LIFTING UNITS IN SELF-INJECTIVE RINGS AND AN INDEX THEORY FOR RICKART C*-ALGEBRAS

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In this paper we study the following question: If $R$ is a right self-injective ring and $I$ an ideal of $R$, when can the units of $R/I$ be lifted to units of $R$?

We answer this question in terms of $K_0(I)$. For a purely infinite regular right self-injective ring $R$ we obtain an isomorphism between $K_1(R/I)$ and $K_0(I)$ which can be viewed as an analogue of the index map for Fredholm operators.

By giving a purely algebraic description of the connecting map $K_1(A/I) \to K_0(I)$ in the case where $A$ is a Rickart $C^*$-algebra, we are able to extend the classical index theory to Rickart $C^*$-algebras in a way which also includes Breuer's theory for $W^*$-algebras.

0. Preliminary results. Throughout this paper $R$ will denote an associative ring with 1. By a *rng* we mean a ring which does not necessarily have a 1.

We write $M_n(R)$ for the ring of all $n \times n$ matrices over $R$, and $\text{GL}_n(R)$ for the group of units of $M_n(R)$, though we shall write $U(R)$ rather than $\text{GL}_1(R)$. For $1 \leq i, j \leq n$ let $e_{ij} \in M_n(R)$ be the usual matrix units. Define $E_n(R)$ to be the subgroup of $\text{GL}_n(R)$ generated by all the matrices of the form $1 + re_{ij}$, $r \in R$, $i \neq j$; and $GE_n(R)$ to be the subgroup of $\text{GL}_n(R)$ generated by $E_n(R)$ together with the subgroup $D_n(R)$ of all invertible diagonal matrices. If $GE_n(R) = \text{GL}_n(R)$, then we say that $R$ is a $GE_n$-ring; if $R$ is a $GE_n$-ring for all $n > 1$ then $R$ is said to be a $GE$-ring.

If $R$ is a $GE_n$-ring, then $E_n(R)$ is a normal subgroup of $\text{GL}_n(R)$ and hence $\text{GL}_n(R) = D_n(R)E_n(R)$.

Let $\text{GL}(R)$ denote the direct limit of the directed system $U(R) \to \text{GL}_2(R) \to \text{GL}_3(R) \to \cdots$ where each $a \in \text{GL}_n(R)$ is mapped to

\[
\begin{pmatrix}
a & 0 \\
0 & 1
\end{pmatrix}
\]
in $GL_{n+1}(R)$. Then $K_1(R)$ is defined to be $GL(R)^{ab}$, that is $GL(R)$ abelianized.

Note that the canonical map $U(R) \to K_1(R)$ is onto in the case where $R$ is a $GE$-ring.

Let $I$ be a rng and $R$ a ring containing $I$ as an ideal. Let $P(I)$ denote the class of all finitely generated projective right $R$-modules $A$ such that $AI = A$. We say that $A, B \in P(I)$ are equivalent if $A \oplus C \cong B \oplus C$ for some $C \in P(I)$. Denote by $[A]$ the equivalence class of $A \in P(I)$. Thus the set $\{[A] | A \in P(I)\}$ with the operation $[A] + [B] = [A \oplus B]$ is a cancellative abelian semigroup. We write $G(I)$ for its associated universal abelian group. Then every element of $G(I)$ has the form $[A] - [B]$ for suitable $A, B \in P(I)$. Denote by $[A]$ the equivalence class of $A \in P(I)$.

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It is not difficult to show that $P(I)$ consists of all $R$-modules $A$ such that $A \cong e(R^n)$ for some idempotent $n \times n$ matrix $e$ with entries in $I$. Thus, we see that $G(I)$ depends only on the structure of the rng $I$ and not on the involving ring $R$. Note that $G$ is a functor from the category of rngs into the category of abelian groups such that preserves direct limits.

For a ring $R$, $G(R)$ is simply $K_0(R)$. Recall that Bass and Milnor have defined a functor $K_0$ on the category of rngs; following Milnor [14, §4], we consider any ring $R$ containing $I$ as an ideal, let $\pi : R \to R/I$ be the natural surjection, and form the pullback

$$
\begin{array}{ccc}
D(R) & \xrightarrow{p_2} & R \\
\downarrow p_1 & & \downarrow \pi \\
R & \xrightarrow{\pi} & R/I.
\end{array}
$$

Then $K_0(I, R)$ is defined as the kernel of $K_0(p_1) : K_0(D(R)) \to K_0(R)$. In [2] it is proved that $K_0(I, R)$ depends only on $I$. Furthermore, there is an exact sequence, cf. [14, §4]:

$$K_1(R) \to K_1(R/I) \xrightarrow{\delta} K_0(I, R) \to K_0(R) \to K_0(R/I).$$

Let $I$ be a rng that is an $F$-algebra, where $F$ is either $\mathbb{Z}$ or a commutative field. Consider $I^1 = I \oplus F$, the unitification of $I$ by $F$; by applying the above exact sequence we obtain

$$K_0(I, I^1) = \text{Ker}(K_0(I^1) \to K_0(F)).$$

When we write $K_0(I)$ we will have $K_0(I, I^1)$ in mind.
If $I$ is a ring with unit $e$, then there is a ring decomposition $I^1 = I \times (1 - e)F$. Therefore $K_0(I^1) = K_0(I) \oplus K_0(F)$ and so $K_0(I, I^1) = K_0(I)$. Hence we see that $K_0(I)$ agrees with the corresponding $K_0$ of $I$, where $I$ is viewed as a ring.

Let $I$ be a rng. With each $A \in P(I)$ we can associate its class in $K_0(I)$. In this way we obtain a group homomorphism $\phi: G(I) \to K_0(I)$. In the case where $\phi$ is an isomorphism we shall write $G(I) = K_0(I)$. When this occurs there is a very simple form for the elements in $K_0(I, R)$. More precisely, if $A \in P(I)$, then $0 \times A$ is a projective $D(R)$-module, and one easily obtains a group isomorphism

$$K_0(I) = G(I) \to K_0(I, R)$$

in which $[A] \mapsto [0 \times A]$.

In general we do not know whether $K_0(I) = G(I)$ but the following easy result will be enough for our purposes.

**PROPOSITION 0.1.** *Let $I$ be an ideal of an $F$-algebra $R$, where $F$ is either $\mathbb{Z}$ or a commutative field. Suppose there exists a set $E$ of idempotents of $I$ such that for each pair $e, f \in E$ there exists $g \in E$ such that $eRe + fRf \subseteq gRg$, so the subrings $eRe + F \cdot 1$ form a directed system. If the induced map*

$$\lim_{\to} K_0(eRe + F1) \to K_0(I + F1)$$

*is a group isomorphism then $K_0(I) = G(I)$.***

**Proof.** There is an obvious commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \lim_{\to} K_0(eRe) & \to & \lim_{\to} K_0(eRe + F1) & \to & K_0(F) & \to & 0 \\
& & \downarrow \alpha & & \downarrow & & \| \\
0 & \to & K_0(I) & \to & K_0(I + F1) & \to & K_0(F) & \to & 0
\end{array}
$$

with exact rows, and by hypothesis the middle column is an isomorphism so $\alpha$ is also. On the other hand $G$ preserves direct limits, so we have a map $\beta: \lim_{\to} e \in E G(eRe) \to G(I)$. As $G(eRe) = K_0(eRe)$ for all $e \in E$, it follows that $\beta^{-1}: K_0(I) \to G(I)$ provides an inverse for $\phi$. Therefore $K_0(I) = G(I)$. \qed

We shall need another result. First recall Milnor's definition of the connecting map $\delta: K_1(R/I) \to K_0(I, R)$. Consider any element $\mu$ of $K_1(R/I)$; it lies in the image of $GL_n(R/I)$ for some $n$ and so can be
represented as the image of a matrix \( u \in M_n(R) \) for which there exists \( v \in M_n(R) \) such that the elements \( i = uv - 1, \; j = vu - 1 \) lie in \( M_n(I) \). Write

\[
M = \{(x, y) \in R \times R | u(x) - y \in I\}.
\]

In [14, Theorem 2.1] it is proved that \( M \) is a finitely generated projective \( D(R) \)-module. Now \( \delta(\mu) \) is defined as \( [M] - [\pi D] \) and this gives the connecting map. In this situation we have:

**Lemma 0.2** As \( D \)-modules \( \pi D \oplus (0 \times \pi(R)) \approx M \oplus (0 \times \pi(R)) \).

**Proof.** By using the Morita equivalence between \( \text{Mod-} D \) and \( \text{Mod-} M_n(D) \) we see that the claimed isomorphism is equivalent to an \( M_n(D) \)-module isomorphism

\[
M_n(D) \oplus (0, i) M_n(D) \approx \pi M \oplus (0, j) M_n(D).
\]

It is clear that

\[
\pi M \approx \{(x, y) \in M_n(D) \times M_n(D) | ux - y \in M_n(I)\}
\]

This shows that without loss of generality we may assume that \( n = 1 \). Now any element of \( M \) can be expressed in the form

\[
(x, y) = (1, u)(x, vy) - (0, i)(y, y)
\]

so \( M = (1, u)D + (0, i)D \). Now define a \( D \)-module homomorphism

\[
\alpha: D \oplus (0, i)D \rightarrow M, \quad ((x, y), (0, i)d) \mapsto (1, u)(x, y) - (0, i)d.
\]

Clearly \( \alpha \) is onto, and \( \ker \alpha = \{((0, y), (0, iy')) \in D \oplus (0, i)D | uy - iy' = 0\} \). But if \( uy - iy' = 0 \) then from the relation

\[
\begin{pmatrix}
-j & v \\
-u & 1
\end{pmatrix}
\begin{pmatrix}
1 & -v \\
u & -i
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

we obtain

\[
\begin{pmatrix}
-j \\
-u
\end{pmatrix}
(vy' - y) = \begin{pmatrix}
y \\
y'
\end{pmatrix}
\]

So \( \ker \alpha = ((0, j), (0, iu))D \approx (0, j)D \). Since \( M \) is \( D \)-projective, \( \alpha \) splits and the result follows.

\[\Box\]

1. **Regular rings.** Let \( R \) be a ring.

Recall that \( R \) is said to be **regular** if for every \( x \in R \) there exists \( y \in R \) such that \( x = xyx \). An element \( x \) of \( R \) is called **unit-regular in** \( R \) if there exists a unit \( u \) of \( R \) such that \( x = xux \). We say that \( R \) is **unit-regular** if every element in \( R \) is unit-regular.
An ideal \( I \) of \( R \) has stable range 1 if for all \( a, b \in I \), if \((1 + a)R + bR = R \) then there exists \( c \in I \) such that \((1 + a + bc)R = R \); cf. \([17, 18]\). Vasershtein \([17]\) proves that \( I \) having stable range 1 depends only on the rng structure of \( I \), and not on the ambient ring \( R \). Now one can see that for a ring \( R \) the stable range 1 condition is equivalent to saying that for all \( a, b \in R \), \( aR + bR = R \) implies \( a + bc \) is a unit for some \( c \in R \), cf. \([18, \text{Theorem 2.6}]\), \([2, \text{p. 231}]\).

A theorem of Fuchs and Kaplansky \([7, \text{Proposition 4.12}]\) asserts that the unit-regular rings are precisely those regular rings with stable range 1. We shall use Evans' theorem \([7, \text{Proposition 4.13}]\): if the endomorphism ring of a right \( R \)-module \( M \) has stable range 1, then \( M \) can be cancelled from direct sums of right \( R \)-modules, that is, \( M \oplus N \approx M \oplus N' \) for some right \( R \)-modules \( N \) and \( N' \) implies \( N \approx N' \). By \([18, \text{Theorem 2.4, Theorem 3.9}]\) the stable range condition carries over to corners and it is Morita-invariant. Hence, if \( R \) has stable range 1 then all finitely-generated projective right \( R \)-modules cancel from direct sums.

Now we shall give a description for \( K_0(I) \) in the case where \( I \) is an ideal of a regular ring \( R \). In an earlier version of this paper we had obtained such a description in the case where \( R \) is unit-regular, and then Goodearl provided us with the general case.

First we need a more-or-less known lemma.

**Lemma 1.1.** Let \( R \) be a regular ring and let \( e, f \in R \) be idempotents. Then

(i) If \( eR \subseteq fR \), then there exists an idempotent \( g \) in \( R \) with \( gR = fR \) and \( ge = eg = e \).

(ii) Let \( I \) be an ideal of \( R \). If \( e, f \in I \) then there exist idempotents \( g, h \in I \) such that \( eRf \subseteq hRh \) and \( eRe + fRf \subseteq gRg \). Moreover if \( a \in I \) then there exists an idempotent \( k \) in \( I \) such that \( a \in kRk \).

**Proof.** (i) Define \( g = (1 + ef(1 - e))f(1 - ef(1 - e)) \).

(ii) Let \( h \) be an idempotent such that \( eR + fR = hR \). Clearly \( h \in I \) and, by (i), we can choose \( h \) such that \( fh = hf = f \). Then \( eRf \subseteq hRh \).

Let \( c \) and \( d \) be idempotents in \( I \) such that \( eR + fR = cR \) and \( Re + Rf = Rd \). Then \( eRe + fRf \subseteq (eR + fR) \cap (Re + Rf) = cRd \). It follows from the above that \( cRd \subseteq gRg \), for some idempotent \( g \) in \( I \).

If \( a \in I \), then by regularity there exists \( x \in R \) such that \( a = axa \), so \( e = ax \) and \( f = xa \) are idempotents in \( I \) and \( a \in eRf \). Now the result follows from the above.

