $X_i = V(1 - e_i)$ *for all* i, *then* $\overline{\bigcup_{i \in I} X_i} = V(1 - \bigvee_{i \in I} e_i)$.

Proof. Set $e = \bigvee_{i \in I} e_i$ and $Y = \overline{\bigcup_{i \in I} X_i}$. Since Y is a clopen set there exists f in $B(R)$ such that $Y = V(1 - f)$. It is easily seen that the inclusion $X_i = V(1 - e_i) \subseteq Y = V(1 - f)$ implies $e_i \leq f$ for all index i. So $e \leq f$. On the other hand we have $\bigcup_{i \in I} X_i \subseteq V(1 - e)$. Because $V(1 - e)$ is clopen it contains Y. So $f \leq e$. $\square$

Let $f\colon X \to [-\infty, \infty]$ be a continuous map of X into the extended real interval $[-\infty, \infty]$. We say that f is *almost finite* if it is finite in a dense open subset of X. We denote by $\mathscr{C}(X, [-\infty, \infty])$ the set of all almost finite continuous maps of X into $[-\infty, \infty]$. Assume $f, g \in \mathscr{C}(X, [-\infty, \infty])$ and let U be a dense open set in X such that f and g are finite in U. Consider the continuous map $f + g$ of U into $[-\infty, \infty]$ defined with pointwise addition. Since X is Stonian, $\overline{U} = X$ is the Stone-Čech compactification of U (see [**19**, 1.14 Theorem]), then, in particular, by [**19**, 1.11 Theorem] $f + g$ can be extended to a unique continuous map, also denoted by $f + g$, of X into $[-\infty, \infty]$. With this addition and the natural order, $\mathscr{C}(X, [-\infty, \infty])$ becomes an ordered abelian group.

Let G be a partially ordered abelian group and let H be a subgroup of G. Recall that H is said to be *directed* if it is upward directed, and *convex* if whenever $x_1, x_2 \in H$ and $y \in G$ such that $x_1 \leq y \leq x_2$, then

$y \in H$. It is known (see for example [7, Proposition 15.17]) that the set of all directed convex subgroups of G ordered by inclusion forms a lattice denoted by $L(G)$.

For any rng I we denote by $L_2(I)$ the lattice of ideals of I.

For the definition of the relative dimension functions on the nonsingular injective right modules over regular right self-injective rings we refer to [7, Chapter 11].

THEOREM 2.5. *Let R be a regular right self-injective ring of Type* II_∞ *and let e_0 be a faithful directly finite idempotent in R. Then*

(i) *The rule*

$$[eR] \mapsto \varphi_e, \quad \varphi_e(M) = d_M(eR : e_0R)$$

defines an isomorphism of partially ordered abelian groups.

$$\varphi\colon K_0(R_0) \overset{\approx}{\to} \mathscr{C}(X, [-\infty, \infty]).$$

(ii) *The map*

$$L_2(R_0) \to L(\mathscr{C}(X, [-\infty, \infty]), \qquad J \mapsto \varphi(K_0(J)))$$

is a lattice isomorphism.

Proof. (i) Denote by $\mathscr{C}(X, [0, \infty])$ the set of all almost finite continuous maps of X into the extended real interval $[0, \infty]$. By [7, Lemma 11.16] if e is an idempotent in R_0 then the map

$$\varphi_e\colon X \to [0, \infty], \qquad M \mapsto d_M(eR\colon e_0R)$$

is continuous. Now we prove that in fact φ_e belongs to $\mathscr{C}(X, [0, \infty])$. Set $U = \varphi_e^{-1}([0, \infty))$, which is an open set. Because X is Stonian, $\overline{U}$ is clopen and so $\overline{U} = V(f)$ for some f in $B(R)$. Suppose $eg \lesssim ne_0g$ for some positive integer n and some central idempotent g. If $fg \neq 0$ then there exists a maximal ideal M in $B(R)$ such that $fg \notin M$, thus $d_M(eR\colon e_0R) \leq n$ and so $M \in V(f)$, which is a contradiction. Then $fg = 0$. Let m be a positive integer. By the general comparability axiom there exists a central idempotent h such that $efh \lesssim me_0fh$ and $(1 - h)me_0f \lesssim ef(1 - h)$. Then by the above $fh = 0$ and so $me_0f \lesssim ef$. Since this holds for all m we see that $e_0f = 0$, cf [7, Corollary 9.23]. Therefore $f = 0$ and $\overline{U} = X$.

Since R is purely infinite, for every finitely projective right R-module A there exists an idempotent e in R such that $A \approx eR$. Thus we have a well-defined map

$$\varphi\colon K_0(R_0)^+ \to \mathscr{C}(X, [0, \infty]), \qquad [eR] \mapsto \varphi_e$$

where e is any idempotent of R_0.

Now we prove that φ is onto. For this let α be an element in $\mathscr{C}(X, [0, \infty])$. Let X_0 denote the closure of the set $\{M \in X \mid \alpha(M) > 0\}$. For any integers m and n such that $m \geq 0$ and $n \geq 1$ let X_{mn} denote the closure of the set $\{M \in X \mid \alpha(M) > m/2^n\}$. Note that $X_{0n} = X_0$ and $X_{mn} \subseteq X_{m-1,n}$ for all m and n. It is easily seen that $\overline{X_0 - \alpha^{-1}(\infty)} = \bigcup_{m=1}^{\infty}(X_{m-1,n} - X_{mn})$ for a fixed n. Since α is almost finite, $\overline{X_0 - \alpha^{-1}(\infty)} = X_0$. Suppose $X_{m-1,n} - X_{mn} = \{M \in X \mid e_{mn} \notin M\}$ for all m and n and for some e_{mn} in $B(R)$. It is clear that for each n the e_{mn}'s are orthogonal because the sets $X_{m-1,n} - X_{mn}$ are disjoint. It follows from Lemma 2.4 that $X_0 = \{M \in X \mid 1 - e \notin M\}$, where $1 - e = \bigvee_{m,n} e_{mn}$.

For any m and n, $X_{m-1,n} = X_{2m-2,n+1}$ and $X_{m-1,n} - X_{mn}$ is the disjoint union of $X_{2m-2,n+1} - X_{2m-1,n+1}$ and $X_{2m-1,n+1} - X_{2m,n+1}$. So $e_{mn} = e_{2m-1,n+1} + e_{2m,n+1}$. Let f_{mn} be an idempotent such that $2^n f_{mn} R \approx m e_0 R$. Since $e_0 R$ is directly finite it follows from [7, Proposition 11.3(e)] that $d_M(f_{mn}R\colon e_0R) = m/2^n$ for all M in X. By Lemma 2.2 we can assume that $f_{mn} \leq f_{st}$ if $m/2^n \leq s/2^t$.

Let $A_n = \bigvee_{m \geq 1}(e_{mn} f_{m-1,n} R)$ and note that A_n is an injective hull of $\bigoplus_{m \geq 1} e_{mn} f_{m-1,n} R$. It is easily seen that A_n is directly finite. Now we have

$$\begin{aligned}\bigoplus_{m\geq 1} e_{mn} f_{m-1,n} R &= \bigoplus_{m\geq 1} (e_{2m-1,n+1} + e_{2m,n+1}) f_{m-1,n} R \\ &\subseteq \left(\bigoplus_{m\geq 1} f_{2m-2,n+1} e_{2m-1,n+1} R\right) \oplus \left(\bigoplus_{m\geq 1} f_{2m-1,n+1} e_{2m,n+1} R\right) \\ &= \bigoplus_{m\geq 1} f_{m-1,n+1} e_{m,n+1} R.\end{aligned}$$

So $A_n \subseteq A_{n+1}$ for all n. Set $A = \bigcup_{n \geq 1} A_n$.

