QUOTIENTS OF NEST ALGEBRAS WITH TRIVIAL COMMUTATOR

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The main result of this paper is to show that every operator $T$ commuting with a nest algebra modulo a two-sided ideal $\mathcal{J}$ of $\mathcal{L}(H)$ is of the form $T = \lambda I + J$ for some $\lambda \in \mathbb{C}$, $J \in \mathcal{J}$.

Introduction. The structure of commutators of non-selfadjoint operator algebras has received considerable interest in recent years [4, 5, 6, 8, 9, 13, 16 and their references] ([7] contains a good survey of known results). However, results for perturbed algebras in general and finite perturbations in particular are not available except for the special case of the ideal $\mathcal{K}$ of all compact operators. To put the results proven here into perspective, we mention two well known and particularly useful special cases. For any subalgebra $\mathcal{A}$ of $\mathcal{L}(H)$ and any subset $\mathcal{M}$ of $\mathcal{L}(H)$, denote by $C(\mathcal{A}, \mathcal{M})$ the collection $\{T \in \mathcal{L}(H): AT - TA \in \mathcal{M} \text{ for every } A \in \mathcal{A}\}$. We now state:

I. (Calkin [3].) Given any ideal $\mathcal{J}$ (two-sided) of $\mathcal{L}(H)$, $C(\mathcal{L}(H), \mathcal{J}) = CI + \mathcal{J}$.

Using the results of Johnson and Parrott [11] on $C(\mathcal{B}, \mathcal{K})$ for $\mathcal{B}$, a type I von Neumann algebra, Christensen and Peligrad were able to show the following.

II. (Christensen and Peligrad [5].) For any nest algebra $\mathcal{A}$, $C(\mathcal{A}, \mathcal{K}) = CI + \mathcal{K}$.

It should be mentioned that II was shown to have an extension to the von Neumann-Schatten $p$-classes in [7].

The central result of this paper is to show that I and II above are "endpoints" of a very general theorem concerning commutators. This result can be stated as:

III. For any nest algebra $\mathcal{A}$ and any ideal $\mathcal{J}$ of $\mathcal{L}(H)$, $C(\mathcal{A}, \mathcal{J}) = CI + \mathcal{J}$. 
Combining III with the main result of [4], we obtain:

IV. Any derivative of a nest algebra into an ideal (two-sided) $\mathcal{I}$ of $\mathcal{L}(H)$ is implemented by an operator from $\mathcal{I}$.

I would like to thank C. Apostol for his helpful conversation.

For the purpose of this paper, $\mathcal{A}$ will denote the nest algebra of all operators acting on a fixed separable Hilbert space $H$ leaving invariant a (complete) totally ordered nest of subspaces $N$. Denote by $\mathfrak{e}$ the corresponding totally ordered nest of orthogonal projections onto the members of $\mathcal{N}$. If $\mathfrak{e} = \{ E_n \}_{n \in \mathbb{Z}}$, let $\Delta_i$ be the orthogonal projection $E_i - E_{i-1}$. $\mathcal{I}$ will denote an arbitrary but non-zero two-sided ideal of $\mathcal{L}(H)$. It is well known [10] that $\mathcal{F} \subseteq \mathcal{I} \subseteq \mathcal{K}$, where $\mathcal{F}$ denotes the ideal of all finite rank operators. (Note that all the results below are obviously true for $\mathcal{I} = (0)$.)

Essential use will be made of the identification between such an ideal $\mathcal{I}$ and its corresponding ideal set $\hat{\mathcal{I}}$ of $s$-numbers in $c_0(N)$ satisfying:

(i) $\{ \lambda_i \}, \{ \mu_i \}$ in $\hat{\mathcal{I}}$ implies $\{ \lambda_i + \mu_i \}$ in $\hat{\mathcal{I}}$.

(ii) $\{ \lambda_i \} \in \hat{\mathcal{I}}$ and $0 \leq \mu_i \leq \lambda_i$ for every $i \in N$ implies $\{ \mu_i \} \in \hat{\mathcal{I}}$.