**Proposition 1.2 (Goodearl).** If \( I \) is an ideal of a regular ring \( R \) then \( G(I) = K_0(I) \).
Proof. Let \( E \) be the set of all idempotents in \( I \). By Lemma 1.1 (ii), \( I + Z \cdot 1 \) is the directed union of the subrings \( eRe + Z \cdot 1 \). Therefore 
\[
\lim_{e \in E} K_0(eRe + Z \cdot 1) = K_0(I + Z \cdot 1).
\]
By Proposition 0.1 the result follows.

Now we shall obtain a tidier expression for the connecting map 
\( \delta : K_1(R/I) \to K_0(I) \) in the case where \( R \) is a regular ring.

If \( a \) is an \( n \times n \) matrix over \( R \) we write \( \text{Ker} \ a \) for the set of elements \( x \in {}^nR \) such that \( a(x) = 0 \). We define \( \text{Coker} \ a \) to be any complement of \( a({}^nR) \) in \( {}^nR \), so \( \text{Coker} \ a \) is determined up to isomorphism.

**Proposition 1.3** (with Goodearl). Let \( R \) be a regular ring and \( I \) an ideal of \( R \). Then the connecting map 
\[
\delta : K_1(R/I) \to K_0(I)
\]
satisfies 
\[
\delta(\bar{a}) = [\text{Coker} \ a] - [\text{Ker} \ a],
\]
where \( a \) is any matrix over \( R \) representing \( \bar{a} \in K_1(R/I) \).

Proof. Suppose \( a \in M_n(R) \). By regularity there exists an \( n \times n \) matrix \( b \) over \( R \) such that \( a = ab \). Since \( a \) is a unit modulo \( I \), we have 
\[
ab - 1 \in M_n(I)
\]
\[
ba - 1 \in M_n(I).
\]
Now \( j({}^nR) = \text{Ker} \ a \) and \( i({}^nR) \oplus a({}^nR) = {}^nR \). With the same notation as in Lemma 0.2 we have 
\[
\delta(\bar{a}) = [M] - [{}^nD] = [0 \times \text{Coker} \ a] - [0 \times \text{Ker} \ a] \in K_0(I, R).
\]
Hence 
\[
\delta(\bar{a}) = [\text{Coker} \ a] - [\text{Ker} \ a] \in K_0(I).
\]

We now use the preceding propositions to obtain some results on lifting units.

**Lemma 1.4.** If \( R \) is a regular ring and \( I \) is an ideal of \( R \) then the following are equivalent

(i) For each idempotent \( e \) in \( I \) the corner ring \( eRe \) is unit-regular.

(ii) \( I + Z \) is a unit-regular subring of \( R \), where \( Z \) is the centre of \( R \).

(iii) \( I \) has stable range 1.

Proof. (i) \( \Rightarrow \) (ii) By Lemma 1.1, \( I + Z \) is the directed union of the subrings \( eRe + Z \), where \( e \) is an idempotent in \( I \). Now \( eRe + Z = eRe \times (1 - e)Z \) is the direct product of two unit-regular rings, so \( eRe + Z \) is unit-regular. Since unit regularity is preserved by taking direct limits we see that \( I + Z \) is unit-regular.
(ii) ⇒ (iii) By hypothesis $I + Z$ is unit-regular and so has stable range 1. It follows from [18, Theorem 3.6 (g)] that $I$ has stable range 1.

(iii) ⇒ (i) Every corner of a rng with stable range 1 also has stable range 1 cf. [18, Theorem 3.9].

It follows from [17, Theorem 4] that the sum of two ideals with stable range 1 has stable range 1. Hence there is a unique largest ideal $R_0$ of $R$ having stable range 1, namely, the sum of all ideals of $R$ with stable range 1.

If an $R$-module $A$ is isomorphic to a direct summand of an $R$-module $B$ then we write $A \leq B$. Two idempotents $e$ and $f$ of $R$ are said to be isomorphic if the modules $A = eR$, $B = fR$ are isomorphic. The notations $e \leq f$ and $e \leq f$ mean $eR \subseteq fR$ and $eR \leq fR$ respectively.

**Lemma 1.5.** If $R$ is a regular ring then $R_0$ coincides with the ideal $I$ generated by all idempotents of $R$ whose corner is unit-regular.

**Proof.** By Lemma 1.1 any ideal of $R$ is the directed union of its corners, so by Lemma 1.4 (i) ⇒ (iii), we see that $R_0 \subseteq I$.

Conversely, if $e$ is an idempotent in $I$ then $e = \sum x_ie_i y_i$, where $x_i, y_i \in R$ and the $e_i$'s are idempotents with $e_i Re_i$ unit-regular. From the $R$-linear map $\oplus e_i R \rightarrow R$, $\sum e_i r_i \mapsto \sum x_i e_i r_i$ we see that $eR \leq \oplus e_i R$. It follows from [12, Corollary 10(ii)] that the endomorphism ring of the $R$-module $\oplus e_i R$ has stable range 1 and since $eRe$ is a corner of this endomorphism ring it also has stable range 1. \hfill \Box

If $I$ is an ideal of $R$ write $\bar{x}$ for $x + I \in R/I$ and denote by $\pi$ the natural projection $R \rightarrow R/I$.

**Proposition 1.6.** Let $R$ be a regular ring and $I$ an ideal of $R$ with stable range 1, then the map

$$\alpha: U(R/I) \rightarrow K_0(I), \quad \bar{a} \mapsto [\text{Coker } a] - [\text{Ker } a],$$

is a group homomorphism. Moreover

$$\text{Ker } \alpha = \pi(U(R)) = \{ \bar{a} \in U(R/I) : a \text{ is unit-regular} \}.$$

**Proof.** By Proposition 1.3 we see that $\alpha$ is the composition of the maps $U(R/I) \rightarrow K_1(R/I)$ and $\delta: K_1(R/I) \rightarrow K_0(I)$ and so it is a group homomorphism.

If $Z$ is the centre of $R$ then $K_0(I)$ is a subgroup of $K_0(I + Z)$. Notice that $\bar{a}$ lies in $\text{Ker } \alpha$ if and only if $[\text{Coker } a] = [\text{Ker } a]$ in $K_0(I + Z)$. 

Since $I$ and so $I + Z$ has stable range 1, we have that $\bar{a} \in \ker \alpha$ if and only if $\coker a \cong \ker a$ and this occurs if and only if $a$ is unit-regular, cf. [7, Proof of Theorem 4.1].

Conversely, let $\bar{a} \in \ker \alpha$, If $a$ is a representative in $R$ for $\bar{a}$, then $a = au\bar{a}$ for some unit $u$ in $R$. Now since $\bar{a} \in U(R/I)$, $(\bar{a} - \bar{u}^{-1})\bar{u} = 0$ so $\bar{a} = \bar{u}^{-1}$ and $\bar{a}$ belongs to $\pi(U/R))$.

Now we consider regular right self-injective rings. The reader is referred to [7] for background. We mention, however, that every regular right self-injective ring can be uniquely expressed as a direct product of a unit-regular ring and a purely infinite regular ring (recall that an idempotent $e$ of a ring $R$ is said to be purely infinite if $(eR) \cong (eR)^2$, so $R$ is a purely infinite regular right self-injective ring if 1 is a purely infinite idempotent in $R$).

**Lemma 1.7.** If $R$ is a purely infinite regular right self-injective ring and $I$ is an ideal of $R$, then $\pi(U(R)) = U(R/I)^\prime$.

**Proof.** By [13, Corollary 2.8] $U(R)$ is a perfect group. Hence $\pi(U(R)) \subseteq U(R/I)^\prime$.

Conversely, take $u$ in the commutator group $U(R/I)^\prime$. Since $R \cong R^2$ there exist matrices $X \in R^2$ and $Y \in R^2$ such that $XY = 1$ and $YX = I_2$. Then $\bar{Y}u\bar{X}$ is a $2 \times 2$ invertible matrix. By [13, Theorem 2.2] $\bar{Y}u\bar{X} \in E_2(R/I)$, hence there exists $Z \in GL_2(R)$ such that $\bar{Z} = \bar{Y}u\bar{X}$. Therefore $v = XZ\bar{Y}$ is a unit of $R$ with $v = u$. The result follows. $\square$

If $e$ is an idempotent of a regular right self-injective ring, then we denote by $cc(e)$ its central cover, that is, the minimum central idempotent such that $cc(e)e = e$.

**Proposition 1.8.** Let $R$ be a purely infinite regular right self-injective ring and $I$ an ideal. If $A, B \in P(I)$, then

(i) $[A] = [B] \in K_0(I)$ if and only if there exists a purely infinite idempotent $e$ in $I$ such that $A \oplus eR \cong B \oplus eR$.

(ii) (with Goodearl) $K_0(I) = 0$ if and only if every idempotent in $I$ is sub-isomorphic to a purely infinite idempotent in $I$.

(iii) $[A] = [B] \in K_0(I)$ if and only there exists a purely infinite idempotent $e$ in $I$ such that $A \oplus cc(e)R \cong B \oplus cc(e)R$.

(iv) $[A] = [B] \in K_0(I)$ if and only if there exists a purely infinite idempotent $e$ in $I$ such that $(1 - cc(e))A \cong (1 - cc(e))B$. 
Proof. (i) By Proposition 1.2, $K_0(I) = G(I)$. Thus $[A] = [B]$ if and only if $A \oplus C \simeq B \oplus C$ for some $C \in P(I)$. It follows from [7, Theorem 10.32] that $C$ can be written as $C_1 \oplus C_2$, where $C_2$ is purely infinite and the endomorphism ring of $C_1$ has stable range 1. But then $C_1$ cancels from direct sums and we have $A \oplus C_2 \simeq B \oplus C_2$. Since $R \simeq R^2$, $C_2$ is cyclic and so $C_2 \simeq eR$, for some purely infinite idempotent $e$ in $I$.

(ii) Since $R$ is purely infinite, we see that every finitely generated right $R$-module is cyclic.

Suppose $K_0(I) = 0$ and let $e$ be an idempotent in $I$. By (i) there exists a purely infinite idempotent $f$ in $I$ such that $eR \oplus fR \simeq fR$. Thus $eR \leq fR$ as desired.

Conversely, let $e$ be an idempotent in $I$. By hypothesis $eR \leq fR$ for some purely infinite idempotent $f$ in $I$. Then, since $fR \leq eR \oplus fR \leq (fR)^2$, by [7, Theorem 10.14] we have $eR \oplus fR \simeq fR$. So $[eR] = 0$ in $K_0(I)$.

(iii) Suppose $[A] = [B] \in K_0(I)$. By (i), $A \oplus eR \simeq B \oplus eR$ for some purely infinite idempotent $e$ in $I$ and a fortiori $A \oplus cc(e)R \simeq B \oplus cc(e)R$.

Conversely, if $A \oplus cc(e)R \simeq B \oplus cc(e)R$ then we have $(1 - cc(e))A \simeq (1 - cc(e))B$. Hence it suffices to prove that $[cc(e)A] = [cc(e)B] = [0]$. By cutting down to $cc(e)R$ we may assume $e$ faithful and we need only verify $[A] = [0]$. Thus we are reduced to the case $A$ directly finite. By the general comparability axiom there exists a central idempotent $h$ such that $heR \leq hA$ and $(1 - h)A \leq (1 - h)eR$. Since $hA$ is directly finite and $heR$ purely infinite we deduce that $heR = 0$. But $e$ is faithful so $h = 0$. Then $A \leq eR$ and the result follows from the proof of (ii).

(iv) The relation $A \oplus cc(e)R \simeq B \oplus cc(e)R$ is equivalent to $(1 - cc(e))A \simeq (1 - cc(e))B$ and $cc(e)A \oplus cc(e)R \simeq cc(e)B \oplus cc(e)R$. Since $cc(e)$ is purely infinite the latter relation always holds. So the result follows from (iii).

Theorem 1.9. Let $R$ be a purely infinite, regular, right self-injective ring and let $I$ be an ideal of $R$. Then

(i) The map

$$\alpha : U(R/I) \to K_0(I), \quad \alpha(\bar{a}) = [\text{Coker } a] - [\text{Ker } a]$$

is a group homomorphism which induces an isomorphism

$$K_1(R/I) = U(R/I)^{ab} \cong K_0(I).$$

(ii) A unit $\bar{a} \in U(R/I)$ can be lifted to a unit in $R$ if and only if $[\text{Coker } a] = [\text{Ker } a] \in K_0(I)$. 

Proof. (i) Let \( f: U(R/I) \to K_1(R/I) \) be the natural map. It follows from [13, Theorem 1.2 (iii) and Theorem 2.2] that \( f \) is onto and \( \text{Ker} f = U(R/I)^\text{ab} \). So \( K_1(R/I) = U(R/I)^\text{ab} \). By [13, Theorem 2.7 (ii)] \( K_1(R) = 0 \) and it follows from [7, Proposition 15.6] that \( K_0(R) = 0 \). Thus (i) follows from Proposition 1.3.

(ii) This is an immediate consequence of (i) and Lemma 1.7. \( \square \)

**Lemma 1.10.** Let \( R \) be a regular right self-injective ring and \( I \) an ideal of \( R \). If \( e \) is an idempotent of \( I \), then the following are equivalent

(i) \( e \leq f \) for some purely infinite idempotent \( f \) in \( I \).

(ii) \( e \leq f \) for some purely infinite idempotent \( f \) in \( I \).

