Pacific Journal of Mathematics

THE DUAL PAIR $(U(3), U(1))$ OVER A p-ADIC FIELD

Courtney Hughes Moen

Vol. 127, No. 1 January 1987

PACIFIC JOURNAL OF MATHEMATICS
Vol. 127, No. 1, 1987

THE DUAL PAIR ($U(3)$, $U(1)$) OVER A p-ADIC FIELD

COURTNEY MOEN

This paper considers some aspects of the oscillator representation of the dual reductive pair $(U(3), U(1))$ over a p-adic field, with p odd.

1. Introduction. One of the simplest examples of a dual pair arising from Howe's general construction [**Ho**] is that of $(U(3), U(1))$. We will study this pair in the p-adic case, with $p \neq 2$. We first give the details of what the construction provides in this case and review some necessary results from [**M**] concerning the dual pair $(U(1), U(1))$. We then consider the irreducible constituents of the oscillator representation restricted to $U(3)$. We first determine which of these constituents embed in principal series, and we find some explicit information concerning these embeddings. We next show that each irreducible supercuspidal constituent is induced from a representation of a maximal compact subgroup of $U(3)$. A surprising feature is that in all cases except that in which $U(3)$ is defined over an unramified extension and we are considering representations of conductor one, the group over the ring of integers does not suffice, and we must use the other class of maximal compact subgroups.

The results in this paper concerning principal series were discovered originally by Howe and Piatetskiĭ-Shapiro and appear in [**GPS**], where they play a role in some of the authors' important results concerning automorphic forms in $U(3)$. The methods of this paper borrow heavily from [**A**] and [**Ho**]. I would also like to thank C. Asmuth for a useful conversation.

2. Basic construction. Let F be a p-adic field, with $p \neq 2$. Let $\mathcal{O}$ be the ring of integers, P the prime ideal, U the units, ν the additive valuation, and π a prime element. $E = F(\sqrt{\alpha})$ will be a quadratic extension of F, with $\mathcal{O}_E$, P_E, U_E, ν_E, and π_E the corresponding objects for E. Let q be the order of $\mathcal{O}/P$.

Let h_1 be the 3-dimensional Hermitian form over E defined by

$$h_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let h_2 be the 1-dimensional form defined by $h_2(x, y) = xy^\sigma$, where $y \to \bar{y} = y^\sigma$ is the Galois action of E/F. Let $U(3)$ and $U(1)$ be the associated unitary groups.

We will now recall the formalism of reductive dual pairs as applied in this case [**Ho**]. Consider two unitary spaces (W_1, h_1) and (W_2, h_2) over E. Let $V = W_1 \otimes_E W_2$ have the Hermitian form $h(w_1 \otimes w_2, w_1' \otimes w_2') = h_1(w_1, w_1')h_2(w_2, w_2')$. Define a skew-symmetric form j on V over F by $j(v_1, v_2) = \beta(h(v_1, v_2) - h(v_2, v_1))$, where $\beta \in E$ satisfies $\beta + \beta^\sigma = 0$. In our case, we will identify $W_1 \otimes_E W_2$ with W_1.

Let W_1 have basis $\{e_1, e_0, e_{-1}\}$ over E, and let a basis for V over F be $\{\sqrt{\alpha}\, e_1, e_1, e, -\sqrt{\alpha}\, e, e_{-1}, -\sqrt{\alpha}\, e_{-1}\}$. With respect to this basis, the matrix of the symplectic form j is

$$J = 2\alpha \left|\begin{array}{c|cc|c} 0 & \multicolumn{2}{c|}{0} & I \\ \hline 0 & 0 & 1 & 0 \\ & -1 & 0 & \\ \hline I & \multicolumn{2}{c|}{0} & 0 \end{array}\right|$$

Recall that as generators of $U(3)$ we may take elements of the following form:

(1) $$m(a) = \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-\sigma} \end{pmatrix}, \qquad a \in E^x;$$

(2) $$t(\tau) = \begin{pmatrix} 1 & & \\ & \tau & \\ & & 1 \end{pmatrix}, \qquad \tau \in N^1_{E/F};$$

(3) $$n(b) = \begin{pmatrix} 1 & 0 & b\sqrt{\alpha} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b \in F;$$

(4) $$u(c) = \begin{pmatrix} 1 & -c^\sigma & -\dfrac{N(c)}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \qquad c \in E;$$

(5) $$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have an injection $\phi\colon U(3) \to \mathrm{Sp}(V)$ which acts on the generators as follows:

(1) $$\phi(m(a)) = \left|\begin{array}{c|c|c} \begin{matrix} a_1 & a_2 \\ \alpha a_2 & a_1 \end{matrix} & & \\ & I & \\ \hline & & \begin{matrix} \dfrac{a_1}{Na} & \dfrac{-\alpha a_2}{Na} \\ \dfrac{-a_2}{Na} & \dfrac{a_1}{Na} \end{matrix} \end{array}\right|, \qquad a = a_1 + \sqrt{\alpha}\, a_2;$$

(2) $$\phi(t(\tau)) = \left|\begin{array}{c|c|c} I & & \\ \hline & \begin{matrix} \tau_1 & -\alpha\tau_2 \\ -\tau_2 & \tau_1 \end{matrix} & \\ \hline & & I \end{array}\right|, \qquad \tau = \tau_1 + \sqrt{\alpha}\,\tau_2;$$

(3) $$\phi(n(b)) = \left|\begin{array}{c|c|c} I & & \begin{matrix} b & 0 \\ 0 & -b\alpha \end{matrix} \\ \hline & I & \\ \hline & & I \end{array}\right|;$$

(4) $$\phi(u(c)) = \left|\begin{array}{c|c|c} I & \begin{matrix} c_2 & c_1 \\ -c_1 & -\alpha c_2 \end{matrix} & \begin{matrix} 0 & Nc/2 \\ -Nc/2 & 0 \end{matrix} \\ \hline & I & \begin{matrix} c_1 & -\alpha c_2 \\ -c_2 & c_1 \end{matrix} \\ \hline & & I \end{array}\right|, \qquad c = c_1 + \sqrt{\alpha}\,c_2;$$

(5) $$\phi(w) = \left|\begin{array}{c|c|c} & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \\ \hline & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \\ \hline \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \end{array}\right|.$$

We will identify $U(3)$ and $\phi(U(3))$.

3. The oscillator representation. Let $\tilde{W}_1$ be the lattice model of $\mathrm{SL}_2(F)$ on $\mathscr{F} \subset \mathscr{S}(E)$ as outlined in **[Ma]**, where $\mathscr{S}(E)$ is the space of locally constant, compactly supported functions on E. Let $\tilde{W}$ be the oscillator representation of $\mathrm{Sp}(V)$ on $\mathscr{S}(F^3)$ as given in **[PS]**. According to **[Ra]**, the cocyle associated to $\tilde{W}$ splits upon restriction to $U(3) \subset \mathrm{Sp}(V)$. We thus obtain by restriction a representation of $U(3)$.

We now give explicit operators for the generators of $U(3)$. Let χ be a character of F^+, $\eta(x) = \chi(\frac{1}{2}x)$, and $\omega(\eta) =$ conductor of η. $\kappa(\eta) = 1$ if $\omega(\eta)$ is even and $\kappa(\eta) = G(\eta)$ if $\omega(\eta)$ is odd, where $G(\eta) = q^{-1/2}\Sigma_{x \in O/P}\eta(\pi^{\omega(n)}x^2)$. Let ϕ be an element of $\mathscr{S}(E, \mathscr{F})$, the space of locally constant, compactly supported functions on E which take values of $\mathscr{F}$.

(1) $(\tilde{W}(m(a), 1)\phi(x) = (\kappa(\eta)/\kappa(\eta_{Na}))|Na|^{1/2}\phi(xa)$, where $|\cdot|$ is the absolute value on F, and $\eta_s(x) = \eta(sx)$.

(2) $(\tilde{W}(n(b), 1)\phi)(x) = \chi(\alpha b N(x))\phi(x)$.

(3)′ $(\tilde{W}(t(\tau), 1)\phi)(x) = \tilde{W}_1(\tau, 1)(\phi(x))$, where here we consider $\tau \in U(1)$ as an element of the corresponding torus in $\mathrm{SL}(2, F)$.

(4) $(\tilde{W}(u(c),1)\phi)(x) = \rho(-cx)(\phi(x))$, where ρ is the restriction to E of the representation with central character χ of the Heisenberg group attached to E.

(5) Write $\phi(w) = g_1 g_2$, where

$$g_1 = \begin{pmatrix} A & & \\ & I & \\ & & {}^tA^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} & & I \\ & I & \\ -I & & \end{pmatrix},$$

with $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. $\tilde{W}(g_1 g_2, 1) = \beta(g_1, g_2)\tilde{W}(g_1, 1)\tilde{W}(g_2, 1)$, where β is the cocycle attached to $\tilde{W}$.