(iii) For any permutation $\pi: N \to N$, $\{ \lambda_i \}$ in $\hat{\mathcal{I}}$ implies that $\{ \lambda_{\pi(i)} \}$ is in $\hat{\mathcal{I}}$.

This identification is given by $s: T \to \sigma((T^*T)^{1/2})$. We will use the standard notation $s_j(T)$ for the $j$th eigenvalue of $(T^*T)^{1/2}$. Given $T$ in $\mathcal{L}(H)$, denote by $\delta_T$ the map from $\mathcal{A}$ to $\mathcal{L}(H)$ given by $\delta_T(A) = AT - TA$. Let $x \otimes y$ be the rank one operator $(x \otimes y)z = \langle z, x \rangle y$. By c.l.s. $\{ S \}$ will be meant the closed linear span in the norm topology of the set $S$.

**Commutants of nest algebras modulo $\mathcal{I}$**. In order to prove III, we initially divide the problem into three cases:

(i) There exists a projection $E$ into $\mathfrak{e}$ with infinite range and kernel.

(ii) There exists an increasing sequence $\{ E_n \}_{n=0}^{\infty}$ of finite dimensional projections in $\mathfrak{e}$, with $E = \sup E_n$ having finite dimensional kernel.

(iii) There exists a decreasing sequence $\{ E_n \}_{n=0}^{\infty}$ of finite co-dimensional projections in $\mathfrak{e}$, with $E = \inf E_n$ having finite range.

**Case** (i). As in [5] we note that there will exist a partial isometry $V$ in $\mathcal{A}$ with $VV^* = E$ and $V^*V = I - E$. Thus both $E\mathcal{L}(H)EV$ and $V(I - E)\mathcal{L}(H)(I - E)$ are subsets of $\mathcal{A}$. Let $\delta_K$ be a (bounded) derivation from $\mathcal{A}$ into $\mathcal{I}$. For any $X$ in $\mathcal{L}(H)$, $\delta_K(EXEV) = \delta_K(EXE)V + EXE\delta_K(V)$, it will immediately follow that $\delta_K(EXE)E$ is
in \mathcal{J}. Define the ideal \mathcal{J}_1 of \mathcal{L}(EH) to be

\[ \mathcal{J}_1 = \{ ETE : T \in \mathcal{J} \}. \]

Consider the action of \( \delta_{EKE} \) on \( \mathcal{L}(EH) \). For any \( X \) in \( \mathcal{L}(H) \),

\[ \delta_{EKE}(EXE) = E(XEK - KEX)E = E\delta_K(EXE)E. \]

Thus \( \delta_{EKE} \) derives \( \mathcal{L}(EH) \) into \( \mathcal{J}_1 \). An application of I above will show that \( EKE = \lambda E + T \) for some \( T \) in \( \mathcal{J}_1 \). An exactly similar argument will show that \( (I - E)K(I - E) \) is of the form \( \mu(I - E) + T \), where \( T_2 = (I - E)T_2(I - E) \) for some \( T_2 \in \mathcal{J} \). In addition, \( EK(I - E) = E\delta_K(E)(I - E) = ET_3(I - E) \) with \( T_3 \in \mathcal{J} \). Similarly, \( (I - E)KE = (I - E)\delta_K(I - E)E = (I - E)ET_4E \) with \( T_4 \in \mathcal{J} \). Therefore, \( K \) can be written as:

\[ K = \begin{bmatrix} \lambda & \mu \\ T_1 & T_4 \\ T_3 & T_2 \end{bmatrix}, \]

where the second term \( T \) is in \( \mathcal{J} \). All that remains is to show \( \lambda = \mu \).

Note, however, that since \( V \in \mathcal{A} \), we have

\[ (\lambda E + \mu(I - E) + T)V = V(\lambda E + \mu(I - E) + T) \in \mathcal{J}. \]

It immediately follows that \( (\lambda - \mu)E \in \mathcal{J} \), showing \( \lambda = \mu \).