**Proof.** Clearly (ii) \( \Rightarrow \) (i). Conversely, by [7, Theorem 10.32] there exists a central idempotent \( h \) in \( R \) such that \( heR \) is purely infinite and \( (1 - h)eR \) is directly finite. So without loss of generality we may assume that \( eR \) is directly finite. We have \( eR \cong e'R \subseteq fR \) for some idempotent \( e' \). Since \( eRe \) has stable range 1, \( (1 - e)R \cong (1 - e')R \), so there exists a unit \( u \) in \( R \) such that \( e = u^{-1}e'u \). The idempotent \( u^{-1}fu \) is a purely infinite idempotent in \( I \) and \( e \leq u^{-1}fu \). \( \square \)

**Corollary 1.11.** Let \( R \) be a regular right self-injective ring. Let \( e_1 \) be the central idempotent in \( R \) such that \( e_1R \) is purely infinite and \( (1 - e_1)R \) is directly finite. Then the following are equivalent

(i) Every unit in \( R/I \) can be lifted to a unit in \( R \).

(ii) For every idempotent \( e \in e_1 I \) there exists a purely infinite idempotent \( f \in I \) such that \( e \leq f \).

(iii) \( K_0(e_1I) = 0 \).

**Proof.** \( R \) decomposes into the direct product of the rings \( R_1 = e_1R \) and \( R_2 = (1 - e_1)R \). Since \( R_2 \) is unit-regular it is clear that a unit in a factor ring of \( R_2 \) can be lifted to a unit in \( R_2 \). Thus without loss of generality we may assume that \( R \) is purely infinite, that is, \( e_1 = 1 \).

The equivalence (ii) \( \Leftrightarrow \) (iii) follows from Proposition 1.8 (ii) and Corollary 1.10. It is clear from Theorem 1.9 (ii) that (i) \( \Leftrightarrow \) (iii). \( \square \)

**Corollary 1.12.** If \( R \) is a regular right self-injective ring of Type III and \( I \) is an ideal of \( R \), then every unit in \( R/I \) can be lifted to a unit in \( R \).

**Proof.** Since \( R \) is Type III every idempotent is purely infinite. The result follows from Corollary 1.11. \( \square \)
Now it is a simple matter to extend Corollary 1.11 to arbitrary right self-injective rings. For this we first need a lemma.

For any ring $R$ denote by $J = J(R)$ its Jacobson radical. We shall use the fact that an element of $R$ is a unit if and only if so is modulo $J$. Recall that if $R$ is right self-injective then $R/J$ is regular and right self-injective. Moreover every idempotent in $R/J$ can be lifted to an idempotent in $R$.

We denote by $R_\infty$ the right ideal generated by all purely infinite idempotents in $R$.

**Lemma 1.13.** If $R$ is right self-injective, then $R_\infty$ is an ideal of $R$.

**Proof.** If $e$ is a purely infinite idempotent in $R$ then it suffices to prove that $xe \in R_\infty$ for all $x$ in $R$. In the case $x$ is a unit we have that $xex^{-1}$ is a purely infinite idempotent, hence $xex^{-1} \in R_\infty$ and so $xe \in R_\infty$. Now write $R/J = R_1 \times R_2$ where $R_1$ is purely infinite and $R_2$ is unit-regular. Let $S_1$ and $S_2$ be the ideals of $R$ such that $S_1/J = R_1$ and $S_2/J = R_2$. Since $R = S_1S_2$ it suffices to consider separately the cases $x \in S_1$ and $x \in S_2$.

Suppose first $x \in S_1$. Since $R_1$ is purely infinite $R_1 \approx M_2(R)$ and hence every element of $R_1$ is a sum of an even number of units in $R_1$. But then, every element of $R_1$ is a sum of units in $R_1 \times R_2$ and so every element of $S_1$ is a sum of units in $R$. Now it is clear that $xe \in R_\infty$.

Assume now $x \in S_2$. Since $R_2$ is unit-regular we can find an idempotent $f$ and a unit $u$ in $R$ such that $xu - f \in J$. So $x - fu^{-1}$ is a sum of two units. On the other hand $fRe \subseteq J$ so also $fu^{-1}e$ is a sum of two units. Therefore $xe = (x - fu^{-1})e + fu^{-1}e \in R_\infty$. \quad \square

**Theorem 1.14** If $R$ is a right self-injective ring and $I$ is an ideal of $R$, then the following are equivalent.

(i) Every unit in $R/I$ can be lifted to a unit in $R$.

(ii) If $e$ is an idempotent in $I$ which is contained in a purely infinite idempotent in $R$, then there exists a purely infinite idempotent in $I$ containing $e$.

(iii) $K_0(IR_\infty) = 0$.

**Proof.** Write $\bar{R} = R/J$ and denote images in $\bar{R}$ by overbars. Note that $R/(I + J)$ is a factor ring of the regular ring $R/J$. So $J(R/(I + J)) = 0$. Therefore $J(R/I) = (I + J)/I$. Now we have the following commutative diagram

\[
\begin{array}{ccc}
R & \rightarrow & R/I \\
\downarrow & & \downarrow \\
\bar{R} & \rightarrow & (R/I)/J(R/I) \approx \bar{R}/I
\end{array}
\]
where the rows and columns are the natural projections. Now it is easily seen that $U(R) \rightarrow U(R/I)$ is onto if and only if $U(R) \rightarrow U(R/I)$ so is.

If $e_1\bar{R}$ is the purely infinite part of $\bar{R}$, then $\bar{IR}_\infty = e_1\bar{I}$. Thus $K_0(e_1\bar{I}) \approx K_0(\bar{IR}_\infty)$ (for this notice that the kernel of the natural projection $IR_\infty \rightarrow e_1\bar{I}$ is contained in $J$). Now it follows from Corollary 1.11, applied to the pair $(\bar{R}, \bar{I})$ that (i) $\Leftrightarrow$ (iii). The result will follow by using Corollary 1.11 and noting that (ii) holds for the pair $(R, I)$ if and only if it holds for $(\bar{R}, \bar{I})$.

Suppose first that $(\bar{R}, \bar{I})$ satisfies (ii). Let $e$ be an idempotent in $I$ such that $e \leq f$ for some purely infinite idempotent $f$ in $R$. Then $\bar{e} \leq \bar{f}$ and so there exists a purely infinite idempotent $g$ in $R$ such that $\bar{e} \leq \bar{g}$ and $\bar{g}$ belonging to $\bar{I}$. In fact $g \in I + J$ and thus $g \in I$.

Now we have $\bar{g}e = \bar{e}$ so $ge - e = j \in J$. From this we easily obtain $g(1 + j)e = (1 + j)e$. But then $g_1 = (1 + j)^{-1}g(1 + j)$ is a purely infinite idempotent in $I$ such that $e = g_1e \leq g_1$.

Conversely, let $\bar{e}$ be an idempotent in $\bar{I}$ such that $\bar{e} \leq \bar{f}$ for some purely infinite idempotent $\bar{f}$ in $\bar{R}$. Clearly we may assume $f$ is a purely infinite idempotent in $R$ and $e$ is an idempotent in $I$. Then $fe - e = j \in J$. As in the preceding paragraph we obtain $e \leq f_1 = (1 + j)^{-1}f(1 + j)$. Clearly $f_1$ is purely infinite and so, by hypothesis, there exists a purely infinite idempotent $g$ in $I$ with $e \leq g$. Therefore $g \in I$ is a purely infinite idempotent such that $\bar{e} \leq \bar{g}$.

**Corollary 1.15.** If $R$ is a prime, regular, right self-injective ring, and $I$ is an ideal of $R$, then

(i) If $I = R_0$, then a unit $\bar{a} \in R/I$ can be lifted to a unit in $R$ if and only if $a$ is unit regular or equivalently $\text{Ker } a \approx \text{Coker } a$.

(ii) If $I \neq R_0$, then every unit in $R/I$ can be lifted to a unit in $R$.

**Proof.** (i) It follows from Proposition 1.6.

(ii) If $I \neq R_0$ then, by Lemma 1.5, there exists an idempotent $e$ in $I$ such that $eRe$ is not unit-regular, but $R$ being prime, regular, right self-injective this implies that $e$ is purely infinite. By Theorem 1.14 we must prove that every idempotent $f$ in $I$ is contained in a purely infinite idempotent in $I$. Without loss of generality we may assume that $f$ is directly finite. Since $R$ satisfies the comparability axiom we have either $e \leq f$ or $f \leq e$. Since $e \neq 0$ we must have $f \leq e$, as desired.

**Example.** Let $R = \text{End}_k(V)$ where $V$ is an infinite-dimensional $K$-vector space. In this case $R_0 = \{ x \in R | \dim_K x(V) < \infty \}$. If we associate with each $[eR] \in K_0(R_0)$ the $K$-dimension of $e(V)$, we obtain an
isomorphism \( K_0(R_0) \cong \mathbb{Z} \). By Theorem 1.9 \( U(R/R_0)^{ab} \cong \mathbb{Z} \), furthermore a unit \( \bar{a} \) in \( R/R_0 \) can be lifted to a unit in \( R \) if and only if \( \dim_K \text{Coker} \, a = \dim_K \text{Ker} \, a \).

\[ \square \]

2. Computation of \( K_0(I) \). Let \( R \) be a purely infinite regular right self-injective ring and let \( I \) be an ideal of \( R \). Our goal now is to realize \( K_0(I) \) as a group of continuous functions. This has been motivated by Olsen's work in \( W^* \)-algebras [15].

The starting point in Olsen's proof is Wils' characterization of the closed ideals of \( W^* \)-algebras. Although in the regular case such a characterization is not our disposal, we can obtain our results by extending some computations due to Goodearl and Boyle.

If \( M \) is a right \( R \)-module and \( n \geq 0 \) is an integer we shall write \( nM \).

**Lemma 2.1** Let \( R \) be a regular ring. Let \( A \) and \( B \) be nonsingular injective right \( R \)-modules such that the endomorphism ring \( \text{End}_R A \) is Type II and \( pA \approx qB \) for some positive integers \( p, q \). Let \( r \) be a positive integer.

(i) If \( r \leq p \) then there exists a right \( R \)-module \( D \) such that \( D \subseteq B \) and \( qD \approx rA \).

(ii) Assume \( A \) is directly finite. Let \( C \) be a finitely generated projective right \( R \)-module such that \( A, B \subseteq C \) and \( rA \leq qC \). If \( r \geq p \) then there exists a right \( R \)-module \( D \) such that \( B \subseteq D \subseteq C \) and \( qD \approx rA \).

**Proof.** (i) We have \( rA \approx B_1 \subseteq qB \) for some \( B_1 \). Since \( \text{End}_R B_1 \) is Type II (see [7, Theorem 10.10]) by [7, Proposition 10.28] \( B_1 \approx qB_2 \) for some \( B_2 \). So \( qB_2 \leq qB \) and by [7, Theorem 10.34] there exists a right \( R \)-module \( D \) such that \( B_2 \approx D \subseteq B \).

(ii) As in (i) there exists \( A_1 \subseteq C \) such that \( rA \approx qA_1 \). Now consider the submodule of \( C, B + A_1 \), which is finitely generated and so projective. Then \( B + A_1 \leq B \oplus A_1 \) and by [7, Corollary 9.20] \( B + A_1 \) is a directly finite nonsingular injective right \( R \)-module. Thus \( \text{End}_R(B + A_1) \) is unit regular.

On the other hand \( qB \approx pA \subseteq rA \approx qA_1 \), so \( B \approx B_1 \subseteq A_1 \) for some \( B_1 \). Then by [7, Corollary 4.4] there are decompositions \( B + A_1 = B \oplus B' = B_1 \oplus B' \) and thus \( D = B \oplus (A_1 \cap B') \) is the desired \( R \)-module.

Finally note that (i) follows for any ring \( R \).

\[ \square \]

**Lemma 2.2.** Let \( R \) be a regular right self-injective ring. Let \( A \) be a principal right ideal of \( R \) such that \( \text{End}_R A \) is Type II. Let \( \{ p_n, q_n \}_{n \in \mathbb{N}} \) be a set of positive integers such that \( p_n A \leq q_n R \) for every \( n \). Then there exist
principal right ideals of \( R; \ B_1, B_2, \ldots \) such that \( q_nB_n \approx p_nA \) for every \( n \) and \( B_n \subseteq B_m \) whenever \( p_n/q_n \leq p_m/q_m \).

**Proof.** We are going to construct the right ideals \( B_n \) by induction on \( n \). Since \( p_xA \leq q_xR \) and \( \text{End}_R A \) is Type II there exists a principal right ideal \( A_1 \) such that \( p_1A \approx p_1q_1A_1 \leq q_1R \) cf. [7, Proposition 10.28]. Then by [7, Theorem 10.34] \( p_1A_1 \approx B_1 \subseteq R \) for some right ideal \( B_1 \).

Now suppose we have constructed \( B_1, \ldots, B_n \). Set \( \lambda_n = p_n/q_n \) for each \( n \). Assume for simplicity that \( \lambda_1 > \lambda_2 > \cdots > \lambda_n \). Now there are three possibilities: (1) \( \lambda_{n+1} \leq \lambda_n \), (2) \( \lambda_1 \leq \lambda_{n+1} \) and (3) \( \lambda_i \geq \lambda_{n+1} \geq \lambda_{i+1} \) for some \( i \in \{1, \ldots, n-1\} \).

(1) By the induction hypothesis we have \( q_nB_n \approx p_nA \), so \( q_{n+1}B_n \approx q_{n+1}p_nA \) and then, by applying Lemma 2.1(i), there exists a principal right ideal \( B_{n+1} \) with \( B_{n+1} \subseteq B_n \) and \( q_{n+1}B_{n+1} \approx p_{n+1}A \).