We have

$$\left(\left(\tilde{W}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, 1\right)\phi\right)(x) = \frac{1}{[\kappa(n)]^2}\int_{F^2} \chi\left(\left\langle y, x\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right\rangle\right)\phi(y)\,dy$$

$$= \hat{\phi}(x).$$

Our symplectic form is $J = 2\alpha\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, so $\langle v, w\rangle = vJ{}^tw = 2\alpha(x \cdot y)$. Also,

$$\left(\tilde{W}\left(\begin{pmatrix} A & & \\ & I & \\ & & {}^tA^{-1} \end{pmatrix}, 1\right)\phi\right)(x) = \phi(x_2, -x_1)$$

where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $x = (x_1, x_2)$. Therefore,

$$(\tilde{W}(w,1)\phi)(x) = \pm\frac{1}{[\kappa(\eta)]^2}\hat{\phi}(x_2, -x_1).$$

$U(1) \cong N^1$ acts on E by multiplication, so $\tau_1 + \sqrt{\alpha\tau_2} = \tau \in U(1)$ embeds in $\mathrm{Sp}(V)$ as

$$\begin{bmatrix} \tau_1 & \tau_2 & & & & \\ \alpha\tau_2 & \tau_1 & & & & \\ & & \tau_1 & -\alpha\tau_2 & & \\ & & -\tau_2 & \tau_1 & & \\ & & & & \tau_1 & -\alpha\tau_2 \\ & & & & -\tau_2 & \tau_1 \end{bmatrix},$$

which is just $\phi\begin{pmatrix} 1 & & \\ & \tau & \\ & & 1 \end{pmatrix}\phi\begin{pmatrix} \tau & & \\ & 1 & \\ & & \tau \end{pmatrix} = \phi\begin{pmatrix} \tau & & \\ & \tau & \\ & & \tau \end{pmatrix}$.

We will thus identify $U(1)$ with

$$\left\{\begin{pmatrix} \tau & & \\ & \tau & \\ & & \tau \end{pmatrix} \,\middle|\, \tau \in N^1\right\}.$$

Let $s: U(3) \to \mathbf{Z}_2$ be the coboundary which splits the restriction of β to $U(3)$. Setting $T(g) = s(g)\tilde{W}(g, 1)$ for $g \in U(3)$, we see that T is a representation of $U(3)$. Let $H = \mathscr{S}(E, \mathscr{F})$ and for any character θ of $U(1)$, let

$$H^\theta = \left\{ \phi \in H \,|\, T\begin{pmatrix} \tau & & \\ & \tau & \\ & & \tau \end{pmatrix} = \theta(\tau)\phi \right\}.$$

The following result is proved in $[H_0]$:

THEOREM 3.1. *H^θ is an irreducible representation of $U(3)$.*

COROLLARY 3.2. *H^θ is irreducible upon restriction to* SU(3)

Proof. The proof of the theorem uses only operators corresponding to elements of SU(3).

4. Results concerning the dual pair $(U(1), U(1))$. Recall that $\tilde{W}_1$ is the oscillator representation of $\mathrm{SL}_2(F)$. We have the dual pair $(U(1), U(1))$ in $\mathrm{SL}_2(F)$. Let T be the compact torus of $\mathrm{SL}_2(F)$ which is isomorphic to $U(1)$. The following facts concerning the restriction of $\tilde{W}_1$ to T are proved in [M].

Let us first assume E/F is unramified. Choose $\theta \in \hat{T}$. Let $\omega(\theta)$, the conductor of θ, be the smallest integer n such that θ is trivial on $T_n = \{t \in T \,|\, t \equiv I(P^n)\}$. The conductor of the trivial character is zero. Let θ_0 be the unique nontrivial character whose square is 1. The conductor of θ_0 is 1. Recall that $\chi \in \hat{F}^+$, and $\omega(\chi) =$ conductor of χ.

PROPOSITION 4.1. (1) *If $\omega(\chi)$ is even, then θ appears in $\tilde{W}_1|_T \Leftrightarrow \omega(\theta)$ is even.*

(2) *If $\omega(\chi)$ is odd, then θ appears in $\tilde{W}_1|_T \Leftrightarrow \theta = 1$, or $\omega(\theta)$ is odd with $\theta \neq \theta_0$.*

Now we assume E/F is ramified, $E = F(\sqrt{\pi})$. For $n \geq 0$, let

$$T_n = \left\{ \begin{pmatrix} a & b \\ \pi b & a \end{pmatrix} \middle| a \in 1 + P^{2n+1},\ b \in P^n \right\}$$

be the filtration of T which defines $\omega(\theta)$. Let θ_0 be the unique nontrivial character of T which is one on T_0, and let 1 denote the trivial character.