**Case (ii).** In order to prove case (ii), it will be necessary to further subdivide case (ii) into (ii) (a) \( \mathcal{J} \neq \mathcal{F} \) and (ii) (b) \( \mathcal{J} = \mathcal{F} \). Before beginning the proof of either, we note that it may as well be assumed that \( \mathcal{E} \) is the classical nest of one-dimensional jumps on \( l^2(N) \). That is, with respect to the usual basis \( \{ e_j \}_{n=1}^{\infty} \), \( E_n \) is given as the projection onto the closed linear span of \( \{ e_j \}_{j=1}^{n} \).

**Case (ii)a.** Let \( \delta_K : \text{Alg}(E_n) \to \mathcal{J} \). It follows from II that we can assume \( K \) is compact. Fix a \( c_0(N) \) sequence \( \{ \varepsilon_i \} \) in \( \mathcal{J} \) satisfying \( \varepsilon_1 > \varepsilon_2 > \cdots > 0 \). Define a partial isometry \( A \) in \( \mathcal{A} \) by \( A* e_i = e_n \), where \( n_i > n_{i-1} \) and \( \| \Delta_{n_i}AK \| < 2^{-i}\varepsilon_i \). That this is possible follows from the compactness of \( K \) and the observation that \( (I - E_n) \downarrow 0 \) strongly. It can now be seen that \( AK \) is the operator with the property that \( \Delta_n AK = \Delta_n K \). We claim that \( s(AK) \) is dominated by \( \{ \varepsilon_i \} \), and thus \( AK \in \mathcal{F} \) by (iii). That this holds is an application of [1]. Indeed we have

\[ s_{n+1}(AK) \leq \|(I - E_n)AK\| \leq \sum_{j=n+1}^{\infty} \| \Delta_j AK \| < \varepsilon_{n+1} \]

since, in particular, rank \( E_n AK \leq n \).
Thus, necessarily $KA$ is also in $\mathcal{J}$. Moreover,

$$s(KA) = s(A^*K^*) = \sigma[(KAA^*K^*)^{1/2}] = \sigma[(KK^*)^{1/2}] = s(K),$$

showing $K$ is also in $\mathcal{J}$.

**Case (ii)b.** It is not too difficult to show that this result follows from case (ii)a using the fact that $\delta_T(A) \not\in \mathcal{F}$ for a given $T \in CI + \mathcal{F}$. However, the following proof is of independent interest in that it provides a concrete example of an operator $A$ such that $\{ \delta_T(A) \not\in \mathcal{F} \}$ for a given $T \in CI + \mathcal{F}$. Since $\delta^O(A) \subseteq \mathcal{F}$ if and only if $\delta^O(A^*) \subseteq \mathcal{F}$, it may as well be assumed that $\mathcal{A}$ is the algebra of all (bounded) lower triangular matrices with respect to the basis $\{ e_n \}$. Let $\delta_T : \mathcal{A} \to \mathcal{F}$. Suppose, contrary to the assertion of III, that $T \not\in CI + \mathcal{F}$. We shall construct sequences $\{ x_n \}$, $\{ y_n \}$ of unit vectors together with associated projections $E_{m(n)}$ and $E_{j(n)}$ satisfying

1. $\langle x_j, x_k \rangle = \langle Tx_j, x_k \rangle = 0$ for $j \neq k$.
2. $x_n = E_{m(n)}x_n$ and $y_n = (E_{j(n)} - E_{m(n)})y_n$.
3. $\{ Ty_k - \langle Tx_k, x \rangle y_k \}_{k=1}^n$ are linearly independent vectors for each $n \in \mathbb{N}$. The construction is an inductive one.

$k = 1$. Let $x_1 = e_1$. If for every $e_j$, $j > 1$, $Te_j = \langle Te_1, e_1 \rangle e_j$, it will immediately follow that $T = \langle Te_1, e_1 \rangle I + K$ for $K$, a rank two operator, contrary to our assumption. Take $y_1 = e_k$, where $k$ is the first integer with $Te_k \not= \langle Te_1, e_1 \rangle e_k$. It is easily seen that $(x_1, y_1)$ satisfies (i), (ii) and (iii) above.