(2) Let \( A_1 \) be a submodule of \( A \) such that \( A \approx q_{n+1}A_1 \). Now \( q_1B_1 \approx p_1A \approx p_1q_{n+1}A_1 \). On the other hand \( p_{n+1}q_{n+1}A_1 \leq q_{n+1}R \) implies \( p_{n+1}A_1 \leq Q \). By Lemma 2.1 (ii) there exists \( B_{n+1} \) with \( B_1 \subseteq B_{n+1} \) and \( q_1B_{n+1} \approx q_{n+1}q_{n+1}A_1 \), thus \( q_{n+1}B_{n+1} \approx p_{n+1}A \).

(3) As in the case (1), there exists a submodule of \( B_i \), say \( B \), such that \( q_{n+1}B \approx p_{n+1}A \). From the relation \( \lambda_{n+1} \geq \lambda_{i+1} \) we obtain \( p_{n+1}q_{n+1}B_{i+1} \approx p_{n+1}q_{n+1}B_{i+1} \approx p_{i+1}B_{i+1} \approx p_{n+1}q_{n+1}B \), so there exists \( B_{i+1}^* \) with \( B_{i+1} \approx B_{i+1}^* \subseteq B \). Then by [7, Corollary 4.4] there are decompositions \( B + B_{i+1} = B_{i+1} \oplus B^* = B_{i+1}^* \oplus B^* \).

Now write \( B_{n+1} \) for the module \( B_{i+1} \oplus (B \cap B^*) \). Then \( B_i \supseteq B_{n+1} \supseteq B_{i+1} \) and \( q_{n+1}B_{n+1} \approx p_{n+1}A \).

Let \( R \) be a regular right self-injective ring. If \( e \) is a directly finite idempotent of \( R \), then \( eRe \) is unit-regular cf [7; Corollary 1.23, Theorem 9.17]. By Lemma 1.5 we see that \( R_0 \) coincides with the ideal of \( R \) generated by all directly finite idempotents of \( R \).

**Lemma 2.3.** Let \( R \) be a regular right self-injective ring and \( I \) an ideal of \( R \) contained in \( R_0 \). If \( J \) is an ideal of \( R \) contained in \( I \), then the natural homomorphism \( K_0(J) \to K_0(I) \), induced by the inclusion \( J \subseteq I \), is injective.

**Proof.** By Proposition 1.2 every element in \( K_0(J) \) can be written in the form \([A] - [B] \) for some finitely generated projective right \( R \)-modules in \( P(I) \). If \([A] = [B] \) in \( K_0(I) \), then there exists a finitely projective right \( R \)-module \( C \subseteq P(I) \) with \( A \oplus C \approx B \oplus C \). Since every idempotent in \( I \)
is directly finite, by [7, Corollary 9.20] C is directly finite and then by [7, Corollary 9.18] \( A \cong B \). So \([A] = [B]\) in \( K_0(J) \).

From now on we shall identity \( K_0(J) \) with its image in \( K_0(I) \).

Let \( B(R) \) be the set of all central idempotents of \( R \). If \( \{ e_i \}_{i \in I} \) is a family of elements in \( R \) we denote by \( \bigvee_{i \in I} e_i \) and by \( \bigwedge_{i \in I} e_i \) its supremum and its infimum respectively. If \( R \) is regular and right self-injective then by [7, Proposition 9.9] \( B(R) \) is a complete Boolean algebra.

Let \( X = BS(R) \) be the Boolean spectrum of \( R \), that is, \( X \) is the set of all maximal ideals of \( B(R) \). Recall that the closed sets in \( X \) are of the form \( V(S) = \{ M \in BS(R) | S \subseteq M \} \), where \( S \subseteq B(R) \). Recall that with this topology, \( X \) is an Stonian space, that is, \( X \) is a compact Hausdorff space such that the closure of every open set is open. If \( Y \subseteq X \) then we denote the closure of \( Y \) in \( X \) by \( \overline{Y} \).

We shall need the following simple lemma.

**Lemma 2.4.** Suppose \( \{ e_i \}_{i \in I} \) is a family of elements in \( B(R) \). If \( X_i = V(1 - e_i) \) for all \( i \), then \( \bigcup_{i \in I} X_i = V(1 - \bigvee_{i \in I} e_i) \).

**Proof.** Set \( e = \bigvee_{i \in I} e_i \) and \( Y = \overline{\bigcup_{i \in I} X_i} \). Since \( Y \) is a clopen set there exists \( f \) in \( B(R) \) such that \( Y = V(1 - f) \). It is easily seen that the inclusion \( X_i = V(1 - e_i) \subseteq Y = V(1 - f) \) implies \( e_i \leq f \) for all index \( i \). So \( e \leq f \). On the other hand we have \( \bigcup_{i \in I} X_i \subseteq V(1 - e) \). Because \( V(1 - e) \) is clopen it contains \( Y \). So \( f \leq e \). \( \Box \)

Let \( f : X \to [-\infty, \infty] \) be a continuous map of \( X \) into the extended real interval \( [-\infty, \infty] \). We say that \( f \) is *almost finite* if it is finite in a dense open subset of \( X \). We denote by \( \mathcal{C}(X, [-\infty, \infty]) \) the set of all almost finite continuous maps of \( X \) into \( [-\infty, \infty] \). Assume \( f, g \in \mathcal{C}(X, [-\infty, \infty]) \) and let \( U \) be a dense open set in \( X \) such that \( f \) and \( g \) are finite in \( U \). Consider the continuous map \( f + g \) of \( U \) into \( [-\infty, \infty] \) defined with pointwise addition. Since \( X \) is Stonian, \( \overline{U} = X \) is the Stone-Čech compactification of \( U \) (see [19, 1.14 Theorem]), then, in particular, by [19, 1.11 Theorem] \( f + g \) can be extended to a unique continuous map, also denoted by \( f + g \), of \( X \) into \( [-\infty, \infty] \). With this addition and the natural order, \( \mathcal{C}(X, [-\infty, \infty]) \) becomes an ordered abelian group.

Let \( G \) be a partially ordered abelian group and let \( H \) be a subgroup of \( G \). Recall that \( H \) is said to be *directed* if it is upward directed, and *convex* if whenever \( x_1, x_2 \in H \) and \( y \in G \) such that \( x_1 \leq y \leq x_2 \), then
y ∈ H. It is known (see for example [7, Proposition 15.17]) that the set of all directed convex subgroups of G ordered by inclusion forms a lattice denoted by L(G).

For any rng I we denote by L_2(I) the lattice of ideals of I.

For the definition of the relative dimension functions on the nonsingular injective right modules over regular right self-injective rings we refer to [7, Chapter 11].

**Theorem 2.5.** Let R be a regular right self-injective ring of Type II_∞ and let e_0 be a faithful directly finite idempotent in R. Then

(i) The rule

\[ [eR] \mapsto \varphi_e, \quad \varphi_e(M) = d_M(eR : e_0 R) \]

defines an isomorphism of partially ordered abelian groups.

\[ \varphi : K_0(R_0) \to \mathcal{C}(X, [-\infty, \infty]). \]

(ii) The map

\[ L_2(R_0) \to L(\mathcal{C}(X, [-\infty, \infty]), J \mapsto \varphi(K_0(J))) \]

is a lattice isomorphism.

**Proof.** (i) Denote by \( \mathcal{C}(X, [0, \infty]) \) the set of all almost finite continuous maps of X into the extended real interval \([0, \infty]\). By [7, Lemma 11.16] if \( e \) is an idempotent in \( R_0 \) then the map

\[ \varphi_e : X \to [0, \infty], \quad M \mapsto d_M(eR : e_0 R) \]

is continuous. Now we prove that in fact \( \varphi_e \) belongs to \( \mathcal{C}(X, [0, \infty]) \). Set \( U = \varphi_e^{-1}([0, \infty]) \), which is an open set. Because \( X \) is Stonian, \( \bar{U} \) is clopen and so \( \bar{U} = V(f) \) for some \( f \) in \( B(R) \). Suppose \( eg \leq ne_0 g \) for some positive integer \( n \) and some central idempotent \( g \). If \( fg \neq 0 \) then there exists a maximal ideal \( M \) in \( B(R) \) such that \( fg \notin M \), thus \( d_M(eR : e_0 R) \leq n \) and so \( M \in V(f) \), which is a contradiction. Then \( fg = 0 \). Let \( m \) be a positive integer. By the general comparability axiom there exists a central idempotent \( h \) such that \( efh \leq me_0 fh \) and \( (1 - h)me_0 f \leq ef(1 - h) \). Then by the above \( fh = 0 \) and so \( me_0 f \leq ef \). Since this holds for all \( m \) we see that \( e_0 f = 0 \), cf [7, Corollary 9.23]. Therefore \( f = 0 \) and \( \bar{U} = X \).

Since \( R \) is purely infinite, for every finitely projective right \( R \)-module \( A \) there exists an idempotent \( e \) in \( R \) such that \( A \cong eR \). Thus we have a well-defined map

\[ \varphi : K_0(R_0)^+ \to \mathcal{C}(X, [0, \infty]), \quad [eR] \mapsto \varphi_e \]

where \( e \) is any idempotent of \( R_0 \).
Now we prove that $\varphi$ is onto. For this let $\alpha$ be an element in $\mathcal{C}(X, [0, \infty))$. Let $X_0$ denote the closure of the set $\{M \in X | \alpha(M) \geq 0\}$. For any integers $m$ and $n$ such that $m \geq 0$ and $n \geq 1$ let $X_{mn}$ denote the closure of the set $\{M \in X | \alpha(M) > m/2^n\}$. Note that $X_{0n} = X_0$ and $X_{mn} \subseteq X_{m-1,n}$ for all $m$ and $n$. It is easily seen that $X_0 - \alpha^{-1}(\infty) = \bigcup_{n=1}^{\infty} (X_{m-1,n} - X_{mn})$ for a fixed $n$. Since $\alpha$ is almost finite, $X_0 - \alpha^{-1}(\infty) = X_0$. Suppose $X_{m-1,n} - X_{mn} = \{M \in X | e_{mn} \notin M\}$ for all $m$ and $n$ and for some $e_{mn}$ in $B(R)$. It is clear that for each $n$ the $e_{mn}$'s are orthogonal because the sets $X_{m-1,n} - X_{mn}$ are disjoint. It follows from Lemma 2.4 that $X_0 = \{M \in X | 1 - e \notin M\}$, where $1 - e = \bigvee_{m,n} e_{mn}$.

For any $m$ and $n$, $X_{m-1,n} = X_{2m-2,n+1}$ and $X_{m-1,n} - X_{mn}$ is the disjoint union of $X_{2m-2,n+1} - X_{2m-1,n+1}$ and $X_{2m-1,n+1} - X_{2m,n+1}$. So $e_{mn} = e_{2m-1,n+1} + e_{2m,n+1}$. Let $f_{mn}$ be an idempotent such that $2^n f_{mn} R \approx me_0 R$. Since $e_0 R$ is directly finite it follows from [7, Proposition 11.3(e)] that $d_M(f_{mn} R: e_0 R) = m/2^n$ for all $M$ in $X$. By Lemma 2.2 we can assume that $f_{mn} \leq f_{st}$ if $m/2^n \leq s/2^t$.

Let $A_n = \bigvee_{m \geq 1} (e_{mn} f_{m-1,n} R)$ and note that $A_n$ is an injective hull of $\bigoplus_{m \geq 1} e_{mn} f_{m-1,n} R$. It is easily seen that $A_n$ is directly finite. Now we have

$$\bigoplus_{m \geq 1} e_{mn} f_{m-1,n} R = \bigoplus_{m \geq 1} \left( (e_{2m-1,n+1} + e_{2m,n+1}) f_{m-1,n} R \right) 
\subseteq \left( \bigoplus_{m \geq 1} f_{2m-2,n+1} e_{2m-1,n+1} R \right) \bigoplus \left( \bigoplus_{m \geq 1} f_{2m-1,n+1} e_{2m,n+1} R \right) 
= \bigoplus_{m \geq 1} f_{m-1,n+1} e_{m,n+1} R.$$ 

So $A_n \subseteq A_{n+1}$ for all $n$. Set $A = \bigcup_{n \geq 1} A_n$.

For any integer $t \geq 1$ define $A_t^* = (\bigvee_{m \geq 1} f_{mt} e_{mt} R)$. As above $A_t^*$ is directly finite and we have

$$\bigoplus_{m \geq 1} f_{m,t+1} e_{m,t+1} R = \left( \bigoplus_{j \geq 1} f_{2j-1,t+1} e_{2j-1,t+1} R \right) \bigoplus \left( \bigoplus_{j \geq 1} f_{2j,t+1} e_{2j,t+1} R \right) 
\subseteq \bigoplus_{j \geq 1} f_{2j,t+1} (e_{2j-1,t+1} + e_{2j,t+1}) R 
= \bigoplus_{j \geq 1} f_{2j,t+1} e_{j,t} R = \bigoplus_{j \geq 1} f_{j,t} e_{jt} R.$$ 

So $A_{t+1}^* \subseteq A_t^*$. Now $A_t \subseteq A_t^*$ and then $A \subseteq A_t^*$.