PROPOSITION 4.2. (1) 1 *appears in* $\tilde{W}_1 \mid_T \Leftrightarrow \omega(\chi)$ *is even or* $(-1/p) = 1$, *where for* $a \in F^x$, $(a/p) = \pm 1$ *as a is or is not a square.*

(2) θ_0 *appears* $\Leftrightarrow$ 1 *does not appear.*

(3) *Exactly half of the characters of a given conductor appear.*

5. Embeddings in principal series. In this section we will consider the restriction of T to SU(3) since the reducibility criteria for this group have been explicitly worked out in [**K**].

Since

$$m = \begin{pmatrix} a & & \\ & a^\sigma a^{-1} & \\ & & a^{-\sigma} \end{pmatrix} = m_1 m_2, \quad \text{where } m_1 = \begin{pmatrix} a^2 a^\sigma & & \\ & 1 & \\ & & a^{-2\sigma} \end{pmatrix} \text{ and } m_2 = \begin{pmatrix} \tau & & \\ & \tau & \\ & & \tau \end{pmatrix},$$

with $\tau = a^\sigma a^{-1}$, we have

$$T(m) = s(m)\tilde{W}(m,1) = s(m)\tilde{W}((I,\beta(m_1,m_2))(m_1,1)(m_2,1)).$$

Recall that s was the splitting coboundary. If we assume $\phi \in H^\theta$, then $T(m)\phi = s(m)\beta(m_1,m_2)\tilde{W}(m_1,1)\theta(\tau)\phi$. But

$$(\tilde{W}(m_1,1)\phi)(x) = \frac{\kappa(n)}{\kappa(\eta_{Na})}|Na|^{1/2}\phi(xa^2a^{-\sigma}),$$

so

$$(T(m)\phi)(x) = \theta(t)s(m)\beta(m_1,m_2)\frac{\kappa(\eta)}{\kappa(\eta_{Na})}|Na|^{1/2}\phi(xa^2a^{-\sigma}).$$

Let us denote the quantity $s(m)\beta(m_1,m_2)(\kappa(\eta)/\kappa(\eta_{Na}))$ by $\rho(a)$, so that we write $(T(m)\phi)(x) = \theta(a^\sigma a^{-1})\rho(a)|Na|^{1/2}\phi(xa^2a^{-\sigma})$. Since T is a homomorphism, ρ is a character of E^x.

Recall that if $\lambda \in \hat{E}^x$, the principal series $\mathrm{Ind}_P^G\lambda = \{f\colon \mathrm{SU}(3) \to \mathbf{C} \mid f(nmg) = |Na|\lambda(a)f(g)\}$. Define a map $\mathscr{A}$ on H^θ by $(\mathscr{A}\phi)(g) = (T(g)\phi)\binom{0}{0}$ for $g \in \mathrm{SU}(3)$. Then $\mathscr{A}$ maps H^θ into $\mathrm{Ind}_P^G\lambda \Leftrightarrow \lambda(a) = \rho(a)\theta(a^\sigma a^{-1})|Na|^{-1/2}$.

PROPOSITION 5.1. *There exists* $\phi \in H^\theta$ *such that* $\mathscr{A}\phi \neq 0 \Leftrightarrow \theta$ *appears in* $\tilde{W}_1$ *restricted to* pT.

Proof. If θ appears, $\exists f \neq 0$ in $\mathscr{F}$ such that $\tilde{W}_1(t)f = \theta(t)f\ \forall t \in T$. To construct ϕ, define $\phi(y) = f$ for $y \in \mathscr{O}_E$, $\phi(y) = 0$ for $y \notin \mathscr{O}_E$. Note that $\phi \in H^\theta \Leftrightarrow \tilde{W}_1(t)\phi(ty) = \theta(t)\phi(y)$ for all $t \in T$, $y \in E$, which is satisfied since $\tilde{W}_1(t)\phi(ty) = \tilde{W}_1(t)f = \theta(t)f = \theta(t)\phi(y)$ if $y \in \mathscr{O}_E$, and both sides are zero if $y \notin \mathscr{O}_E$. Now suppose $\exists\phi \in H^\theta$ with $\mathscr{A}\phi \neq 0$.

Choose g with $(\mathscr{A}\phi)(g) = (T(g)\phi)(0) \neq 0$. Let $\phi' = T(g)\phi \in H^\theta$. Then $\tilde{W}_1(t)\phi'(ty) = \theta(t)\phi'(y)$ for all $t \in T$, $y \in E$. Letting $y = 0$ and $f = \phi'(0)$, we see $f \neq 0$ and $\tilde{W}_1(t)f = \theta(t)f$ $\forall t \in T$, showing that θ appears in $\tilde{W}_1 \mid_T$.

LEMMA 5.2. *The character* ρ *of* $E^\times$ *is of order* 2.