$k = n$ implies $k = n + 1$. Suppose that $\{ x_i \}_{i=1}^n$ and $\{ y_i \}_{i=1}^n$ have been chosen to satisfy (i) through (iii). Let $H_n$ be c.l.s. $\{ x_1, \ldots, x_n, Tx_1^*, \ldots, Tx_n^* \}$ and note that $E_{2n+1}(H_n) \subsetneq E_{2n+1}(H)$. From this we deduce the existence of a unit vector $x_{n+1} = E_{2n+1}x_{n+1}$ satisfying (i) for $j, k \leq n + 1$. Take $E_{m(n+1)} = E_{2n+1}$.

Define $\tilde{H}_n$ to be c.l.s. $\{ y_1, \ldots, y_n, Ty_1^*, \ldots, Ty_n^* \}$ and $\lambda = \langle Tx_{n+1}, x_{n+1} \rangle$. Suppose that, for every $I > E \geq E_{m(n+1)}$ and $y \in (E - E_{m(n+1)})\tilde{H}_n$, $Ty - \lambda y$ is in $\tilde{H}_n$. It would immediately follow that $(T - \lambda)(I - E_{m(n+1)}) \in \mathcal{F}$. That is, $T = \lambda I + F$ for some $F$ in $\mathcal{F}$, contrary to our assumption. Thus, for some $j(n + 1) > m(n + 1)$, we have both $y_{n+1} \in (E_{j(n+1)} - E_{m(n+1)})H$ and $Ty_{n+1} - \lambda y_{n+1} \in \tilde{H}_n$.

Let $A$ be the operator

$$A = \sum_{n=1}^{\infty} x_n \otimes y_n.$$
Now each $x_n \otimes y_n$ is in $\mathcal{A}$ and $\mathcal{A}$ is strongly closed; therefore, $A \in \mathcal{A}$. Consider the vector $w_k = (TA - AT)x_k = Ty_k - \langle Tx_k, x_k \rangle y_k$. From (iii) it follows that, for each $n$, $\{ w_k \}_{k=1}^n$ are linearly independent vectors in the range of $\delta_T(A)$.

Case (iii). If $X$ derives $\mathcal{A}$ into $\mathcal{J}$, then $X^*$ derives $\mathcal{A}^*$ into $\mathcal{J}$. Since $\mathcal{A}^* = \text{Alg}\{ I - E_n \}$, where $\{ I - E_n \}$ satisfies the hypotheses of case (ii), we obtain case (iii).

In order to prove IV, we simply combine III with the main result of [4], which says that any derivation of a nest algebra into $\mathcal{L}(H)$ is inner.

**Corollary.** It easily follows that for any generalized commutator pair $AB$, with $AT - TB$ in $\mathcal{J}$ for all $T$ in $\mathcal{A}$ implies $A, B$ are both in $CI + \mathcal{J}$.

**Remark.** There has been considerable recent interest in automorphisms of perturbed algebras [14], determining under which circumstances an automorphism of $\mathcal{A} + \mathcal{J}$ is inner. For nests indexed by $\mathbb{N}$ and $\mathcal{J} = \mathcal{N}$, it is shown in [14] that every automorphism is inner. In the general situation there will exist outer automorphisms (for example, the bilateral shift acting on the classical nest of one-dimensional jumps indexed by $\mathbb{Z}$). Indeed, it is shown in [16] and [6] that these have a rather rich structure being isomorphic to the group of all dimension preserving order isomorphisms of the underlying nest. However, a key to all these results is the fact [2] that $\mathcal{A} + \mathcal{N}$ is precisely all operators $T$ in $\mathcal{L}(H)$ such that $E \to (I - E)TE$ is continuous from $\mathcal{E}$ (strong operator topology) to $\mathcal{N}$ (norm topology). In the situation of arbitrary (two sided) ideals, this does not hold even for tractable classes such as symmetrically normed ideals [12].

**References**


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