We shall prove that $\varphi([A]) = \alpha$. Since $\alpha$ is almost finite we must show that $\varphi([A])(M) = \alpha(M)$ for all $M \in X - \alpha^{-1}(\infty)$. If $e \notin M$ then $d_M(A: e_0 R) = d_M(Ae: e_0 R) = 0$ because $Ae = 0$. Now suppose that $e
belongs to $M$. Then for each $n$ we have that there exists an $m$ such that $M \in X_{m-1,n} - X_{m,n}$. So $m - 1/2^n \leq \alpha(M) \leq m/2^n$. Since $A_n e_{mn} R = f_{m-1,n} e_{mn} R$ then $d_M(A_n; e_0 R) = m - 1/2^n$ and so $d_M(A; e_0 R) \geq (m - 1)/2^n$. Similarly $d_M(A; e_0 R) \leq d_M(A_n^*; e_0 R) = d_M(A_n^* e_{mn}; e_0 R) = d_M(f_{mn} R; e_0 R) = m/2^n$. Then $\varphi([A])(M) - 1/2^n \leq \alpha(M) \leq \varphi([A])(M) + 1/2^n$ for all $n$. So $\alpha(M) = \varphi([A])(M)$.

Now by [7, Theorem 11.11] the map

$$\varphi: K_0(R_0) \rightarrow C(X, [-\infty, \infty]), \quad [eR] - [fR] \rightarrow \varphi_e - \varphi_f$$

is a group homomorphism. By [7, Theorem 11.15 (a)] $\varphi$ is an order preserving homomorphism. Because any element of $C(X, [-\infty, \infty])$ can be written as a difference of two elements of $C(X, [0, \infty])$, by the preceding paragraph it is clear that $\varphi$ is onto. To prove injectivity suppose $\varphi_e = \varphi_f$ for some idempotents $e$ and $f$ in $K_0(R_0)$. Then by [7, Theorem 11.15 (b)] $eR \cong fR$ and so $[eR] = [fR]$ in $K_0(R_0)$.

(ii) If $J$ is an ideal of $R$ contained in $R_0$, by Lemma 2.3 $K_0(J)$ is a subgroup of $K_0(R_0)$. Now, as in the proof of [7, Theorem 15.20] one can see that the correspondence $J \rightarrow K_0(J)$ defines a lattice isomorphism of $L(R_0)$ onto $L(K_0(R_0))$. Since $\varphi$ is an order group isomorphism, the result follows.

**Corollary 2.6.** If $R$ is a prime regular right self-injective ring of Type $\Pi_{\infty}$, then $K_1(R/R_0) = U(R/R_0)^{ab} \cong K_0(R_0) \cong R$.

**Proof.** It follows from Theorem 1.9 and Theorem 2.5. □

Now we shall consider almost finite continuous functions on $X$ taking its values on $\mathbb{Z} \cup \{\pm \infty\}$. As above we shall write $C(X, \mathbb{Z} \cup \{\pm \infty\})$ for the group of all this functions.

**Lemma 2.7.** Let $R$ be a regular ring. Let $A$ and $B$ be finitely generated right $R$-modules such that $\text{End}_R A$ is unit-regular. If $A/AP \cong B/BP$ for all prime ideals $P$ of $R$ then $A \cong B$.

**Proof.** In [7, Theorem 4.19] this lemma is proved under the hypothesis of unit-regularity. But, with the notation of [7, Lemma 4.18], it is only necessary that the $R$-module $A_1/A_1 K$ cancels from direct sums, and it is easily seen that this also occurs if $\text{End}_R A$ is unit-regular. □
The proof of the next result is quite similar to Theorem 2.5.

**Theorem 2.8.** Let $R$ be a regular right self-injective ring of Type $I_\infty$ and let $e_0$ be a faithful abelian idempotent in $R$. Then

(i) the rule

$$[eR] \mapsto \varphi_e, \quad \varphi_e(M) = d_M(eR: e_0R)$$

defines a partially ordered abelian group isomorphism

$$\varphi: K_0(R_0) \rightarrow \mathcal{U}(X, \mathbb{Z} \cup \{\pm \infty\})$$

(ii) the map

$$L_2(R_0) \rightarrow L(\mathcal{U}(X, \mathbb{Z} \cup \{\pm \infty\}), \quad J \mapsto \varphi(K_0(J))$$

is a lattice isomorphism.

**Proof.** (i) First we prove that if $e$ is an idempotent in $R_0$ then $d_M(eR: e_0R)$ is either an integer or $\infty$. For this we need only prove that if $nfR \leq mgR$, where $m, n$ are positive integers and $f, g$ are idempotents with $g$ abelian, then there exists an integer $s, s \leq m/n$, such that $fR \leq sgR$.

Let $P$ be a prime ideal in $R$ and let $\bar{f}, \bar{g} \in R/P$ the images of $f$ and $g$ in $R/P$ respectively. Then $n\bar{f}R/P \leq m\bar{g}R/P$. Since $R/P$ is prime and $\bar{g}$ is abelian in $R/P$ we see that $\bar{g}R/P$ is a simple module and so $\bar{f}R/P \cong r\bar{g}R/P$ for some $r \in \mathbb{N}$. Hence $\bar{f}R/P \leq [m/n]\bar{g}R/P$, where $[m/n]$ denotes the integer part of $m/n$. By Lemma 2.7 we obtain $fR \leq [m/n]gR$ as desired.

As in the proof of Theorem 2.5 (i) we derive that $\varphi$ is a well-defined injective map.

Now we are going to prove that $\varphi$ is onto. Like Theorem 2.5 (i) it suffices to prove that for every positive $\alpha$ in $\mathcal{U}(X, \mathbb{Z} \cup \{\pm \infty\})$ there exists $A$ in $P(R_0)$ such that $\varphi([A]) = \alpha$. For each natural $k$, set $X_k = \{ M \in X | \alpha(M) = k \}$. Certainly $X_k$ is a clopen set in $X$. Hence $X_k = \{ M \in X | e_k \notin M \}$, for some suitable $e_k$ in $B(R)$. Since the $X_k$'s are pairwise disjoint we have that the corresponding $e_k$'s are orthogonal.

For a given natural number $n$, we have, since $R$ is purely infinite, that $ne_0R \leq R$. Thus $\bigoplus_k ke_k e_0R \leq R$. Let $A$ denote a principal right ideal of $R$ that is isomorphic to the injective hull of $\bigoplus_k ke_k e_0R$. There is no difficulty in proving that $A$ belongs to $P(R_0)$. Clearly $e_kA \cong ke_k e_0R$ and, by [7, Proposition 11.3] we have

$$\varphi([A])(M) = d_M(A: e_0R) = d_M(ke_k e_0R: e_0R) = k = \alpha(M),$$

for all $M \in X_k$. Since $\alpha$ is almost finite we see $\varphi([A]) = \alpha$.

(ii) It follows similarly to Theorem 2.5 (ii).
LEMMA 2.9. Let $R$ be a regular right self-injective ring and $I$ an ideal of $R$. If $C \in P(I)$ is purely infinite then $C \cong eR$ for some (purely infinite) idempotent $e$ in $I$.

Proof. Suppose $C = A \oplus B$ for some directly finite right $R$-module $A$ and some purely infinite right $R$-module $B$. Now we prove that $C \cong B$.

By [7, Theorem 9.14] there exists $h \in B(R)$ such that $Ah \leq Bh$ and $B(1 - h) \leq A(1 - h)$. Then, since $B$ is purely infinite, we have $B(1 - h) = 0$. So $C(1 - h) \approx A(1 - h)$ and thus also $A(1 - h) = 0$. Then $B \leq A \oplus B \leq B \oplus B \approx B$ and so, by [7, Theorem 10.14] $C = A \oplus B \approx B$.

Now, suppose $C = e_1R \oplus \cdots \oplus e_nR$ for some idempotents $e_1, \ldots, e_n$ in $I$. By [7, Theorem 10.32] there exists $h_i \in B(R)$ such that $h_i e_i R$ is directly finite and $(1 - h_i)e_i R$ is purely infinite for $i = 1, \ldots, n$. Then by the preceding paragraph we can assume that each $e_i$ is purely infinite.

Since $R$ satisfies general comparability, there exists $h \in B(R)$ such that $he_1 \leq he_2$ and $(1 - h)e_2 \leq (1 - h)e_1$. Then it is clear that $e = (1 - h)e_1 + he_2$ is a purely infinite idempotent in $I$ such that $e_1 R \oplus e_2 R \approx eR \oplus eR$. By induction on $n$ the result follows.

For each ideal $I$ of $R$ we denote by $I_0$ the ideal of $R$ generated by all directly finite idempotents in $I$ and by $I_1$ the ideal of $R$ generated by all directly finite idempotents in $I$ that are contained in some purely infinite idempotent in $I$.

If $S \in L(C(X, K))$, where $K$ is either $[-\infty, \infty]$ or $\mathbb{Z} \cup \{\pm \infty\}$, and $\Gamma$ is a closed set in $X$, then we write $S\Gamma$ for the quotient $S/\{\alpha \in S: \alpha = 0$ in some open set in $X$ containing $\Gamma\}$.

THEOREM 2.10. Let $R$ be a regular right self-injective ring and $I$ an ideal of $R$. Then

(i) $K_0(I) \approx K_0(I_0)/K_0(I_1)$.

(ii) Let $\Gamma(I) = V(\{cc(g) | g is a purely infinite idempotent in I\})$. If $R$ is either Type $\Pi_\infty$ or $I_\infty$ then $K_0(I) \approx \varphi(K_0(I_0))\Gamma(I)$ where $\varphi: K_0(R_0) \to C(X, K)$ is the map defined in Theorem 2.5 or Theorem 2.8, respectively.

Proof. (i) First we prove that the natural map $\Psi: K_0(I_0) \to K_0(I)$ is onto. Let $A \in P(I)$. By [7, Theorem 10.32] there exists a central idempotent $h$ in $R$ such that $Ah$ is directly finite and $A(1 - h)$ is purely infinite. Then $[A(1 - h)] = 0$ in $K_0(I)$ and so we can assume that $A$ is directly finite, but in this case it is clear that $A$ belongs to $P(I_0)$.

Now we prove that $\text{Ker} \, \Psi = K_0(I_1)$. Let $A \in P(I_1)$. Since $A$ is isomorphic to a direct sum of principal right ideals, each of which is generated by an idempotent in $I_1$, it is clear that in order to prove
[A] ∈ Ker Ψ we may assume \( A = eR \) for some idempotent \( e \) in \( I_1 \). Then there exists a purely infinite right \( R \)-module \( B \) in \( P(I) \) such that \( A \preceq B \). Thus \( A \oplus B \cong B \). Then \([A] = 0 \) in \( K_0(I) \) and so \( K_0(I_1) \subseteq Ker \Psi \).

Conversely, let \([A] - [B] \in Ker \Psi \). Then by Proposition 1.2 and the proof of Proposition 1.8 (i) there exists a purely infinite right \( R \)-module \( C \) in \( P(I) \) such that \( A \oplus C \equiv B \oplus C \). Now, by the general comparability axiom there exists \( h \in B(R) \) such that \( Bh \leq Ch \) and \( C(1 - h) \leq B(1 - h) \). Since \( B(1 - h) \) is directly finite and \( C(1 - h) \) is purely infinite we see \( C(1 - h) = 0 \). From the relation \( A \oplus C \equiv B \oplus C \) we have \( A(1 - h) \approx B(1 - h) \) so \([A] - [B] = [Ah] - [Bh] \). Then we may assume \( B \leq C \) and since \( C \) is purely infinite also \( A \leq C \). By Lemma 2.9 \( A \leq eR \) for some purely infinite idempotent \( e \) in \( I \). Then by Lemma 1.10 \( A \subseteq fR \) for some purely infinite idempotent \( f \) in \( I \). Hence \( A \in P(I_1) \). Similarly \( B \in P(I_1) \) and then \([A] - [B] \in K_0(I_1) \).

(ii) Since \( R \) is purely infinite, every element in \( K_0(I) \) can be written in the form \([eR] - [fR]\) for some idempotents \( e, f \) in \( I \).

By (i) it suffices to show that \( \varphi(K_0(I_1)) = \{ \alpha \in \varphi(K_0(I_0)) : \alpha = 0 \) in some open set in \( X \) containing \( \Gamma(I) \}. If \([eR] \in K_0(I_1) \) then there exists a purely infinite idempotent \( g \) in \( I \) such that \( e \leq g \). Then \( eR \oplus gR \approx gR \) and by [7, Theorem 11.11] \( d_M(eR; e_0R) \leq d_M(gR; e_0R) \) for all \( M \in X \) (here \( e_0 \) is as in Theorem 2.5 or Theorem 2.8). By [7, Proposition 11.3] \( d_M(gR; e_0R) = 0 \) if \( M \in V(cc(g)) \). So, since \( \Gamma(I) \subseteq V(cc(g)) \), we have \( \varphi(K_0(I_1)) \subseteq \{ \alpha \in \varphi(K_0(I_0)) : \alpha = 0 \) in an open set in \( X \) containing \( \Gamma(I) \}. \)

Now we prove the reverse inclusion. For simplicity here we denote by \( E \) the set of all purely infinite idempotents in \( I \). First we shall note that the set \( S = \{ cc(g) : g \in E \} \) is an ideal of \( B(R) \). If \( x \in B(R) \) and \( g \in E \), then by [7, Lemma 11.4 (c)] \( xcc(g) = cc(xg) \). Since \( xg \in E \), we see that \( xcc(g) \in S \). Let \( g_1, g_2 \in E \) and let \( k = cc(g_1) + cc(g_2) - 2cc(g_1)cc(g_2) \). By [7, Lemma 11.4(c)] and observing that \( g_1(1 - cc(g_2)) \) and \( g_2(1 - cc(g_1)) \) are orthogonal idempotents we have

\[
cc(g_1(1 - cc(g_2)) + g_2(1 - cc(g_1))) = cc(g_1(1 - cc(g_2))) + cc(g_2(1 - cc(g_1))) = cc(g_1)(1 - cc(g_2)) + cc(g_2)(1 - cc(g_1)) = k.
\]

By noting that \( g_1(1 - cc(g_2)) + g_2(1 - cc(g_1)) \in E \) we obtain that \( k \in S \). Then \( S \) is an ideal of \( B(R) \).