Proof. Since $s(m)$ and $\beta(m_1, m_2)$ are each ± 1, it remains only to show $\kappa(\eta)/\kappa(\eta_{Na}) = \pm 1$, and this is an easy calculation.

PROPOSITION 5.3. *Let* ε_E *be a non-square unit in* E, *and let* $(x, y)_E$ *be the Hilbert symbol on* E.

(1) *If* E/F *is unramified*, $\rho(a) = (\varepsilon_E, a)_E$ *or* $\rho(a) = (\pi\varepsilon_E, a)_E$.

(2) *If* E/F *is ramified*, $\rho(a)$ *is one of the two ramified characters of order two.*

Proof. Recall that H^θ is unitarizable. If θ appears in $\tilde{W}_1 \mid_T$, then H^θ embeds in $\mathrm{Ind}_P^G \lambda$, $\lambda(a) = \rho(a)(a^\sigma/a)|Na|^{-1/2}$. Suppose E/F is unramified. Assume $\rho(a) = 1$ and choose $\theta = 1$. Then H^θ embeds in $\mathrm{Ind}_P^G |Na|^{-1/2}$. But it is shown in **[K]** that $\mathrm{Ind}_P^G |Na|^{-1/2}$ is irreducible and nonunitarizable, so ρ cannot be the trivial character. Assume $\rho(a) = (\pi, a)_E$. Note that any ramified character λ with $\mathrm{Ind}_P^G \lambda$ reducible satisfies $\lambda(\pi) = -q^{-1}$. Choosing $\theta = 1$ shows H^θ embeds in $\mathrm{Ind}_P^G \lambda$ with $\lambda(a) = \rho(a)|Na|^{-1/2}$, which implies $\mathrm{Ind}_P^G \lambda$ is reducible. But for this choice of λ, $\lambda(\pi) = q$, which is a contradiction. A similar argument works for the ramified case.

PROPOSITION 5.4. *If* $T_\lambda = \mathrm{Ind}_P^G \lambda$ *is reducible, then* $\exists \theta \in \hat{N}^1$ *and* $\chi \in \hat{F}^+$ *such that* H^θ, *constructed with* χ, *embeds in* T_λ *with nonzero image, with the following exceptions*:

(1) *If* E/F *is unramified*, $\lambda(a) \neq |a|_E^{-1}$ *and* $\lambda(a) \neq \rho(a)(\pi, a)_E |a|_E^{-1/2}$.

(2) *If* E/F *is ramified*, λ *must be ramified.*

Proof. (1) Suppose E/F is unramified. We already know that no H^θ can embed in T_λ if $\lambda(a) = |a|_E^{-1}$. If $\lambda(a) \neq |a|_E^{-1}$ and T_λ is reducible, then writing $\lambda(a) = \lambda_0(a)|a|_E^s$, $\lambda_0|_{F^\times} = 1$, $s = -1/2 + (\pi i/2 \ln q)$, H^θ embeds in $T_\lambda \Leftrightarrow \theta(a^\sigma/a) = \rho(a)\lambda_0(a)(\varepsilon_E, a)_E$. Given $t \in N^1$, choose a such that $t = a^\sigma a^{-1}$ and define $\theta(t) = \rho(a)\lambda_0(a)(\varepsilon_E, a)_E$. Then θ is well-defined and θ appears in $\tilde{W}_1 \mid_T$ for some $\chi \in \hat{F}^+ \Leftrightarrow \theta \neq \theta_0$. Note that $\theta_0(a) = (\pi, a)_E$, so $\theta = \theta_0 \Leftrightarrow \lambda_0(a) = \rho(a)(\pi, a)_E |a|_E^{-1/2}$, which we therefore exclude.

(2) Suppose E/F is ramified, λ is ramified, and T_λ is reducible. Then $\lambda|_U$ is of order 2 and $s = -1/2$, so $\lambda(a) = \lambda_0(a)|a|_E^{-1/2}$. We must have $\theta(a^\sigma/a) = \rho(a)\lambda_0(a)$, so given $t \in N^1$, choose a such that $t = a^\sigma a^{-1}$ and let $\theta(t) = \rho(a)\lambda_0(a)$. θ is then well-defined, and a calculation shows that θ must appear in $\tilde{W}_1|_T$. Note that no H^θ can embed in T_λ if $\lambda = |a|_E^{-1}$.

PROPOSITION 5.5. *If H^θ embeds in T_λ, then T_λ is reducible.*

Proof. We need only compare the λ for which an H^θ embeds in T_λ with the list of reducible principal series given in [**K**].