For any integer $t \geq 1$ define $A_t^* = (\bigvee_{m \geq 1} f_{mt} e_{mt} R)$. As above A_t^* is directly finite and we have

$$\begin{aligned}\bigoplus_{m\geq 1} f_{m,t+1} e_{m,t+1} R &= \left(\bigoplus_{j\geq 1} f_{2j-1,t+1} e_{2j-1,t+1} R\right) \oplus \left(\bigoplus_{j\geq 1} f_{2j,t+1} e_{2j,t+1} R\right) \\ &\subseteq \bigoplus_{j\geq 1} f_{2j,t+1}(e_{2j-1,t+1} + e_{2j,t+1}) R \\ &= \bigoplus_{j\geq 1} f_{2j,t+1} e_{j,t} R = \bigoplus_{j\geq 1} f_{jt} e_{jt} R.\end{aligned}$$

So $A_{t+1}^* \subseteq A_t^*$. Now $A_t \subseteq A_t^*$ and then $A \subseteq A_t^*$.

We shall prove that $\varphi([A]) = \alpha$. Since α is almost finite we must show that $\varphi([A])(M) = \alpha(M)$ for all $M \in X - \alpha^{-1}(\infty)$. If $e \notin M$ then $d_M(A\colon e_0R) = d_M(Ae\colon e_0R) = 0$ because $Ae = 0$. Now suppose that e

belongs to M. Then for each n we have that there exists an m such that $M \in X_{m-1,n} - X_{m,n}$. So $m - 1/2^n \leq \alpha(M) \leq m/2^n$. Since $A_n e_{mn} R = f_{m-1,n} e_{mn} R$ then $d_M(A_n\colon e_0R) = m - 1/2^n$ and so $d_M(A\colon e_0R) \geq (m-1)/2^n$. Similarly $d_M(A\colon e_0R) \leq d_M(A_n^*\colon e_0R) = d_M(A_n^* e_{mn}\colon e_0R) = d_M(f_{mn}R\colon e_0R) = m/2^n$. Then $\varphi([A])(M) - 1/2^n \leq \alpha(M) \leq \varphi([A])(M) + 1/2^n$ for all n. So $\alpha(M) = \varphi([A])(M)$.

Now by [7, Theorem 11.11] the map

$$\varphi\colon K_0(R_0) \to \mathscr{C}(X, [-\infty, \infty]), \qquad [eR] - [fR] \to \varphi_e - \varphi_f$$

is a group homomorphism. By [7, Theorem 11.15 (a)] φ is an order preserving homomorphism. Because any element of $\mathscr{C}(X, [-\infty, \infty])$ can be written as a difference of two elements of $\mathscr{C}(X, [0, \infty])$, by the preceding paragraph it is clear that φ is onto. To prove injectivity suppose $\varphi_e = \varphi_f$ for some idempotents e and f in $K_0(R_0)$. Then by [7, Theorem 11.15 (b)] $eR \approx fR$ and so $[eR] = [fR]$ in $K_0(R_0)$.

(ii) If J is an ideal of R contained in R_0, by Lemma 2.3 $K_0(J)$ is a subgroup of $K_0(R_0)$. Now, as in the proof of [7, Theorem 15.20] one can see that the correspondence $J \mapsto K_0(J)$ defines a lattice isomorphism of $L(R_0)$ onto $L(K_0(R_0))$. Since φ is an order group isomorphism, the result follows. □

COROLLARY 2.6. *If R is a prime regular right self-injective ring of Type* II_∞, *then* $K_1(R/R_0) = U(R/R_0)^{\mathrm{ab}} \approx K_0(R_0) \approx \mathbf{R}$.

Proof. It follows from Theorem 1.9 and Theorem 2.5. □

Now we shall consider almost finite continuous functions on X taking its values on $\mathbf{Z} \cup \{\pm\infty\}$. As above we shall write $\mathscr{C}(X, \mathbf{Z} \cup \{\pm\infty\})$ for the group of all this functions.

LEMMA 2.7. *Let R be a regular ring. Let A and B be finitely generated right R-modules such that* $\operatorname{End}_R A$ *is unit-regular. If $A/AP \lesssim B/BP$ for all prime ideals P of R then $A \lesssim B$.*

Proof. In [7, Theorem 4.19] this lemma is proved under the hypothesis of unit-regularity. But, with the notation of [7, Lemma 4.18], it is only necessary that the R-module A_1/A_1K cancels from direct sums, and it is easily seen that this also occurs if $\operatorname{End}_R A$ is unit-regular. □

The proof of the next result is quite similar to Theorem 2.5.

THEOREM 2.8. *Let R be a regular right self-injective ring of Type* I_∞ *and let e_0 be a faithful abelian idempotent in R. Then*

(i) *the rule*

$$[eR] \mapsto \varphi_e, \qquad \varphi_e(M) = d_M(eR\colon e_0R)$$

defines a partially ordered abelian group isomorphism

$$\varphi\colon K_0(R_0) \overset{\approx}{\to} \mathscr{C}(X, \mathbf{Z} \cup \{\pm\infty\})$$

(ii) *the map*

$$L_2(R_0) \to L(\mathscr{C}(X, \mathbf{Z} \cup \{\pm\infty\})), \qquad J \mapsto \varphi(K_0(J))$$

is a lattice isomorphism.

Proof. (i) First we prove that if e is an idempotent in R_0 then $d_M(eR\colon e_0R)$ is either an integer or ∞. For this we need only prove that if $nfR \lesssim mgR$, where m, n are positive integers and f, g are idempotents with g abelian, then there exists an integer s, $s \le m/n$, such that $fR \lesssim sgR$.

Let P be a prime ideal in R and let $\bar{f}$, $\bar{g} \in R/P$ the images of f and g in R/P respectively. Then $n\bar{f}R/P \lesssim m\bar{g}R/P$. Since R/P is prime and $\bar{g}$ is abelian in R/P we see that $\bar{g}R/P$ is a simple module and so $\bar{f}R/P \approx r\bar{g}R/P$ for some $r \in N$. Hence $\bar{f}R/P \lesssim [m/n]\bar{g}R/P$, where $[m/n]$ denotes the integer part of m/n. By Lemma 2.7 we obtain $fR \lesssim [m/n]gR$ as desired.

As in the proof of Theorem 2.5 (i) we derive that φ is a well-defined injective map.

Now we are going to prove that φ is onto. Like Theorem 2.5 (i) it suffices to prove that for every positive α in $\mathscr{C}(X, \mathbf{Z} \cup \{\pm\infty\})$ there exists A in $P(R_0)$ such that $\varphi([A]) = \alpha$. For each natural k, set $X_k = \{M \in X \mid \alpha(M) = k\}$. Certainly X_k is a clopen set in X. Hence $X_k = \{M \in X \mid e_k \notin M\}$, for some suitable e_k in $B(R)$. Since the X_k's are pairwise disjoint we have that the corresponding e_k's are orthogonal.

For a given natural number n, we have, since R is purely infinite, that $ne_0R \lesssim R$. Thus $\oplus_k ke_ke_0R \lesssim R$. Let A denote a principal right ideal of R that is isomorphic to the injective hull of $\oplus_k ke_ke_0R$. There is no difficulty in proving that A belongs to $P(R_0)$. Clearly $e_kA \approx ke_ke_0R$ and, by [7, Proposition 11.3] we have

$$\varphi([A])(M) = d_M(A\colon e_0R) = d_M(ke_ke_0R\colon e_0R) = k = \alpha(M),$$

for all $M \in X_k$. Since α is almost finite we see $\varphi([A]) = \alpha$.