Let \( e \in I_0 \) be an idempotent such that \( \varphi([eR]) \) is zero in an open set \( U \) containing \( \Gamma(I) \). For each \( M \in \Gamma(I) \) there exists \( h_M \in B(R) \) with \( M \in V(h_M) \subseteq U \). Since \( \Gamma(I) \) is compact we can find \( M_1, \ldots, M_r \in \Gamma(I) \)
with \( \Gamma(I) \subseteq V(h_{M_1}) \cup \cdots \cup V(h_{M_r}) = V(h_{M_1} \cdots h_{M_r}) \); set \( h = h_{M_1} \cdots h_{M_r} \), then from the inclusion \( V(S) = \Gamma(I) \subseteq V(h) \) we obtain \( h \in S \) and so \( h = cc(g) \) for some \( g \) in \( E \).

Let \( M \in X \). If \( 1 - h \in M \), then by [7, Proposition 11.3 (a)]

\[
d_M(e(1-h)R : e_0 R) = 0.
\]

If \( h \in M \) then, since \( V(h) \subseteq U \), \( d_M(e(1-h)R : e_0 R) = \varphi([eR])(M) = 0 \).

Hence, by [7, Proposition 11.6], \( e(1-h) = 0 \). Let \( t \in B(R) \) such that \( te \leq tg \) and \( (1-t)g \leq (1-t)e \). Because \( (1-t)e \) is directly finite and \( (1-t)g \) is purely infinite, we obtain \( (1-t)g = 0 \) and so \( h = cc(g) \leq t \).

Then by multiplying the relation \( te \leq g \) by \( h \), we obtain \( hte \leq hg = g \), and, because \( ht = h \) and \( he = e \), we have \( e \leq g \). By Lemma 1.10 we may assume \( e \leq g \) and so \( [eR] \subseteq K_0(I) \) as desired. \( \square \)

3. Rickart \( C^* \)-algebras. Recall that a \( C^* \)-algebra \( A \) is said to be Rickart if the right annihilator of each element in \( A \) is generated by a projection. In notation \( r(a) = eA \) where \( e = e^2 = e^* \). If the annihilator condition holds for every subset of \( A \), then \( A \) is called an \( AW^* \)-algebra.

As usual we shall write \( RP(a) \) (the right projection of \( a \)) for \( 1 - e \). The left projection of \( a \), \( LP(a) \), is defined similarly. It is known [3, Proposition 1.3.7 and Lemma 1.8.2] that with the relation \( \leq \) the set of all projections of a Rickart \( C^* \)-algebra is a complemented \( \chi_0 \)-complete lattice. Two projections \( e \) and \( f \) are said to be equivalent, written \( e \sim f \), if \( eA \cong fA \). A projection \( e \) is said to be finite if \( e \sim f \leq e \) implies \( e = f \). We say \( A \) is finite if \( 1 \) is a finite projection. Since \( A \) is a \( C^* \)-algebra \( e \sim f \) if and only if \( e \) and \( f \) are \( * \)-equivalent, that is \( e = xx^* \) and \( f = x^*x \) for some \( x \in eAf \) cf. [9, Proposition 19.1 (a)]. If \( e \) is an idempotent of a \( C^* \)-algebra \( A \), then there exists a unique projection \( f \) in \( A \) such that \( eA = fA \) cf. [9, proof of Proposition 19.1 (b)]. From this we see that Rickart \( C^* \)-algebras are precisely those \( C^* \)-algebras that are principal projective. It seems to be unknown whether Rickart \( C^* \)-algebras are semihereditary.

For background and basic concepts on Rickart \( C^* \)-algebras the reader can consult [3].

**Proposition 3.1.** If \( A \) is a Rickart \( C^* \)-algebra and \( I \) is an ideal of \( A \) then \( K_0(I) = G(I) = K_0(\bar{I}) \), where \( \bar{I} \) is the closure of \( I \).

**Proof.** Let \( E \) be the set of all projections in \( I \). It follows from [3, Proposition 5.22.1] that the sub-\( C^* \)-algebras \( \{ eAe + C1 \}_{e \in E} \) form a directed system. Since \( \bar{I} \) is the closed \( C \)-linear span of its projections [3, p.
we have that $C^*-\text{dir} \lim_{e \in E}(eAe + C1) = I + C1$. Now it follows from [9, Proposition 19.9] that the natural map

$$\text{dir} \lim_{e \in E} K_0(eAe + C) \to K_0(I + C)$$

is a group isomorphism. Since the diagram

$$\text{dir} \lim_{e \in E} K_0(eAe + C) \twoheadrightarrow K_0(I + C)$$

$$\downarrow$$

$$K_0(I + C)$$

is commutative, where the maps are the natural ones, then the map

$$\text{dir} \lim_{e \in E} K_0(eAe + C) \to K_0(I + C)$$

is injective, and onto by [9, Proposition 19.3].

Thus, by Proposition 0.1 we have $K_0(I) = K_0(I) = G(I)$.

Let $A$ be a $C^*$-algebra and let $I$ be an ideal of $A$. If $\pi: A \to A/I$ is the natural surjection, then we set $\mathcal{F}(I, A) = \pi^{-1}(U(A/I))$. An element of $\mathcal{F}(I, A)$ is said to be a Fredholm element of $A$ relative to $I$. In the case where $A = B(H)$ is the ring of all bounded operators on a separable Hilbert space and $I = \mathcal{N}$ is the ideal of compact operators, then the elements of $\mathcal{F}(\mathcal{N}, B(H))$ are the usual Fredholm operators cf. [6, Chapter 5].

Let us recall briefly some basic results on index theory for Fredholm operators. If $T \in \mathcal{F}(\mathcal{N}, B(H))$, then by Atkinson's theorem [6, 5.17 Theorem] $\dim \ker T$ and $\dim \ker T^*$ are both finite and the map $\iota: \mathcal{F}(\mathcal{N}, B(H)) \to \mathbb{Z}$ given by $T \mapsto \dim \ker T^* - \dim \ker T$ (the index map) is a continuous monoid homomorphism [6, 5.36 Theorem]. Furthermore the connected components of $\mathcal{F}(\mathcal{N}, B(H))$ are $\iota^{-1}(n), n \in \mathbb{Z}$ [6, 5.36 Theorem]. Breuer [4] [5] generalizes this result to an arbitrary $W^*$-algebra (here the compact ideal means the closure of the ideal generated by all finite projections in $A$). More recently Olsen [15] has defined an index map for each closed ideal $I$ of a $W^*$-algebra which permits to describe the connected components of $\mathcal{F}(I, A)$.

Next we shall extend Breuer's theory to arbitrary Rickart $C^*$-algebras. In order to obtain an explicit index map for any closed ideal in a Rickart $C^*$-algebra $A$ we will need the following additional axioms on $A$:

(i) $A$ has a projection $e$ such that $e \sim 1 - e \sim 1$

(ii) $A$ satisfies the general comparability axiom (i.e. for each pair of projections $e, f$ there exists a central projection $h$ such that $he \leq hf$ and $h(1 - f) \leq h(1 - e)$).

```
As we shall see this axioms are not an obstacle for constructing an index theory for arbitrary $AW^*$-algebras.

The following lemma is known under the additional hypothesis of general comparability (see [3, Lemma 1.8.3, Theorem 3.17.3]).

If $A$ is a Rickart $C^*$-algebra, then we denote by $\mathcal{K} = \mathcal{K}(A)$ the closure of the ideal generated by all finite projections of $A$. We say that $\mathcal{K}$ is the compact ideal of $A$.

**Lemma 3.2.** Every projection in $\mathcal{K}$ is finite.

**Proof.** Let $I$ be the ideal generated by all finite projections in $A$.

Since $\mathcal{K}$ is the closure of $I$ it is well-known that every projection in $\mathcal{K}$ belongs to $I$ cf [3, Chapter 5 §22 Exercise 6A]. Now let $f$ be a projection in $I$, then $f = \sum x_i e_i y_i$, where $x_i, y_i \in A$ and the $e_i$'s are finite projections. Consider now the map $\psi: \bigoplus e_i A \to fA$ defined by $\psi(\sum e_i r_i) = \sum f x_i e_i r_i$. Clearly $\psi$ is an onto $A$-module homomorphism. Thus $fA \leq \bigoplus e_i A$. Now a finite Rickart $C^*$-algebra has stable range 1 cf [10]. So the endomorphisms rings $e_i A e_i \approx \text{End}_R(e_i A)$ have stable range 1. In particular $\bigoplus e_i A$ cancels from direct sums of right $A$-modules and, since $fA$ is isomorphic to a direct summand of $\bigoplus e_i A$, the same is true for $fA$. Therefore $f$ is finite. 

If $M$ and $N$ are right $A$-modules, then $M \hookrightarrow N$ means that $M$ is isomorphic to a submodule of $N$.

**Lemma 3.3.** If $A$ is a Rickart $C^*$-algebra, then

1. If $e \in A$ is a finite projection, then $eA$ does not contain an infinite direct sum of nonzero pairwise isomorphic right ideals. In particular, every $A$-module $M \hookrightarrow eA$ is directly finite.
2. If $P$ and $Q$ are directly finite cyclic projective right $A$-modules such that $P \hookrightarrow Q$ and $Q \hookrightarrow P$, then $P \approx Q$.
3. If $x$ is an element of $A$ such that $LP(x)$ is finite, then $LP(x) \sim RP(x)$. Further $xA \approx x^*A$.

**Proof.** (i) Let $\{ A_n \}$ be a sequence of pairwise isomorphic right ideals contained in $eA$. Then $\{ A_n e \}$ is a sequence of pairwise isomorphic right ideals of $eA e$. Now $eA e$ is a finite Rickart $C^*$-algebra and so, $R$, its classical ring of quotients [1, Theorem 3.1(i)] [11, Theorem 2.1] is an $\mathfrak{S}_0$-continuous regular ring which contains an infinite direct sum of pairwise isomorphic right ideals. By [8, Proposition 1.1] $A_1 e \otimes_{eA e} R = 0$, hence $A_1 e = 0$. But $A$ is semiprime so $0 = eA_i = A_i$ as desired.
(ii) We may assume $P = eA$ and $Q = fA$ for some finite projections $e, f$ in $A$. Let $g$ be the supremum of $e$ and $f$. By [3, Proposition 5.22.1] $g \in \mathcal{X}^*$ and it follows from Lemma 3.2 that $g$ is finite and so $gAg$ is a finite Rickart C*-algebra. Now $e(gAg) \hookrightarrow f(gAg)$ and $f(gAg) \hookrightarrow e(gAg)$. If $R$ is the classical ring of quotients of $gAg$, then because $R$ is regular we have $eR \leq fR$ and $fR \leq eR$. But $R$ is unit-regular cf. [11, Theorem 3.2] so $eR \cong fR$. Because of the unit regularity one has [7, Corollary 4.23] that $eR$ and $fR$ are perspective in the lattice $L(R)$ of principal right ideals of $R$. By [11, Theorem 2.1(3)] $L(R) = L(gAg)$ so that $e \sim f$ in $gAg$ and so in $A$.

(iii) Since $xA \cong RP(x)A$ and $xA \leq LP(x)A$, we see that $RP(x) \hookrightarrow LP(x)$, similarly $LP(x) \hookrightarrow RP(x)$. By (i) $RP(x)$ is finite and then from (ii) we get $LP(x) \sim RP(x)$ and $xA \approx x*A$. □

**Lemma 3.4.** Let $e$ be a finite projection in a Rickart C*-algebra $A$. If $x \in A$ is such that $xx^*$ and $e$ commute then

$$xA \cap eA \approx exx*A \approx e(xx^*)^{1/2}A.$$  

**Proof.** Since $LP(ex) \leq e$ we see that $LP(ex)$ is a finite projection. By Lemma 3.3 (iii) $exA \approx x*eA$. Since $r(xx^*e) = r(xx^*e)$ and $xx^*$ commutes with $e$ we have $xA \cap eA \subseteq exA \approx exx*A \subseteq xA \cap eA$. By Lemma 3.3(i), $xA \cap eA$ and $exx*A$ are directly finite right $A$-modules. Moreover, left multiplication by $x$ induces an epimorphism from $r((1 - e)x)$ to $xA \cap eA$, then $xA \cap eA$ is a cyclic right ideal and so projective. Thus by Lemma 3.2 (ii) $exx*A \approx xA \cap eA$. Since $r(exx*) = r(e(xx^*)^{1/2})$ left multiplication by $(xx^*)^{1/2}$ gives $(exx*)A \approx e(xx^*)^{1/2}A$. □

Notice that if $A$ has polar decomposition, then by [3, Proposition 4.21.3] $xA = (xx^*)^{1/2}A$ for every $x$ in $A$. Thus in this case the preceding lemma is obvious. It is not known whether Rickart C*-algebras have polar decomposition cf. [3, Chapter 4 §21 Exercise 10D]. In fact we have the following result noted by Handelman.

**Lemma 3.5 (Handelman).** A semihereditary Rickart C*-algebra has polar decomposition.

**Proof.** If $A$ is a semihereditary Rickart C*-algebra, then $M_2(A)$ is also a Rickart C*-algebra cf. [9, Theorem 7.4, Proof of Proposition 19.1 (b)]. Now $M_2(A)$ contains two orthogonal copies of $A$ and by using the same techniques than in the proof of [3, Proposition 4.20.2] we see that partial isometries are $S_0$-addable in $A$. But then, as it is noted in [3, p. 276 Exercise 11 (ii)] $A$ has polar decomposition. □
Lemma 3.6. Let $A$ be a Rickart $C^*$-algebra and let $I$ be an ideal of $A$. If $x$ is an element of $A$ then the following are equivalent

(i) $x \in \mathcal{F}(I, A)$

(ii) There exist a positive unit $\gamma$ and projections $e, f$ in $I$ such that

$$e \gamma xx^* \gamma = \gamma xx^* \gamma e$$

$$(1 - e) \gamma xx^* \gamma (1 - e) = 1 - e$$

$$x^* \gamma (1 - e) \gamma x = 1 - f.$$  

(iii) There exist projections $f, g$ in $I$ such that

$$1 - e \in x^* A$$

and $1 - g \in x A$.

Moreover, if either $I \subseteq \mathcal{K}$ or $A$ is semihereditary, then for any pair of projections $e, f$ satisfying (ii) we have $r(x^*) \oplus f A \cong r(x) \oplus e A$.

Proof. (i) $\Rightarrow$ (ii). If $x \in \mathcal{F}(I, A)$ then $x A + z A = A$ for some $z \in I$ and, since $A$ is a $C^*$-algebra, $xx^* + zz^*$ is a unit. By [3, Proposition 1.8.4], for a given $\varepsilon > 0$, there exists a projection $p \in zz^* A$ with $\|zz^* - pzz^*\| < \varepsilon$. Thus we can choose $p$ such that $xx^* + pzz^*$ is a unit. But then $x A + p A = A$, say $xx^* + p = (\gamma^{-1})^2$ where $\gamma = \gamma^*$ is a unit. Define $e = LP(\gamma p \gamma)$, since $\gamma p \gamma$ is positive $e = RP(\gamma p \gamma)$, moreover $e \in I$. Since $\gamma xx^* \gamma + \gamma p \gamma = 1$ we see that $e$ commutes with $\gamma xx^* \gamma$. By multiplying the latter relation by $1 - e$ we get $(1 - e) \gamma xx^* \gamma (1 - e) = 1 - e$. Therefore $x^* \gamma (1 - e) \gamma x$ is a projection, say $1 - f$. Since $x \in \mathcal{F}(I, A)$, we see that $f \in I$. The proof is complete.

(ii) $\Rightarrow$ (iii) Since $e \gamma xx^* = \gamma xx^* \gamma e$ and $(1 - e) \gamma xx^* \gamma (1 - e) = 1 - e$, we see that $1 - e \in \gamma x A$, that is $\gamma^{-1}(1 - e) \gamma \in x A$. Now $\gamma^{-1}(1 - e) \gamma A = (1 - g) A$, where $g$ is a projection, and because $e \in I$ we see that $g \in I$ cf. [9, proof of Proposition 19.1 (b)]. Hence $1 - g \in x A$. On the other hand is clear that $1 - f \in x^* A$.

Obviously (iii) implies (i).

Suppose now that $e$ and $f$ are projections satisfying (ii). Since $r(x) = r(\gamma x)$ and $r(x^*) \cong r(x^* \gamma)$ we may assume, without loss of generality, that $\gamma = 1$. Now consider the following exact sequences

$$0 \to r(x) \to r((1 - e) x) \to x A \cap e A \to 0$$

$$0 \to r(x^*) \to r((1 - e)(xx^*)^{1/2}) \to (xx^*)^{1/2} A \cap e A \to 0$$

If $I \subseteq \mathcal{K}$, then, by Lemma 3.4, $x A \cap e A \cong (xx^*)^{1/2} A \cap e A$. In the case where $A$ is semihereditary we also have this isomorphism because then $A$ has polar decomposition (Lemma 3.5). Thus in both cases we can apply Schanuel’s lemma to get

$$r(x) \oplus r((1 - e) xx^*) \cong r(x^*) \oplus r((1 - e) x),$$
now
\[ r((1 - e)xx*) = r((1 - e)xx*(1 - e)) = r(x*(1 - e)) = eA \]
and \( r((1 - e)x) = fA. \) The proof is complete. \( \square \)

**Proposition 3.7.** Let \( A \) be a Rickart \( C^* \)-algebra and let \( I \) be an ideal of \( A \). If \( \alpha \) denotes the composite map

\[ \mathcal{F}(I, A) \to U(A/I) \to K_1(A/I) \to K_0(I) \]

then we have

(i) If \( I \subseteq \mathcal{K} \) then\]
\[ \alpha(x) = [r(x^*)] - [r(x)] \]
and \( LP(x) \sim RP(x) \) for all \( x \in \mathcal{F}(I, A) \).

(ii) If \( A \) is semihereditary, then
\[ \alpha(x) = [r(x^*)] - [r(x)] \text{ for all } x \in \mathcal{F}(I, A). \]

**Proof.** Let \( \beta: \mathcal{F}(I, A) \to K_0(I) \) be the map defined by \( \beta(x) = [r(x^*)] - [r(x)] \). Then we must prove that \( \beta = \alpha \).

Let \( x \in \mathcal{F}(I, A) \). Now let \( \gamma, e, f \) as in Lemma 3.6 (ii). Then we have \( \beta(\gamma x) = [r(x^*\gamma)] - [r(\gamma x)] = [r(x^*)] - [r(x)] = \beta(x) \). On the other hand it is clear that \( \alpha(\gamma) = 0 \) so \( \alpha(\gamma x) = \alpha(\gamma) + \alpha(x) = \alpha(x) \). Hence we may assume \( \gamma = 1 \). For simplicity we shall write \( y = (1 - e)x \), then we have
\[ yy^* = 1 - e \]
\[ y^*y = 1 - f. \]
It follows from Lemma 0.2 and the remarks preceding it that
\[ \alpha(y) = [(0, e)D] - [(0, f)D] \in K_0(I, A) \]
\[ = [eA] - [fA] \in K_0(I). \]
Hence
\[ \alpha(x) = \alpha(y) = [eA] - [fA] = [r(x^*)] - [r(x)] = \beta(x). \]
Suppose now \( I \subseteq \mathcal{K} \). Then
\[ 1 - e = LP(y) = LP((1 - e)x) \sim 1 - f \]
and, by Lemma 3.3 (iii), we obtain
\[ e \geq LP(ex) \sim RP(ex). \]
Since \( exx^* = xx^*e \) we then get
\[
LP(x) = LP((1 - e)x) + LP(ex) \sim RP((1 - e)x) + RP(ex) \leq RP(x),
\]
so \( LP(x) \leq RP(x) \), for all \( x \in \mathcal{F}(I, A) \). By symmetry \( RP(x) \leq LP(x) \). Now it follows from the generalized Schröder-Bernstein theorem that \( RP(x) \sim LP(x) \).

**Corollary 3.8.** If \( A \) is a semihereditary Rickart \( C^* \)-algebra and \( I \) is an ideal of \( A \), then the connecting map
\[
\delta: K_1(A/I) \to K_0(I)
\]
is defined by
\[
\delta(\bar{X}) = [r(x^*)] - [r(X)]
\]
where \( X \) is any matrix over \( A \) such that modulo \( I \) is an invertible matrix representing \( \bar{X} \in K_1(A/I) \).

**Proof.** Since \( A \) is semihereditary, matrix rings over \( A \) are also semihereditary Rickart \( C^* \)-algebras. The result follows, by using matrices, as in the proof of Proposition 3.7 (ii).

**Theorem 3.9.** Let \( A \) be a Rickart \( C^* \)-algebra and let \( I \) be a closed ideal in \( A \) consisting of compact elements. Then

(i) Let \( \pi: \mathcal{F}(I, A) \to U(A/I) \) be the natural surjection and let \( \lambda \) be the composite map
\[
U(A/I) \to K_1(A/I) \xrightarrow{\delta} K_0(I).
\]
Denote by \( U(A/I)^0 \) the connected component of \( 1 \in U(A/I) \). Then
\[
U(A/I)^0 = \pi(U(A)) = \ker \lambda.
\]

(ii) If \( K_0(I) \) is considered as a discrete group, then the map
\[
\alpha: \mathcal{F}(I, A) \to K_0(I)
\]
\[
x \mapsto [r(x^*)] - [r(x)]
\]
is a continuous monoid homomorphism.

(iii) \( \alpha(\mathcal{F}(I, A)) \) consists of those elements \( z \in K_0(I) \) such that \( z = [eA] - [fA] \) where \( e \) and \( f \) are projections in \( I \) with \( 1 - e \sim 1 - f \). Moreover, two projections \( e, f \) in \( I \) satisfy \( [eA] = [fA] \in K_0(I) \) if and only if \( e \sim f \).
(iv) \( x, y \in \mathcal{F}(I, A) \) lie in the same connected component if and only if \( \alpha(x) = \alpha(y) \). Further \( \alpha \) induces a group isomorphism

\[
U(A/I)/U(A/I)^0 \cong \alpha(\mathcal{F}(I, A))
\]

(v) \( \alpha(x) = 0 \) if and only if \( LP(x) \) and \( RP(x) \) are unitary equivalent.

(vi) \( \alpha(x) = 0 \) if and only if \( x + I \) contains a unit.


Proof. Consider any \( x \in \mathcal{F}(I, A) \). Say \( eA = r(x^*) \) and \( fA = r(x) \), where \( e \) and \( f \) are projections which belong to \( I \). By Proposition 3.7 (i) \( 1 - e = LP(x) \sim RP(x) = 1 - f \). Conversely let \( z = [eA] - [fA] \) with \( 1 - e \sim 1 - f \). Suppose \( x \in A \) is such that \( xx^* = 1 - e \), \( x^*x = 1 - f \).

Certainly \( x \in \mathcal{F}(I, A) \) and \( r(x^*) = eA \), \( r(x) = fA \). Therefore \( \alpha(x) = z \).

Suppose now \( [eA] = [fA] \in K_0(I) \). If for each projection \( g \) we write \( A_g = gAg + C \), then \( I + C \) is the \( C^* \)-direct limit of the \( A_g \)'s for \( g \) in \( I \).

By [9, Theorem 19.9] \( K_0(I + C \cdot 1) = \text{dir.lim. } K_0(gAg + C) \), so \( K_0(I) = \text{dir.lim. } K_0(gAg) \). By Proposition 3.1 \( K_0(gAg) = G(gAg) \), then there exists a projection \( g \) in \( I \) with \( e, f \leq g \) and a finitely generated projective \( A_g \)-module \( C \) such that \( eA_g \oplus C \approx fA_g \oplus C \).