PROPOSITION 5.6. *θ appears in $\tilde{W}_1|_T \Leftrightarrow H^\theta$ is not supercuspidal.*

Proof. If θ appears, then H^θ embeds in a principal series representation, so H^θ is not supercuspidal. Now suppose θ does not appear. Recall that this implies that $\phi(0) = 0\ \forall\phi \in H^\theta$. We want to show that if $\phi \in H^\theta$, the function $g \to \langle T(g)\phi, \phi\rangle = \int\langle T(g)\phi(x), \phi(x)\rangle\, dx$ is compactly supported. ϕ is K-finite, so choose K' whose action under T fixes ϕ. Let $\{k_1, \dots, k_l\}$ be coset representatives of K' in K. Each $\phi_i = T(k_i)\phi$ is in H^θ, so $\phi_i(0) = 0$. Each ϕ_i is locally constant and compactly supported, so $\phi_i(x) = 0$ for each i if x is in a sufficiently small neighborhood of 0. Choose L to be a compact open set containing the supports of all the ϕ_i. If E/F is unramified, we have the Cartan decomposition

$$G = \bigcup_{r\geq 0} Kd_rK, \quad d_r = \begin{pmatrix} \pi^r & & \\ & 1 & \\ & & \pi^{-r} \end{pmatrix}.$$

Choose $g \in Kd_rK$. Then for some $h, h' \in K'$ and some i, j, we have $g = k_j^{-1}hd_rh'k_i$. Then

$$\langle T(g)\phi, \phi\rangle = \langle T(d_r)\phi_i, \phi_j\rangle = \int \langle T(d_r)\phi_i(x), \phi_j(x)\rangle\, dx$$

$$= q^{-r}\int \langle \phi_i(\pi^r x), \phi_j(x)\rangle\, dx.$$

For r sufficiently large, $\phi_i(\pi^r x) = 0$, so $\langle T(g)\phi, \phi\rangle = 0$ for g in these sets Kd_rK. If E/F is ramified, we have $G = \bigcup_{r\geq 0} Kd_rK$, where

$$d_r = \begin{pmatrix} \pi_E^r & & \\ & (\pi_E^\sigma\pi_E^{-1})^r & \\ & & (\pi_E^r)^{-\sigma} \end{pmatrix}$$

Choosing $\pi_E = \sqrt{\pi}$,

$$d_r = \begin{pmatrix} (\sqrt{\pi})^r & & \\ & (-1)^r & \\ & & (-1)^r(\sqrt{\pi})^r \end{pmatrix}.$$

If r is even, $(T(d_r)\phi)(x) = q^{-r/2}\phi(\pi^{r/2}x)$ and if r is odd,

$$\big(T(d_r)\phi\big)(x_1, x_2) = \pm q^{-r/2}\phi\big(-\pi^{(r-1)/2}x_2, -\pi^{(r+1)/2}x_1\big).$$

For r sufficiently large, both of these quantities are zero, so the argument proceeds as in the unramified case.

6. Supercuspidal components. We will first consider the case when E/F is unramified. Let $K = G(\mathcal{O})$.

LEMMA 6.1. *K is generated by the following elements*: (1) $m(a)$, $a \in U_E$; (2) $t(\tau)$, $\tau \in N^1$; (3) $n(b)$, $b\sqrt{\alpha} \in \mathcal{O}_E$; (4) $u(c)$, $c \in \mathcal{O}_E$; (5) w.

Proof. We have the Bruhat decomposition $G = NA \cup NAwN$. $g \in NA \Rightarrow g = u(c)n(b)m(a)t(\tau)$. If all entries of g are in $\mathcal{O}_E$, then a calculation shows that $c, b\sqrt{\alpha} \in \mathcal{O}_E$, $a \in U_F$, and $\tau \in N^1$. A similar argument applies for $g \in K \cap NAwN$.

For $n \geq 2$, let $\theta \neq \theta_0$ be a character of N^1 with conductor $\omega(\theta) = n - 1$, and choose $\chi \in \hat{F}^+$ with $\omega(\chi) = n$. for $\theta = \theta_0$, choose χ with $\omega(\chi) = 1$. Then θ does not appear in $\tilde{W}_1\,|_T$, so H^θ is supercuspidal. Let

$$H_K^\theta = \left\{\phi \in \mathcal{S}_{\mathcal{F}}\big(\mathcal{O}_E, P_E^{\omega(\chi)}\big) \cap H^\theta \,|\, \phi(x) \in \mathcal{S}_{\mathrm{C}}\big(\mathcal{O}_E, P_E^{\omega(\chi)}\big)\ \forall x \in E\right\}.$$

Note that we are not considering the case $\theta = 1$, since this choice never produces a supercuspidal, regardless of the choice of χ.