(ii) It follows similarly to Theorem 2.5 (ii). □

LEMMA 2.9. *Let R be a regular right self-injective ring and I an ideal of R. If $C \in P(I)$ is purely infinite then $C \approx eR$ for some (purely infinite) idempotent e in I.*

Proof. Suppose $C = A \oplus B$ for some directly finite right R-module A and some purely infinite right R-module B. Now we prove that $C \approx B$. By [7, Theorem 9.14] there exists $h \in B(R)$ such that $Ah \lesssim Bh$ and $B(1-h) \lesssim A(1-h)$. Then, since B is purely infinite, we have $B(1-h) = 0$. So $C(1-h) \approx A(1-h)$ and thus also $A(1-h) = 0$. Then $B \lesssim A \oplus B \lesssim B \oplus B \approx B$ and so, by [7, Theorem 10.14] $C = A \oplus B \approx B$.

Now, suppose $C \approx e_1 R \oplus \cdots \oplus e_n R$ for some idempotents $e_1, \ldots, e_n$ in I. By [7, Theorem 10.32] there exists $h_i \in B(R)$ such that $h_i e_i R$ is directly finite and $(1-h_i)e_i R$ is purely infinite for $i = 1, \ldots, n$. Then by the preceding paragraph we can assume that each e_i is purely infinite. Since R satisfies general comparabilaity, there exists $h \in B(R)$ such that $he_1 \lesssim he_2$ and $(1-h)e_2 \lesssim (1-h)e_1$. Then it is clear that $e = (1-h)e_1 + he_2$ is a purely infinite idempotent in I such that $e_1 R \oplus e_2 R \approx eR \oplus eR \approx eR$. By induction on n the result follows. □

For each ideal I of R we denote by I_0 the ideal of R generated by all directly finite idempotents in I and by I_1 the ideal of R generated by all directly finite idempotents in I that are contained in some purely infinite idempotent in I.

If $S \in L(\mathscr{C}(X, K))$, where K is either $[-\infty, \infty]$ or $\mathbf{Z} \cup \{\pm\infty\}$, and Γ is a closed set in X, then we write S_Γ for the quotient $S/\{\alpha \in S\colon \alpha = 0$ in some open set in X containing $\Gamma\}$.

THEOREM 2.10. *Let R be a regular right self-injective ring and I an ideal of R. Then*

(i) $K_0(I) \approx K_0(I_0)/K_0(I_1)$.

(ii) *Let $\Gamma(I) = V(\{\mathrm{cc}(g) \mid g$ is a purely infinite idempotent in $I\})$. If R is either Type II_∞ or I_∞ then $K_0(I) \approx \varphi(K_0(I_0))_{\Gamma(I)}$ where $\varphi\colon K_0(R_0) \to \mathscr{C}(X, K)$ is the map defined in Theorem 2.5 or Theorem 2.8, respectively.*

Proof. (i) First we prove that the natural map $\Psi\colon K_0(I_0) \to K_0(I)$ is onto. Let $A \in P(I)$. By [7, Theorem 10.32] there exists a central idempotent h in R such that Ah is directly finite and $A(1-h)$ is purely infinite. Then $[A(1-h)] = 0$ in $K_0(I)$ and so we can assume that A is directly finite, but in this case it is clear that A belongs to $P(I_0)$.

Now we prove that $\operatorname{Ker}\Psi = K_0(I_1)$. Let $A \in P(I_1)$. Since A is isomorphic to a direct sum of principal right ideals, each of which is generated by an idempotent in I_1, it is clear that in order to prove

$[A] \in \operatorname{Ker}\Psi$ we may assume $A = eR$ for some idempotent e in I_1. Then there exists a purely infinite right R-module B in $P(I)$ such that $A \lesssim B$. Thus $A \oplus B \approx B$. Then $[A] = 0$ in $K_0(I)$ and so $K_0(I_1) \subseteq \operatorname{Ker}\Psi$.

Conversely, let $[A] - [B] \in \operatorname{Ker}\Psi$. Then by Proposition 1.2 and the proof of Proposition 1.8 (i) there exists a purely infinite right R-module C in $P(I)$ such that $A \oplus C \approx B \oplus C$. Now, by the general comparability axiom there exists $h \in B(R)$ such that $Bh \lesssim Ch$ and $C(1-h) \lesssim B(1-h)$. Since $B(1-h)$ is directly finite and $C(1-h)$ is purely infinite we see $C(1-h) = 0$. From the relation $A \oplus C \approx B \oplus C$ we have $A(1-h) \approx B(1-h)$ so $[A] - [B] = [Ah] - [Bh]$. Then we may assume $B \lesssim C$ and since C is purely infinite also $A \lesssim C$. By Lemma 2.9 $A \lesssim eR$ for some purely infinite idempotent e in I. Then by Lemma 1.10 $A \subseteq fR$ for some purely infinite idempotent f in I. Hence $A \in P(I_1)$. Similarly $B \in P(I_1)$ and then $[A] - [B] \in K_0(I_1)$.

(ii) Since R is purely infinite, every element in $K_0(I)$ can be written in the form $[eR] - [fR]$ for some idempotents e, f in I.

By (i) it suffices to show that $\varphi(K_0(I_1)) = \{\alpha \in \varphi(K_0(I_0))\colon \alpha = 0$ in some open set in X containing $\Gamma(I)\}$. If $[eR] \in K_0(I_1)$ then there exists a purely infinite idempotent g in I such that $e \lesssim g$. Then $eR \oplus gR \approx gR$ and by [7, Theorem 11.11] $d_M(eR\colon e_0R) \leq d_M(gR\colon e_0R)$ for all M in X (here e_0 is as in Theorem 2.5 or Theorem 2.8). By [7, Proposition 11.3] $d_M(gR\colon e_0R) = 0$ if $M \in V(\mathrm{cc}(g))$. So, since $\Gamma(I) \subseteq V(\mathrm{cc}(g))$, we have $\varphi(K_0(I_1)) \subseteq \{\alpha \in \varphi(K_0(I_0)) \mid \alpha = 0$ in an open set in X containing $\Gamma(I)\}$.

Now we prove the reverse inclusion. For simplicity here we denote by E the set of all purely infinite idempotents in I. First we shall note that the set $S = \{\mathrm{cc}(g) \mid g \in E\}$ is an ideal of $B(R)$. If $x \in B(R)$ and $g \in E$, then by [7, Lemma 11.4 (c)] $x\mathrm{cc}(g) = \mathrm{cc}(xg)$. Since $xg \in E$, we see that $x\mathrm{cc}(g) \in S$. Let $g_1, g_2 \in E$ and let $k = \mathrm{cc}(g_1) + \mathrm{cc}(g_2) - 2\mathrm{cc}(g_1)\mathrm{cc}(g_2)$. By [7, Lemma 11.4(c)] and observing that $g_1(1 - \mathrm{cc}(g_2))$ and $g_2(1 - \mathrm{cc}(g_1))$ are orthogonal idempotents we have

$$\begin{aligned}
&\mathrm{cc}(g_1(1 - \mathrm{cc}(g_2)) + g_2(1 - \mathrm{cc}(g_1))) \\
&\qquad = \mathrm{cc}(g_1(1 - \mathrm{cc}(g_2))) + \mathrm{cc}(g_2(1 - \mathrm{cc}(g_1))) \\
&\qquad = \mathrm{cc}(g_1)(1 - \mathrm{cc}(g_2)) + \mathrm{cc}(g_2)(1 - \mathrm{cc}(g_1)) = k.
\end{aligned}$$

By noting that $g_1(1 - \mathrm{cc}(g_2)) + g_2(1 - \mathrm{cc}(g_1)) \in E$ we obtain that $k \in S$. Then S is an ideal of $B(R)$.