Since \( A_g \) has stable range 1, \( C \) cancels from the direct sums and so \( eA_g \approx fA_g \). Therefore \( e \sim f \). Thus (iii) follows.

(i) Now we compute \( \text{Ker } \lambda \). If \( x \in F(I, A) \) then we shall denote \( \pi(x) \) by \( \bar{x} \). Note that \( \pi(U(A)) \subset \text{Ker } \lambda \). Conversely, if \( \lambda(\bar{x}) = 0 \), then by (iii) \( r(x^*) \approx r(x) \) and with the notation of Lemma 3.6 we have

\[
(1 - e)\gamma xx^*\gamma (1 - e) = 1 - e
\]

and \( e \sim f \). Let \( u \) be a unitary such that \( f = ueu^* \). Then it is easily seen that

\[
((1 - e)\gamma x + u^*f)(x\gamma(1 - e) + fu) = 1
\]

and \( (x\gamma(1 - e) + fu)((1 - e)\gamma x + u^*f) = 1 \),

so \( (1 - e)\gamma x + u^*f \in U(A) \). Hence \( \gamma x - (e\gamma x + u^*f) \in U(A) \). Putting \( i = \gamma^{-1}(e\gamma x + u^*f) \in I \) we have that \( x - i \in U(A) \) and so \( \bar{x} \in \pi(U(A)) \).

Since the unit group of a Rickart \( C^* \)-algebra is connected \( \pi(U(A)) \) also is. If we prove that \( \pi(U(A)) \) is open, then it is clear that \( \pi(U(A)) = U(A/I)^0 \). For this let \( \bar{u} \in \pi(U(A)) \) such that \( \|\bar{u} - 1\| < 1 \). This means that \( \inf_{i \in I} \|(u + i) - 1\| < 1 \). Thus there exists \( i \in I \) with \( \|(u + i) - 1\| < 1 \), then \( u + i \) is a unit and therefore \( \bar{u} \in \pi(U(A)) \).
By Proposition 3.7 (i) \( \alpha = \lambda \pi \). So (ii) and the isomorphism 
\( U(A/I)U(A/I)^0 \cong \alpha(F(I, A)) \) of (iv) follow. In order to end the proof 
of (iv) note that \( \alpha(x) = \alpha(y) \) if and only if \( x \) and \( y \) lie in the same 
connected component of \( U(A/I) \). Since the map \( \pi \) is open and onto the 
result follows.

(v) Suppose \( \alpha(x) = 0 \), then by (iii) \( r(x) \approx r(x^*) \) and since \( LP(x) \sim 
RP(x) \) we see that \( LP(x) \) and \( RP(x) \) are unitary equivalent.

(vi) By (i) it is clear that \( \alpha(x) = 0 \) if and only if \( x \in \pi(U(A)) \). So 
\( \alpha(x) = 0 \) if and only if \( x + I \) contains a unit. \( \square \)

Lemma 3.10. Let \( M \) be a \( 2 \times 2 \) matrix over a ring \( R \). If for some entry 
a in \( M \) there exist \( b, c \) in \( R \) such that \( bac = 1 \), then \( M \) can be reduced by 
elementary transformations to a diagonal matrix.

Proof. There is no loss of generality in assuming that \( M \) is of the form 
\[
M = \begin{pmatrix} * & a \\ * & * \end{pmatrix}
\]
and \( bac = 1 \). Now notice that the matrices 
\[
P = \begin{pmatrix} b & 0 \\ 1 - acb & ac \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} ba & 0 \\ 1 - cba & c \end{pmatrix}
\]
belong to \( GE_2(R) \). But then we have that \( PMQ \) is of the form 
\[
\begin{pmatrix} * & 1 \\ * & * \end{pmatrix},
\]
since this matrix can be reduced to a diagonal one, the same holds for 
\( M \). \( \square \)

Proposition 3.11. Let \( A \) be a Banach algebra satisfying the following 
condition:
For each \( a \in A \) and \( \varepsilon > 0 \) there exists an idempotent \( e \in aA \) and a 
central idempotent \( h \in A \) such that 
(a) \( \|a - ea\| < \varepsilon \)
(b) \( he \sim h \) and \( (1 - h)(1 - e) \sim (1 - h) \). Then \( A \) is a \( GE_2 \)-ring.

Proof. For any Banach algebra [16, Proposition 8.7] we have \( GL_2(A)^0 \) 
\subseteq GE_2(A) \( \subseteq \). Hence \( GE_2(A) \) is clopen. In order to prove that \( GE_2(A) = 
GL_2(A) \) it suffices to note that \( GE_2(A) \) is a dense subset of \( GL_2(A) \). For 
this let 
\[
X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)
\]
and \( \varepsilon > 0 \). Choose an integer \( n \) such that \( n > 1/\varepsilon \). By hypothesis there exist idempotents \( e \) and \( h \), with \( h \) central, such that
\[
\| a - ea \| < 1/n
\]
and
\[
he \sim h \quad \text{while} \quad (1 - h)(1 - e) \sim 1 - h.
\]

Consider now the matrix
\[
M = \begin{pmatrix} ea & b \\ c & d \end{pmatrix}.
\]

Then we have \( \| X - M \| < 1/n < 1/\| X^{-1} \| \). Therefore \( M \in \text{GL}_2(A) \). We claim that \( M \in \text{GE}_2(A) \). Since \( h \) induces a ring decomposition of \( A \), by cutting down to each part we may assume that either (i) \( e \sim 1 \) or (ii) \( 1 - e \sim 1 \). In the first case there exist \( x, y \in A \) such that \( xey = 1 \) and since \( e = az \), for some \( z \in A \), we have \( x( ea ) z y = 1 \). It follows from Lemma 3.10 that \( M \in \text{GE}_2(A) \).

Now suppose that \( 1 - e \sim 1 \). From the relation \( eA + bA = A \) we see that \( 1 - e \in (1 - e)bA \). Hence \( xby = 1 \), for some \( x, y \in A \). The result follows again by using Lemma 3.10. \( \square \)

We say that a Rickart \( C^* \)-algebra \( A \) is purely infinite if 1 is the supremum of a sequence of orthogonal projections all equivalent to 1. It is a simple exercise to see that \( A \) is purely infinite if and only if \( A \approx A^2 \) as right \( A \)-modules.

**Lemma 3.12** (Pere Ara). Let \( A \) be a purely infinite Rickart \( C^* \)-algebra satisfying general comparability. Suppose \( e \) is a projection such that \( e \sim 1 - e \), then \( e \sim 1 \).

**Proof.** Denote by \( \vee \) and \( \wedge \) the operations of taking supremum and infimum respectively. Since \( A \) is purely infinite choose a projection \( f \) such that \( f \sim 1 - f \sim 1 \). Define
\[
g = (1 - e) \wedge f
\]
\[
h = LP(ef) = (1 - e) \vee f - (1 - e).
\]

Since \( A \) satisfies the parallelogram law \([3, \text{Theorem 2.13.1}]\)
\[
(1) \quad hA \oplus gA = ((1 - e) \vee f - (1 - e))A \oplus ((1 - e) \wedge f)A
\]
\[
\approx (f - (1 - e) \wedge f)A \oplus ((1 - e) \wedge f)A
\]
\[
= fA.
\]
Since $h < e$ and $g < 1 - e$, we see that $e - h$ and $(1 - e) - g$ are orthogonal projections, we have

$$ (e - h)A \oplus ((1 - e) - g)A $$

$$ = (e - h)A \oplus ((1 - e) - (1 - e) \wedge f)A $$

$$ \approx (e - h)A \oplus ((1 - e) \vee f - f)A $$

$$ = (e - h)A \oplus (h + 1 - e - f)A = (1 - f)A. $$

Now we shall prove that $e \sim 1$. Since $A$ satisfies general comparability we may assume that either $g \leq e - h$ or $e - h \leq g$. In the first case we have (by using (1)) that

$$ 1 \sim f \sim h + g \leq h + (e - h) = e \leq 1, $$

while in the second case we have (by using (2)) that

$$ 1 \sim 1 - f \sim (e - h) + ((1 - e) - g) \leq g + ((1 - e) - g) $$

$$ = 1 - e \sim e \leq 1. $$

Thus in both cases we see that $1 \leq e \leq 1$. Then the generalized Schröder-Bernstein theorem yields the result. \qed

**Theorem 3.13.** Let $A$ be a purely infinite Rickart $C^*$-algebra satisfying general comparability. If $I$ is an ideal of $A$, then

(i) $K_1(A/I) = U(A/I)/\pi(U(A)) = U(A/I)^{ab}.$

(ii) If $I$ is closed in $A$, then

$$ \pi(U(A)) = U(A/I)^0. $$

(iii) $A/I$ is a GE-ring.

**Proof.** (i) Since $A^2 \approx A$ we have $(A/I)^2 \approx A/I$ as $A/I$-modules. In order to prove that $K_1(A/I) = U(A/I)^{ab}$ it suffices to show cf. [13, Theorem 1.2 (iii)] that $A/I$ is a $GE_2$-ring. In proving this we first assume that $I$ is closed. By noting that the hypotheses in Proposition 3.11 carry over algebra Banach factors, it suffices to verify that the algebra $A$ satisfies (a) and (b) of that proposition. Obviously (a) is an immediate consequence of the spectral theorem [3, Proposition 1.8.4]. For (b), let $e$ be an idempotent in $A$. By general comparability there exists a central idempotent $h$ such that $h(1 - e) \leq he(1)$ and $(1 - h)e \leq (1 - h)(1 - e)$ (2). From the relation (1) we have $hA \leq (heA)^2$. Since $A$ is purely infinite we have also $(heA)^2 \leq hA$. So $hA \approx (heA)^2$ and we can write $hA = e_1A \oplus e_2A$ for some projections $e_1, e_2 \in hA$ such that $e_1 \sim e_2 \sim he$. Then $e_1 \sim h - e_1$ and Lemma 3.12 yields $e_1 \sim h$ so $he \sim h$. Using the relation (2) we have $(1 - h)(1 - e) \sim 1 - h$. Thus we have shown that $A/I$ is a $GE_2$-ring for any closed ideal $I$ of $A$. Now assume $I$ is an arbitrary ideal of $A$. Let $M \in M_2(A)$ such that $M$ is a unit modulo $I$. If $I$ denotes the
closure of $I$ in $A$, then $M$ is a unit modulo $I$ and by the above we may assume, by using elementary transformations, that $M$ is of the form
\[
\begin{pmatrix}
u & 0 \\
0 & *
\end{pmatrix}
\]
where $\nu + I$ is a unit of $A/I$. It is easily seen that $\nu + I$ must be a unit of $A/I$. Now by elementary transformations we can reduce $M$ modulo $I$ to obtain a diagonal matrix. Thus $A/I$ is a $GE_2$-ring. If $A$ is a purely infinite Rickart $C^*$-algebra then $A \cong M_2(A)$ and so $A$ is semihereditary. In particular, by Lemma 3.5, $A$ has polar decomposition.

Now by using that $U(A)$ is a perfect group [13, proof of Theorem 2.10] we can proceed as in the proof of Lemma 1.7 to get $\pi(U(A)) = U(A/I)$' and so (i) follows.

(ii) Since $U(A)$ is connected also is $\pi(U(A))$. As in the proof of Theorem 3.9 we can prove that $\pi(U(A))$ is clopen in $U(A/I)$, so $\pi(U(A)) = U(A/I)^0$.

(iii) Notice that if $I = 0$, then the result follows from [13, Proof of Theorem 2.10] or [16, Theorem 2.10]. Fix $n > 1$. Since $A$ is purely infinite $A \cong M_n(A)$. By applying (i) to $\pi: M_n(A) \rightarrow M_n(A/I)$ we obtain $\pi(GE_n(A)) = GL_n(A/I)'$ and so $GL_n(A/I)' \subseteq GE_n(A/I)$.

Let $M \in GL_n(A/I)$. Since $U(A/I) \rightarrow K_1(A/I)$ is onto, there exists a unit $u \in U(A/I)$ such that
\[
M\begin{pmatrix}
u & 0 \\
1 & . . . . \\
0 & 1
\end{pmatrix} = 0 \in K_1(A/I).
\]
But $K_1(A/I) = U(A/I)^{ab}$ implies
\[
M\begin{pmatrix}
u & 0 \\
1 & . . . . \\
0 & 1
\end{pmatrix} \in GL_n(A/I)',
\]
and by the above we have that $M \in GE_n(A/I)$ as desired. 

**Theorem 3.14.** Let $A$ be a purely infinite Rickart $C^*$-algebra satisfying general comparability. If $I$ is a closed ideal of $A$, then

(i) The map
\[
\alpha: \mathcal{F}(I, A) \rightarrow K_0(I), \quad x \mapsto [r(x^*)] - [r(x)]
\]
is a continuous monoid homomorphism which is onto

(ii) $[r(x^*)] = [r(x)]$ if and only if there exists a projection $e \in I$ such that
\[
r(x^*) \oplus eA \cong r(x) \oplus eA.
\]
(iii) \( x, y \in \mathcal{F}(I, A) \) lie in the same connected component if and only if \( \alpha(x) = \alpha(y) \). Furthermore \( \alpha \) induces a group isomorphism

\[
K_1(A/I) = U(A/I)/U(A/I)^0 \cong K_0(I).
\]

(iv) \( \alpha(x) = 0 \) if and only if \( x + I \) contains a unit.

**Proof.** By Proposition 3.7 (ii) we see that \( \alpha \) is a well-defined monoid homomorphism. Since \( A \) is purely infinite we have [13, Theorem 2.7 (ii) and the proof of Theorem 2.10] that \( K_1(A) = 0 \). Clearly \( K_0(A) = 0 \). Therefore the connecting map \( \delta: K_1(A/I) \to K_0(I) \) is an isomorphism, in particular \( \alpha \) is onto. By Theorem 3.13 \( K_1(A/I) = U(A/I)/U(A/I)^0 \) so \( \alpha \) is continuous. Thus we have shown (i) and a part of (iii). The remainder part of (iii) follows as in Theorem 3.9 (iv).

By Theorem 3.13, (iv) follows.

Now (ii) follows from Proposition 3.1. \( \square \)

If \( A \) is an \( A\mathcal{W}* \)-algebra, then \( A \) decomposes uniquely as a direct product \( A_1 \times A_2 \) where \( A_1 \) is directly finite and \( A_2 \) is purely infinite. Now \( A_1 \) is a ring with stable range 1 so the connecting map associated with each ideal of \( A_1 \) is zero. Therefore we see that Theorem 3.14 is trivially true for \( A_1 \). Since any \( A\mathcal{W}* \)-algebra satisfies general comparability, Theorem 3.14 also holds for \( A_2 \). Thus we have

**Corollary 3.15.** The conclusions of Theorem 3.14 are true for any closed ideal of an \( A\mathcal{W}* \)-algebra. \( \square \)

Finally we remark the following result which is an extension of Corollary 10.7 in [15] to \( A\mathcal{W}* \)-algebras.

**Corollary 3.16.** If \( I \) is an ideal of a \( A\mathcal{W}* \)-algebra \( A \) of Type III, then every unit of \( A/I \) can be lifted to a unit of \( A \). If in addition \( I \) is closed, then \( U(A/I) \) is connected.

**Proof.** Let \( \bar{I} \) be the closure of \( I \) in \( A \). Then since a unit in \( A/\bar{I} \) lifts automatically to a unit of \( A/I \), we may assume without loss of generality that \( I \) is closed. Since \((eA)^2 = eA\) for every idempotent \( e \) in \( I \) we see from Proposition 3.1 that \( K_0(I) = 0 \). By Theorem 3.14 (iii) \( U(A/I) = U(A/I)^0 \) is connected; and by Theorem 3.13 (i) we get \( \pi(U(A)) = U(A/I) \). \( \square \)
Acknowledgment. We would like to express our gratitude to Ken Goodearl for his contributions which led to improvements of an earlier version of this paper.

REFERENCES


Received November 29, 1984 and in revised form December 10, 1985. This work has been supported by a grant from the Comisión Asesora de Investigación Científica y Técnica. Ministerio de Educación y Ciencia, Spain.

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