LEMMA 6.2. *H_K^θ is a K-invariant subspace of H^θ.*

Proof. It is easy to check that for k a generator of K, $\phi \in H_K^\theta \Rightarrow T(k)\phi \in H_K^\theta$.

LEMMA 6.3. *Suppose $\theta \neq 1$, θ_0. Then $\phi \in H^\theta \Rightarrow \phi$ is supported on $\mathcal{O}_E - P_E^2$.*

Proof. Choose $y \in P_E^2$, $t \in N_{n-2}^1 = N^1 \cap (1 + P_E^{n-2})$. Then $ty - y \in P_E^n$, so $\phi(ty) = \phi(y)$. But $\phi \in H^\theta \Rightarrow \tilde{W}_1(t)(\phi(ty)) = \theta(t)\phi(y)$, so $\tilde{W}_1(t)(\phi(y)) = \theta(t)\phi(y)$. Let $f = \phi(y)$. Write $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, where $\mathcal{F}_1$ is the direct sum of eigenspaces of $\mathcal{F}$ corresponding to characters γ of N^1 such that $\omega(\gamma) \leq n - 2$ and γ appears in $\tilde{W}_1\,|_T$, and $\mathcal{F}_2$ corresponds to those γ in $\tilde{W}_1 \mid_T$ which have conductor n. Let $\{f_i\}$ be a basis for $\mathscr{F}_1$, and let $\{g_j\}$ be a basis for $\mathscr{F}_2$. Write $f = \sum a_i f_i + \sum b_j g_j$. We have $\tilde{W}_1(t) f_i = \theta_i(t) f_i$ and $\tilde{W}_1(t) g_j = \gamma_j(t) g_j$. Therefore, $\tilde{W}_1(t) f = \sum a_i \theta_i(t) f_i + \sum b_j \gamma_j(t) g_j$. For $t \in N^1_{n-2}$, $\tilde{W}_1(t) f = \theta(t) f = \sum a_i \theta(t) f_i + \sum b_j \theta(t) g_j$. Also $t \in N^1_{n-2} \Rightarrow \theta_i(t) = 1$, so $\tilde{W}_1(t) f = \sum a_i f_i + \sum b_j \gamma_j(t) g_j$. So $a_i \theta(t) = a_i$ and $b_j \theta(t) = b_j \gamma_j(t)$ for all $t \in N^1_{n-2}$. Choosing $t \in N^1_{n-2}$ with $\theta(t) \neq 1$, we see that all $a_i = 0$. Also, $\theta \equiv 1$ on N^1_{n-1} and each $\gamma_j \not\equiv 1$ on N^1_{n-1}, so all $b_j = 0$. Therefore $f = 0$, so $\phi(y) = 0$ for all $y \in P_E^2$.

LEMMA 6.4. *If $\theta = \theta_0$, $\phi \in H_K^\theta \Rightarrow \theta$ is supported on $\mathcal{O}_E - P_E$.*

Proof. We know $\phi(0) = 0$, so ϕ constant on $P_E \Rightarrow \phi$ vanishes on P_E.

LEMMA 6.5. *Suppose $\phi \in H_K^\theta$ is an eigenfunction for $\{n(b) \mid b\sqrt{\alpha} \in \mathcal{O}_E\} = Z(N)$. Then ϕ vanishes on U_E or on $P_E - P_E^2$.*

Proof. Choose $y_1, y_2 \in \operatorname{supp} \phi$. Suppose $(T(n(b))\phi)(y_i) = \chi(\varepsilon b N y_i)\phi(y_i) = \psi(b)\phi(y_i)$. $\phi(y_i) \neq 0 \Rightarrow \chi(\varepsilon b N y_1) = \chi(\varepsilon b N y_2)$ $\forall b \in \mathcal{O}$, so $Ny_1 - Ny_2 \in P^n$. If $\nu_E(y_1) = 0$ and $\nu_E(y_2) = 1$, then $\nu(Ny_1) = 0$ and $\nu(Ny_2) = 2 \Rightarrow \nu(Ny_1 - Ny_2) = 0$, a contradiction. So $\nu_E(y_1) = 0 \Leftrightarrow \nu_E(y_2) = 0$.

We now proceed to show H_K^θ is irreducible. Let $\mathscr{I}$ be an intertwining operator for H_K^θ.