Let $e \in I_0$ be an idempotent such that $\varphi([eR])$ is zero in an open set U containing $\Gamma(I)$. For each $M \in \Gamma(I)$ there exists $h_M \in B(R)$ with $M \in V(h_M) \subseteq U$. Since $\Gamma(I)$ is compact we can find $M_1, \ldots, M_r \in \Gamma(I)$

with $\Gamma(I) \subseteq V(h_{M_1}) \cup \cdots \cup V(h_{M_r}) = V(h_{M_1} \cdots h_{M_r})$; set $h = h_{M_1} \cdots h_{M_r}$, then from the inclusion $V(S) = \Gamma(I) \subseteq V(h)$ we obtain $h \in S$ and so $h = \mathrm{cc}(g)$ for some g in E.

Let $M \in X$. If $1 - h \in M$, then by [7, Proposition 11.3 (a)]

$$d_M\big(e(1-h)R\colon e_0R\big) = 0.$$

If $h \in M$ then, since $V(h) \subseteq U$, $d_M(e(1-h)R\colon e_0R) = \varphi([eR])(M) = 0$. Hence, by [7, Proposition 11.6], $e(1-h) = 0$. Let $t \in B(R)$ such that $te \lesssim tg$ and $(1-t)g \lesssim (1-t)e$. Because $(1-t)e$ is directly finite and $(1-t)g$ is purely infinite, we obtain $(1-t)g = 0$ and so $h = \mathrm{cc}(g) \leq t$. Then by multiplying the relation $te \lesssim g$ by h, we obtain $hte \lesssim hg = g$, and, because $ht = h$ and $he = e$, we have $e \lesssim g$. By Lemma 1.10 we may assume $e \leq g$ and so $[eR] \in K_0(I_1)$ as desired. □

3. Rickart C^*-algebras. Recall that a C^*-algebra A is said to be *Rickart* if the right annihilator of each element in A is generated by a projection. In notation $r(a) = eA$ where $e = e^2 = e^*$. If the annihilator condition holds for every subset of A, then A is called an *AW^*-algebra*. As usual we shall write $\mathrm{RP}(a)$ (the *right projection of a*) for $1 - e$. The *left projection of a*, *$LP(a)$*, is defined similarly. It is known [3, Proposition 1.3.7 and Lemma 1.8.2] that with the relation $\leq$ the set of all projections of a Rickart C^*-algebra is a complemented χ_0-complete lattice. Two projections e and f are said to be *equivalent*, written $e \sim f$, if $eA \approx fA$. A projection e is said to be *finite* if $e \sim f \leq e$ implies $e = f$. We say A is *finite* if 1 is a finite projection. Since A is a C^*-algebra $e \sim f$ if and only if e and f are $*$-equivalent, that is $e = xx^*$ and $f = x^*x$ for some $x \in eAf$ cf. [9, Proposition 19.1 (a)]. If e is an idempotent of a C^*-algebra A, then there exists a unique projection f in A such that $eA = fA$ cf. [9, proof of Proposition 19.1 (b)]. From this we see that Rickart C^*-algebras are precisely those C^*-algebras that are principal projective. It seems to be unknown whether Rickart C^*-algebras are semihereditary.

For background and basic concepts on Rickart C^*-algebras the reader can consult [3].

PROPOSITION 3.1. *If A is a Rickart C^*-algebra and I is an ideal of A then $K_0(I) = G(I) = K_0(\bar{I})$, where $\bar{I}$ is the closure of I.*

Proof. Let E be the set of all projections in I. It follows from [3, Proposition 5.22.1] that the sub-C^*-algebras $\{eAe + \mathbf{C}1\}_{e \in E}$ form a directed system. Since $\bar{I}$ is the closed $\mathbf{C}$-linear span of its projections [3, p.

142, Exc. 7A] we have that $C^*\text{-dir.lim}_{e\in E}(eAe + \mathbf{C}1) = \bar{I} + \mathbf{C}1$. Now it follows from [**9**, Proposition 19.9] that the natural map

$$\operatorname*{dir.lim}_{e\in E} K_0(eAe + \mathbf{C}) \to K_0(\bar{I} + \mathbf{C})$$

is a group isomorphism. Since the diagram

$$\begin{array}{ccc} \operatorname*{dir.lim}_{e\in E} K_0(eAe + \mathbf{C}) & \to & K_0(I + \mathbf{C}) \\ \wr\downarrow & \swarrow & \\ K_0(\bar{I} + \mathbf{C}) & & \end{array}$$

is commutative, where the maps are the natural ones, then the map

$$\operatorname*{dir.lim.}_{e\in E} K_0(eAe + \mathbf{C}) \to K_0(I + \mathbf{C})$$

is injective, and onto by [**9**, Proposition 19.3].

Thus, by Proposition 0.1 we have $K_0(I) = K_0(\bar{I}) = G(I)$. □

Let A be a C^*-algebra and let I be an ideal of A. If $\pi\colon A \to A/I$ is the natural surjection, then we set $\mathscr{F}(I, A) = \pi^{-1}(U(A/I))$. An element of $\mathscr{F}(I, A)$ is said to be a *Fredholm element of A relative to I*. In the case where $A = B(H)$ is the ring of all bounded operators on a separable Hilbert space and $I = \mathscr{K}$ is the ideal of compact operators, then the elements of $\mathscr{F}(\mathscr{K}, B(H))$ are the usual Fredholm operators cf. [**6**, Chapter 5].

Let us recall briefly some basic results on index theory for Fredholm operators. If $T \in \mathscr{F}(\mathscr{K}, B(H))$, then by Atkinson's theorem [**6**, 5.17 Theorem] $\dim\ker T$ and $\dim\operatorname{Ker} T^*$ are both finite and the map $\imath$: $\mathscr{F}(\mathscr{K}, B(H)) \to \mathbf{Z}$ given by $T \mapsto \dim\operatorname{Ker} T^*\text{-}\dim\operatorname{Ker} T$ (the *index map*) is a continuous monoid homomorphism [**6**, 5.36 Theorem]. Furthermore the connected components of $\mathscr{F}(\mathscr{K}, B(H))$ are $\imath^{-1}(n)$, $n \in \mathbf{Z}$ [**6**, 5.36 Theorem]. Breuer [**4**] [**5**] generalizes this result to an arbitrary W^*-algebra (here the compact ideal means the closure of the ideal generated by all finite projections in A). More recently Olsen [**15**] has defined an index map for each closed ideal I of a W^*-algebra which permits to describe the connected components of $\mathscr{F}(I, A)$.

Next we shall extend Breuer's theory to arbitrary Rickart C^*-algebras. In order to obtain an explicit index map for any closed ideal in a Rickart C^*-algebra A we will need the following additional axioms on A:

(i) A has a projection e such that $e \sim 1 - e \sim 1$

(ii) A satisfies the general comparability axiom (i.e. for each pair of projections e, f there exists a central projection h such that $he \lesssim hf$ and $h(1 - f) \lesssim h(1 - e)$).

As we shall see this axioms are not an obstacle for constructing an index theory for arbitrary AW^*-algebras.

The following lemma is known under the additional hypothesis of general comparability (see [**3**, Lemma 1.8.3, Theorem 3.17.3]).

If A is a Rickart C^*-algebra, then we denote by $\mathscr{K} = \mathscr{K}(A)$ the closure of the ideal generated by all finite projections of A. We say that $\mathscr{K}$ is the *compact ideal of* A.

LEMMA 3.2. *Every projection in* $\mathscr{K}$ *is finite.*

Proof. Let I be the ideal generated by all finite projections in A.

Since $\mathscr{K}$ is the closure of I it is well-known that every projection in $\mathscr{K}$ belongs to I cf [**3**, Chapter 5 §22 Exercise 6A]. Now let f be a projection in I, then $f = \Sigma x_i e_i y_i$, where $x_i,\ y_i \in A$ and the e_i's are finite projections. Consider now the map ψ: $\oplus e_i A \to fA$ defined by $\psi(\Sigma e_i r_i) = \Sigma f x_i e_i r_i$. Clearly ψ is an onto A-module homomorphism. Thus $fA \lesssim \oplus e_i A$. Now a finite Rickart C^*-algebra has stable range 1 cf [**10**]. So the endomorphisms rings $e_i A e_i \approx \operatorname{End}_R(e_i A)$ have stable range 1. In particular $\oplus e_i A$ cancels from direct sums of right A-modules and, since fA is isomorphic to a direct summand of $\oplus e_i A$, the same is true for fA. Therefore f is finite. □

If M and N are right A-modules, then $M \hookrightarrow N$ means that M is isomorphic to a submodule of N.

LEMMA 3.3. *If A is a Rickart C^*-algebra, then*

(i) *If $e \in A$ is a finite projection, then eA does not contain an infinite direct sum of nonzero pairwise isomorphic right ideals. In particular, every A-module $M \hookrightarrow eA$ is directly finite.*

(ii) *If P and Q are directly finite cyclic projective right A-modules such that $P \hookrightarrow Q$ and $Q \hookrightarrow P$, then $P \approx Q$.*

(iii) *If x is an element of A such that* $\mathrm{LP}(x)$ *is finite, then* $\mathrm{LP}(x) \sim \mathrm{RP}(x)$. *Further $xA \approx x^*A$.*

Proof. (i) Let $\{A_n\}$ be a sequence of pairwise isomorphic right ideals contained in eA. Then $\{A_n e\}$ is a sequence of pairwise isomorphic right ideals of eAe. Now eAe is a finite Rickart C^*-algebra and so, R, its classical ring of quotients [**1**, Theorem 3.1(i)] [**11**, Theorem 2.1] is an $\aleph_0$-continuous regular ring which contains an infinite direct sum of pairwise isomorphic right ideals. By [**8**, Proposition 1.1] $A_i e \otimes_{eAe} R = 0$, hence $A_i e = 0$. But A is semiprime so $0 = eA_i = A_i$ as desired.

(ii) We may assume $P = eA$ and $Q = fA$ for some finite projections e, f in A. Let g be the supremum of e and f. By [**3**, Proposition 5.22.1] $g \in \mathcal{K}$ and it follows from Lemma 3.2 that g is finite and so gAg is a finite Rickart C^*-algebra. Now $e(gAg) \hookrightarrow f(gAg)$ and $f(gAg) \hookrightarrow e(gAg)$. If R is the classical ring of quotients of gAg, then because R is regular we have $eR \lesssim fR$ and $fR \lesssim eR$. But R is unit-regular cf [**11**, Theorem 3.2] so $eR \approx fR$. Because of the unit regularity one has [**7**, Corollary 4.23] that eR and fR are perspective in the lattice $L(R)$ of principal right ideals of R. By [**11**, Theorem 2.1(3)] $L(R) = L(gAg)$ so that $e \sim f$ in gAg and so in A.

(iii) Since $xA \approx \mathrm{RP}(x)A$ and $xA \leq \mathrm{LP}(x)A$, we see that $\mathrm{RP}(x) \hookrightarrow \mathrm{LP}(x)$, similarly $\mathrm{LP}(x) \hookrightarrow \mathrm{RP}(x)$. By (i) $\mathrm{RP}(x)$ is finite and then from (ii) we get $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ and $xA \approx x^*A$. □

LEMMA 3.4. *Let e be a finite projection in a Rickart C^*-algebra A. If $x \in A$ is such that xx^* and e commute then*

$$xA \cap eA \approx exx^*A \approx e(xx^*)^{1/2}A.$$

Proof. Since $\mathrm{LP}(ex) \leq e$ we see that $\mathrm{LP}(ex)$ is a finite projection. By Lemma 3.3 (iii) $exA \approx x^*eA$. Since $r(x^*e) = r(xx^*e)$ and xx^* commutes with e we have $xA \cap eA \subseteq exA \approx exx^*A \subseteq xA \cap eA$. By Lemma 3.3(i), $xA \cap eA$ and exx^*A are directly finite right A-modules. Moreover, left multiplication by x induces an epimorphism from $r((1-e)x)$ to $xA \cap eA$, then $xA \cap eA$ is a cyclic right ideal and so projective. Thus by Lemma 3.2 (ii) $exx^*A \approx xA \cap eA$. Since $r(exx^*) = r(e(xx^*)^{1/2})$ left multiplication by $(xx^*)^{1/2}$ gives $(exx^*)A \approx e(xx^*)^{1/2}A$. □

Notice that if A has polar decomposition, then by [**3**, Proposition 4.21.3] $xA = (xx^*)^{1/2}A$ for every x in A. Thus in this case the preceding lemma is obvious. It is not known whether Rickart C^*-algebras have polar decomposition cf. [**3**, Chapter 4 §21 Exercise 10D]. In fact we have the following result noted by Handelman.

LEMMA 3.5 (*Handelman*). *A semihereditary Rickart C^*-algebra has polar decomposition.*

Proof. If A is a semihereditary Rickart C^*-algebra, then $M_2(A)$ is also a Rickart C^*-algebra cf. [**9**, Theorem 7.4, Proof of Proposition 19.1 (b)]. Now $M_2(A)$ contains two orthogonal copies of A and by using the same techniques than in the proof of [**3**, Proposition 4.20.2] we see that partial isometries are $\aleph_0$-addable in A. But then, as it is noted in [**3**, p. 276 Exercise 11 (ii)] A has polar decomposition. □

LEMMA 3.6. *Let A be a Rickart C^*-algebra and let I be an ideal of A. If x is an element of A then the following are equivalent*

(i) $x \in \mathscr{F}(I, A)$

(ii) *There exist a positive unit γ and projections e, f in I such that*

$$e\gamma xx^*\gamma = \gamma xx^*\gamma e$$

$$(1 - e)\gamma xx^*\gamma(1 - e) = 1 - e$$

$$x^*\gamma(1 - e)\gamma x = 1 - f.$$

(iii) *There exist projections f, g in I such that $1 - f \in x^*A$ and $1 - g \in xA$.*

Moreover, if either $I \subseteq \mathscr{K}$ or A is semihereditary, then for any pair of projections e, f satisfying (ii) *we have $r(x^*) \oplus fA \approx r(x) \oplus eA$.*

Proof. (i) $\Rightarrow$ (ii). If $x \in \mathscr{F}(I, A)$ then $xA + zA = A$ for some $z \in I$ and, since A is a C^*-algebra, $xx^* + zz^*$ is a unit. By [3, Proposition 1.8.4], for a given $\varepsilon > 0$, there exists a projection $p \in zz^*A$ with $\|zz^* - pzz^*\| < \varepsilon$. Thus we can choose p such that $xx^* + pzz^*$ is a unit. But then $xA + pA = A$, say $xx^* + p = (\gamma^{-1})^2$ where $\gamma = \gamma^*$ is a unit. Define $e = \mathrm{LP}(\gamma p\gamma)$, since $\gamma p \gamma$ is positive $e = \mathrm{RP}(\gamma p\gamma)$, moreover $e \in I$. Since $\gamma xx^*\gamma + \gamma p\gamma = 1$ we see that e commutes with $\gamma xx^*\gamma$. By multiplying the latter relation by $1 - e$ we get $(1 - e)\gamma xx^*\gamma(1 - e) = 1 - e$. Therefore $x^*\gamma(1 - e)\gamma x$ is a projection, say $1 - f$. Since $x \in \mathscr{F}(I, A)$, we see that $f \in I$. The proof is complete.

(ii) $\Rightarrow$ (iii) Since $e\gamma xx^* = \gamma xx^*\gamma e$ and $(1 - e)\gamma xx^*\gamma(1 - e) = 1 - e$, we see that $1 - e \in \gamma xA$, that is $\gamma^{-1}(1 - e)\gamma \in xA$. Now $\gamma^{-1}(1 - e)\gamma A = (1 - g)A$, where g is a projection, and because $e \in I$ we see that $g \in I$ cf. [9, proof of Proposition 19.1 (b)]. Hence $1 - g \in xA$. On the other hand is clear that $1 - f \in x^*A$.

Obviously (iii) implies (i).

Suppose now that e and f are projections satisfying (ii). Since $r(x) = r(\gamma x)$ and $r(x^*) \approx r(x^*\gamma)$ we may assume, without loss of generality, that $\gamma = 1$. Now consider the following exact sequences

$$0 \to r(x) \to r((1 - e)x) \to xA \cap eA \to 0$$

$$0 \to r(x^*) \to r\left((1 - e)(xx^*)^{1/2}\right) \to (xx^*)^{1/2}A \cap eA \to 0$$

If $I \subseteq \mathscr{K}$, then, by Lemma 3.4, $xA \cap eA \approx (xx^*)^{1/2}A \cap eA$. In the case where A is semihereditary we also have this isomorphism because then A has polar decomposition (Lemma 3.5). Thus in both cases we can apply Schanuel's lemma to get

$$r(x) \oplus r\left((1 - e)xx^*\right) \approx r(x^*) \oplus r\left((1 - e)x\right),$$

now

$$r\big((1-e)xx^*\big) = r\big((1-e)xx^*(1-e)\big) = r\big(x^*(1-e)\big) = eA$$

and $r((1-e)x) = fA$. The proof is complete. □

PROPOSITION 3.7. *Let A be a Rickart C^*-algebra and let I be an ideal of A. If α denotes the composite map*

$$\mathscr{F}(I,A) \to U(A/I) \to K_1(A/I) \xrightarrow{\delta} K_0(I)$$

then we have

(i) *If $I \subseteq \mathscr{K}$ then*

$$\alpha(x) = \big[r(x^*)\big] - \big[r(x)\big]$$

and $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ *for all* $x \in \mathscr{F}(I,A)$.

(ii) *If A is semihereditary, then*

$$\alpha(x) = \big[r(x^*)\big] - \big[r(x)\big] \text{ for all } x \in \mathscr{F}(I,A).$$

Proof. Let β: $\mathscr{F}(I,A) \to K_0(I)$ be the map defined by $\beta(x) = [r(x^*)] - [r(x)]$. Then we must prove that $\beta = \alpha$.

Let $x \in \mathscr{F}(I,A)$. Now let γ, e, f as in Lemma 3.6 (ii). Then we have $\beta(\gamma x) = [r(x^*\gamma)] - [r(\gamma x)] = [r(x^*)] - [r(x)] = \beta(x)$. On the other hand it is clear that $\alpha(\gamma) = 0$ so $\alpha(\gamma x) = \alpha(\gamma) + \alpha(x) = \alpha(x)$. Hence we may assume $\gamma = 1$. For simplicity we shall write $y = (1-e)x$, then we have

$$yy^* = 1 - e$$

$$y^*y = 1 - f.$$

It follows from Lemma 0.2 and the remarks preceding it that

$$\begin{aligned} \alpha(y) &= [(0,e)D] - [(0,f)D] \in K_0(I,A) \\ &= [eA] - [fA] \in K_0(I). \end{aligned}$$

Hence

$$\alpha(x) = \alpha(y) = [eA] - [fA] = \big[r(x^*)\big] - \big[r(x)\big] = \beta(x).$$

Suppose now $I \subseteq \mathscr{K}$. Then

$$1 - e = \mathrm{LP}(y) = \mathrm{LP}((1-e)x) \sim 1 - f = \mathrm{RP}(y) = \mathrm{RP}((1-e)x)$$

and, by Lemma 3.3 (iii), we obtain

$$e \geq \mathrm{LP}(ex) \sim \mathrm{RP}(ex).$$

Since $exx^* = xx^*e$ we then get

$$\mathrm{LP}(x) = \mathrm{LP}((1-e)x) + \mathrm{LP}(ex) \sim \mathrm{RP}((1-e)x) + \mathrm{RP}(ex) \lesssim \mathrm{RP}(x),$$

so $\mathrm{LP}(x) \lesssim \mathrm{RP}(x)$, for all $x \in \mathscr{F}(I, A)$. By symmetry $\mathrm{RP}(x) \lesssim \mathrm{LP}(x)$. Now it follows from the generalized Schröder-Bernstein theorem that $\mathrm{RP}(x) \sim \mathrm{LP}(x)$. □

COROLLARY 3.8. *If A is a semihereditary Rickart C^*-algebra and I is an ideal of A, then the connecting map*

$$\delta\colon K_1(A/I) \to K_0(I)$$

is defined by

$$\delta(\overline{X}) = [r(x^*)] - [r(X)]$$

where X is any matrix over A such that modulo I is an invertible matrix representing $\overline{X} \in K_1(A/I)$.

Proof. Since A is semihereditary, matrix rings over A are also semihereditary Rickart C^*-algebras. The result follows, by using matrices, as in the proof of Proposition 3.7 (ii). □

THEOREM 3.9. *Let A be a Rickart C^*-algebra and let I be a closed ideal in A consisting of compact elements. Then*

(i) *Let $\pi\colon \mathscr{F}(I, A) \to U(A/I)$ be the natrual surjection and let λ be the composite map*

$$U(A/I) \to K_1(A/I) \xrightarrow{\delta} K_0(I).$$

Denote by $U(A/I)^0$ the connected component of $1 \in U(A/I)$. Then

$$U(A/I)^0 = \pi(U(A)) = \ker\lambda.$$

(ii) *If $K_0(I)$ is considered as a discrete group, then the map*

$$\alpha\colon \mathscr{F}(I, A) \to K_0(I)$$

$$x \to [r(x^*)] - [r(x)]$$

is a continuous monoid homomorphism.

(iii) *$\alpha(\mathscr{F}(I, A))$ consists of those elements $z \in K_0(I)$ such that $z = [eA] - [fA]$ where e and f are projections in I with $1 - e \sim 1 - f$. Moreover, two projections e, f in I satisfy $[eA] = [fA] \in K_0(I)$ if and only if $e \sim f$.*

(iv) $x,\ y \in \mathscr{F}(I, A)$ *lie in the same connected component if and only if* $\alpha(x) = \alpha(y)$. *Further* α *induces a group isomorphism*

$$U(A/I)/U(A/I)^0 \approx \alpha(\mathscr{F}(I, A))$$

(v) $\alpha(x) = 0$ *if and only if* $\mathrm{LP}(x)$ *and* $\mathrm{RP}(x)$ *are unitary equivalent.*
(vi) $\alpha(x) = 0$ *if and only if* $x + I$ *contains a unit.*

Proof. Consider any $x \in \mathscr{F}(I, A)$. Say $eA = r(x^*)$ and $fA = r(x)$, where e and f are projections which belong to I. By Proposition 3.7 (i) $1 - e = \mathrm{LP}(x) \sim \mathrm{RP}(x) = 1 - f$. Conversely let $z = [eA] - [fA]$ with $1 - e \sim 1 - f$. Suppose $x \in A$ is such that $xx^* = 1 - e,\ x^*x = 1 - f$. Certainly $x \in \mathscr{F}(I, A)$ and $r(x^*) = eA$, $r(x) = fA$. Therefore $\alpha(x) = z$.

Suppose now $[eA] = [fA] \in K_0(I)$. If for each projection g we write $A_g = gAg + \mathbf{C}$, then $I + \mathbf{C}$ is the C^*-direct limit of the A_g's for g in I. By [**9**, Theorem 19.9] $K_0(I + \mathbf{C} \cdot 1) = \text{dir.lim. } K_0(gAg + \mathbf{C})$, so $K_0(I) = \text{dir.lim. } K_0(gAg)$. By Proposition 3.1 $K_0(gAg) = G(gAg)$, then there exists a projection g in I with $e,\ f \leq g$ and a finitely generated projective A_g-module C such that $eA_g \oplus C \approx fA_g \oplus C$.

Since A_g has stable range 1, C cancels from the direct sums and so $eA_g \approx fA_g$. Therefore $e \sim f$. Thus (iii) follows.

(i) Now we compute $\operatorname{Ker}\lambda$. If $x \in F(I, A)$ then we shall denote $\pi(x)$ by $\bar{x}$. Note that $\pi(U(A)) \subset \operatorname{Ker}\lambda$. Conversely, if $\lambda(\bar{x}) = 0$, then by (iii) $r(x^*) \approx r(x)$ and with the notation of Lemma 3.6 we have

$$(1 - e)\gamma xx^*\gamma(1 - e) = 1 - e$$

$$x^*\gamma(1 - e)\gamma x = 1 - f$$

and $e \sim f$. Let u be a unitary such that $f = ueu^*$. Then it is easily seen that

$$\big((1 - e)\gamma x + u^*f\big)\big(x^*\gamma(1 - e) + fu\big) = 1$$

$$\big(x^*\gamma(1 - e) + fu\big)\big((1 - e)\gamma x + u^*f\big) = 1,$$

so $(1 - e)\gamma x + u^*f \in U(A)$. Hence $\gamma x - (e\gamma x + u^*f) \in U(A)$. Putting $i = \gamma^{-1}(e\gamma x + u^*f) \in I$ we have that $x - i \in U(A)$ and so $\bar{x} \in \pi(U(A))$.

Since the unit group of a Rickart C^*-algebra is connected $\pi(U(A))$ also is. If we prove that $\pi(U(A))$ is open, then it is clear that $\pi(U(A)) = U(A/I)^0$. For this let $\bar{u} \in \pi(U(A))$ such that $\|\bar{u} - \bar{1}\| < 1$. This means that $\inf_{i \in I}\|(u + i) - 1\| < 1$. Thus there exists $i \in I$ with $\|(u + i) - 1\| < 1$, then $u + i$ is a unit and therefore $\bar{u} \in \pi(U(A))$.

By Proposition 3.7 (i) $\alpha = \lambda\pi$. So (ii) and the isomorphism $U(A/I)U(A/I)^0 \approx \alpha(F(I, A))$ of (iv) follow. In order to end the proof of (iv) note that $\alpha(x) = \alpha(y)$ if and only if $\bar{x}$ and $\bar{y}$ lie in the same connected component of $U(A/I)$. Since the map π is open and onto the result follows.

(v) Suppose $\alpha(x) = 0$, then by (iii) $r(x) \approx r(x^*)$ and since $\mathrm{LP}(x) \sim \mathrm{RP}(x)$ we see that $\mathrm{LP}(x)$ and $\mathrm{RP}(x)$ are unitary equivalent.

(vi) By (i) it is clear that $\alpha(x) = 0$ if and only if $\bar{x} \in \pi(U(A))$. So $\alpha(x) = 0$ if and only if $x + I$ contains a unit. □

LEMMA 3.10. *Let M be a 2×2 matrix over a ring R. If for some entry a in M there exist b, c in R such that $bac = 1$, then M can be reduced by elementary transformations to a diagonal matrix.*

Proof. There is no loss of generality in assuming that M is of the form

$$M = \begin{pmatrix} * & a \\ * & * \end{pmatrix}$$

and $bac = 1$. Now notice that the matrices

$$P = \begin{pmatrix} b & 0 \\ 1 - acb & ac \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} ba & 0 \\ 1 - cba & c \end{pmatrix}$$

belong to $GE_2(R)$. But then we have that PMQ is of the form

$$\begin{pmatrix} * & 1 \\ * & * \end{pmatrix},$$

since this matrix can be reduced to a diagonal one, the same holds for M. □

PROPOSITION 3.11. *Let A be a Banach algebra satisfying the following condition*:

For each $a \in A$ and $\varepsilon > 0$ there exists an idempotent $e \in aA$ and a central idempotent $h \in A$ such that

(a) $\|a - ea\| < \varepsilon$

(b) $he \sim h$ *and* $(1 - h)(1 - e) \sim (1 - h)$. *Then A is a GE_2-ring.*

Proof. For any Banach algebra [**16**, Proposition 8.7] we have $\mathrm{GL}_2(A)^0 \subseteq GE_2(A)$. Hence $GE_2(A)$ is clopen. In order to prove that $GE_2(A) = \mathrm{GL}_2(A)$ it suffices to note that $GE_2(A)$ is a dense subset of $\mathrm{GL}_2(A)$. For this let

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A)$$

and $\varepsilon > 0$. Choose an integer n such that $n > 1/\varepsilon$, $\|X^{-1}\|$. By hypothesis there exist idempotents e and h, with h central, such that

$$\|a - ea\| < 1/n$$

and

$$he \sim h \quad \text{while } (1-h)(1-e) \sim 1-h.$$

Consider now the matrix

$$M = \begin{pmatrix} ea & b \\ c & d \end{pmatrix}.$$

Then we have $\|X - M\| < 1/n < 1/\|X^{-1}\|$. Therefore $M \in \mathrm{GL}_2(A)$. We claim that $M \in GE_2(A)$. Since h induces a ring decomposition of A, by cutting down to each part we may assume that either (i) $e \sim 1$ or (ii) $1 - e \sim 1$. In the first case there exist x, $y \in A$ such that $xey = 1$ and since $e = az$, for some $z \in A$, we have $x(ea)zy = 1$. It follows from Lemma 3.10 that $M \in GE_2(A)$.

Now suppose that $1 - e \sim 1$. From the relation $eA + bA = A$ we see that $1 - e \in (1-e)bA$. Hence $xby = 1$, for some x, $y \in A$. The result follows again by using Lemma 3.10. □

We say that a Rickart C^*-algebra A is *purely infinite* if 1 is the supremum of a sequence of orthogonal projections all equivalent to 1. It is a simple exercise to see that A is purely infinite if and only if $A \approx A^2$ as right A-modules.

LEMMA 3.12 (*Pere Ara*). *Let A be a purely infinite Rickart C^*-algebra satisfying general comparability. Suppose e is a projection such that $e \sim 1 - e$, then $e \sim 1$.*

Proof. Denote by $\vee$ and $\wedge$ the operations of taking supremum and infimum respectively. Since A is purely infinite choose a projection f such that $f \sim 1 - f \sim 1$. Define

$$g = (1-e) \wedge f$$
$$h = LP(ef) \qquad (= (1-e) \vee f - (1-e)).$$

Since A satisfies the parallelogram law [**3**, Theorem 2.13.1]

$$\begin{aligned} hA \oplus gA &= ((1-e) \vee f - (1-e))A \oplus ((1-e) \wedge f)A \\ &\approx (f - (1-e) \wedge f)A \oplus ((1-e) \wedge f)A \\ &= fA. \end{aligned} \tag{1}$$

Since $h < e$ and $g < 1 - e$ we see that $e - h$ and $(1 - e) - g$ are orthogonal projections, we have

$$\begin{aligned}(2)\qquad (e-h)A &\oplus((1-e)-g)A\\ &= (e-h)A \oplus((1-e)-(1-e)\wedge f)A\\ &\approx (e-h)A \oplus((1-e)\vee f - f)A\\ &= (e-h)A \oplus(h+1-e-f)A = (1-f)A.\end{aligned}$$

Now we shall prove that $e \sim 1$. Since A satisfies general comparability we may assume that either $g \lesssim e - h$ or $e - h \lesssim g$. In the first case we have (by using (1)) that

$$1 \sim f \sim h + g \lesssim h + (e-h) = e \leq 1,$$

while in the second case we have (by using (2)) that

$$\begin{aligned}1 \sim 1 - f \sim (e-h) + ((1-e)-g) &\lesssim g + ((1-e)-g)\\ = 1 - e \sim e \leq 1.\end{aligned}$$

Thus in both cases we see that $1 \lesssim e \leq 1$. Then the generalized Schröder-Bernstein theorem yields the result. □

THEOREM 3.13. *Let A be a purely infinite Rickart C^*-algebra satisfying general comparability. If I is an ideal of A, then*

(i) $K_1(A/I) = U(A/I)/\pi(U(A)) = U(A/I)^{\mathrm{ab}}$.

(ii) *If I is closed in A, then*

$$\pi(U(A)) = U(A/I)^0.$$

(iii) *A/I is a GE-ring.*

Proof. (i) Since $A^2 \approx A$ we have $(A/I)^2 \approx A/I$ as A/I-modules. In order to prove that $K_1(A/I) = U(A/I)^{\mathrm{ab}}$ it suffices to show cf. [**13**, Theorem 1.2 (iii)] that A/I is a GE_2-ring. In proving this we first assume that I is closed. By noting that the hypotheses in Proposition 3.11 carry over algebra Banach factors, it suffices to verify that the algebra A satisfies (a) and (b) of that proposition. Obviously (a) is an immediate consequence of the spectral theorem [**3**, Proposition 1.8.4]. For (b), let e be an idempotent in A. By general comparability there exists a central idempotent h such that $h(1-e) \lesssim he(1)$ and $(1-h)e \lesssim (1-h)(1-e)$ (2). From the relation (1) we have $hA \lesssim (heA)^2$. Since A is purely infinite we have also $(heA)^2 \lesssim hA$. So $hA \approx (heA)^2$ and we can write $hA = e_1A \oplus e_2A$ for some projections e_1, $e_2 \in hA$ such that $e_1 \sim e_2 \sim he$. Then $e_1 \sim h - e_1$ and Lemma 3.12 yields $e_1 \sim h$ so $he \sim h$. Using the relation (2) we have $(1-h)(1-e) \sim 1 - h$. Thus we have shown that A/I is a GE_2-ring for any closed ideal I of A. Now assume I is an arbitrary ideal of A. Let $M \in M_2(A)$ such that M is a unit modulo I. If $\bar{I}$ denotes the closure of I in A, then M is a unit modulo $\bar{I}$ and by the above we may assume, by using elementary transformations, that M is of the form

$$\begin{pmatrix} u & 0 \\ 0 & * \end{pmatrix}$$

where $u + \bar{I}$ is a unit of $A/\bar{I}$. It is easily seen that $u + I$ must be a unit of A/I. Now by elementary transformations we can reduce M modulo I to obtain a diagonal matrix. Thus A/I is a GE_2-ring. If A is a purely infinite Rickart C^*-algebra then $A \approx M_2(A)$ and so A is semihereditary. In particular, by Lemma 3.5, A has polar decomposition.

Now by using that $U(A)$ is a perfect group [**13**, proof of Theorem 2.10] we can proceed as in the proof of Lemma 1.7 to get $\pi(U(A)) = U(A/I)'$ and so (i) follows.

(ii) Since $U(A)$ is connected also is $\pi(U(A))$. As in the proof of Theorem 3.9 we can prove that $\pi(U(A))$ is clopen in $U(A/I)$, so $\pi(U(A)) = U(A/I)^0$.

(iii) Notice that if $I = 0$, then the result follows from [**13**, Proof of Theorem 2.10] or [**16**, Theorem 2.10]. Fix $n > 1$. Since A is purely infinite $A \approx M_n(A)$. By applying (i) to π: $M_n(A) \to M_n(A/I)$ we obtain $\pi(GE_n(A)) = \mathrm{GL}_n(A/I)'$ and so $\mathrm{GL}_n(A/I)' \subseteq GE_n(A/I)$.

Let $M \in \mathrm{GL}_n(A/I)$. Since $U(A/I) \to K_1(A/I)$ is onto, there exists a unit $u \in U(A/I)$ such that

$$M\begin{pmatrix} u & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = 0 \in K_1(A/I).$$

But $K_1(A/I) = U(A/I)^{\mathrm{ab}}$ implies

$$M\begin{pmatrix} u & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in \mathrm{GL}_n(A/I)',$$

and by the above we have that $M \in GE_n(A/I)$ as desired. □

THEOREM 3.14. *Let A be a purely infinite Rickart C^*-algebra satisfying general comparability. If I is a closed ideal of A, then*

(i) *The map*

$$\alpha\colon \mathcal{F}(I, A) \to K_0(I), \qquad x \mapsto [r(x^*)] - [r(x)]$$

is a continuous monoid homomorphism which is onto

(ii) $[r(x^*)] = [r(x)]$ *if and only if there exists a projection $e \in I$ such that*

$$r(x^*) \oplus eA \approx r(x) \oplus eA.$$

(iii) $x, y \in \mathscr{F}(I, A)$ *lie in the same connected component if and only if* $\alpha(x) = \alpha(y)$. *Furthermore* α *induces a group isomorphism*

$$K_1(A/I) = U(A/I)/U(A/I)^0 \xrightarrow{\approx} K_0(I).$$

(iv) $\alpha(x) = 0$ *if and only if* $x + I$ *contains a unit.*

Proof. By Proposition 3.7 (ii) we see that α is a well-defined monoid homomorphism. Since A is purely infinite we have [**13**, Theorem 2.7 (ii) and the proof of Theorem 2.10] that $K_1(A) = 0$. Clearly $K_0(A) = 0$. Therefore the connecting map δ: $K_1(A/I) \to K_0(I)$ is an isomorphism, in particular α is onto. By Theorem 3.13 $K_1(A/I) = U(A/I)/U(A/I)^0$ so α is continuous. Thus we have shown (i) and a part of (iii). The remainder part of (iii) follows as in Theorem 3.9 (iv).

By Theorem 3.13, (iv) follows.

Now (ii) follows from Proposition 3.1. □

If A is an AW^*-algebra, then A decomposes uniquely as a direct product $A_1 \times A_2$ where A_1 is directly finite and A_2 is purely infinite. Now A_1 is a ring with stable range 1 so the connecting map associated with each ideal of A_1 is zero. Therefore we see that Theorem 3.14 is trivially true for A_1. Since any AW^*-algebra satisfies general comparability, Theorem 3.14 also holds for A_2. Thus we have

COROLLARY 3.15. *The conclusions of Theorem* 3.14 *are true for any closed ideal of an* AW^**-algebra.* □

Finally we remark the following result which is an extension of Corollary 10.7 in [**15**] to AW^*-algebras.

COROLLARY 3.16. *If* I *is an ideal of a* AW^**-algebra* A *of Type* III, *then every unit of* A/I *can be lifted to a unit of* A. *If in addition* I *is closed, then* $U(A/I)$ *is connected.*

Proof. Let $\bar{I}$ be the closure of I in A. Then since a unit in $A/\bar{I}$ lifts automatically to a unit of A/I, we may assume without loss of generality that I is closed. Since $(eA)^2 \approx eA$ for every idempotent e in I we see from Proposition 3.1 that $K_0(I) = 0$. By Theorem 3.14 (iii) $U(A/I) = U(A/I)^0$ is connected; and by Theorem 3.13 (i) we get $\pi(U(A)) = U(A/I)$. □

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