LEMMA 6.6. *$\mathscr{I}$ acts as a scalar on functions in H_K^θ which are supported on a single orbit of the form $\{t y_0 \mid t \in N^1,\ y_0 \in \mathcal{O}_E\}$.*

Proof. The proof is essentially that of [**Ho**], which we summarize here. For $c \in \mathcal{O}_E$, let T_c denote the operator $T(u(c))$. For $\phi \in H_K^\theta$, define $\tilde{\phi}$ on N^1 by $\tilde{\phi}(t) = \phi(t y_0)$. Under the map $\phi \to \tilde{\phi}$, the operator T_c goes to an operator we denote by $\tilde{T}_c$. $\tilde{T}_c$ is of the form $\tilde{W}_1(s)^{-1}\rho(v)\tilde{W}_1(s)$, where $v \in E$. $\mathscr{I}$ is transformed into an operator $\tilde{\mathscr{I}}$ which commutes with all $\tilde{T}_c$, and so $\tilde{\mathscr{I}}$ commutes with all $\tilde{W}_1(s)^{-1}\rho(v)\tilde{W}_1(s)$. But the family of operators $\{\rho(v) \mid v \in E\}$ acts irreducibly on $\mathscr{F}$, so $\tilde{\mathscr{I}}$ is a scalar, and thus $\mathscr{I}$ acts as a scalar on functions supported on a single orbit.

LEMMA 6.7. *$\mathscr{I}$ acts as a scalar on the set of all $\phi \in H_K^\theta$ supported on U_E.*

Proof. First, any eigenfunction ϕ supported on U_E is actually supported on a single orbit in U_E. For, if $y_1, y_2 \in \operatorname{supp}\phi$, $Ny_1 - Ny_2 \in P_E^n$ $\Rightarrow \exists t \in N^1$ such that $t y_1 - y_2 \in P_E^n \Rightarrow y_2 \in t y_1 + P_E^n \Rightarrow \operatorname{supp}(\phi)$ is contained in a single orbit of the form $\{t y_0 + P_E^n\} \subset U_E$. Now, given any

$\phi \in H_K^\theta$, write $\phi = \Sigma\phi_i$ as a sum of eigenfunctions, each supported on a single orbit $\mathcal{O}(x_i)$, on which $\mathcal{I}$ acts by c_i; so that $\mathcal{I}\phi_i = c_i\phi_i$. Given ϕ_i and ϕ_j supported on $\mathcal{O}(x_i)$ and $\mathcal{O}(x_j)$, let $a = x_i/x_j$. Then $T(m(a))\phi_i$ is supported on $\mathcal{O}(x_j)$. Since $T(m(a))$ and $\mathcal{I}$ commute, $c_i = c_j$. The same argument shows:

LEMMA 6.8. *$\mathcal{I}$ acts as a scalar on the set of all $\phi \in H_K^\theta$ supported on $P_E - P_E^2$.*

PROPOSITION 6.9. *$\mathcal{I}$ acts as a scalar on all of H_K^θ, and so H_K^θ is irreducible.*

Proof. It remains only to show $\mathcal{I}$ acts on functions supported on U_E by the same scalar as it acts on those supported on $P_E - P_E^2$. Recall that the trivial character of T appears in $\tilde{W}_1|_T$ regardless of the choice of $\chi \in \hat{F}^+$. Choose $f_0 \in \mathcal{F}$ such that $\tilde{W}_1(t)f_0 = f_0\ \forall t \in N^1$. Choose y_0 with $\nu_E(y_0) = 1$. Consider the orbit $\mathcal{O}(y_0) = \{ty_0 + P_E^N \mid t \in N^1\}$. Define $\phi \in H_K^\theta$ by $\phi(ty_0 + P_E^n) = \theta(t)\tilde{W}_1(t^{-1})f_0$. This gives a well-defined function in H_K^θ which is supported on $P_E - P_E^2$. Now we claim that $\phi \notin \mathcal{S}_{\mathcal{F}}(P_E, P_E^{n-1})$. Suppose $\phi \in \mathcal{S}_{\mathcal{F}}(P_E, P_E^{n-1})$. Choose any $s \in N_{n-2}^1$. Let $x = sy_0$, $y = (1-s)y_0$. Since $s \in N_{n-2}^1$, $(1-s)y_0 = y \in P_E^{n-1} \Rightarrow \phi(x+y) = \phi(x)$. Thus, $\theta(s)\tilde{W}_1(s^{-1})f_0 = f_0\ \ \forall s \in N_{n-2}^1$. But we know that $W_1(t)f_0 = f_0$ $\forall t \in N^1$, so $\theta(t) = 1\ \forall t \in N_{n-2}^1$, which is a contradiction, since $\omega(\theta) = n-1$.

Now, $\phi \notin \mathcal{S}_{\mathcal{F}}(P_E, P_E^{n-1}) \Rightarrow T(w)\phi \notin \mathcal{S}_{\mathcal{F}}(P_E, P_E^{n-1})$. Also, $\phi \in \mathcal{S}_{\mathcal{F}}(P_E, P_E^n) \Rightarrow T(w)\phi \in \mathcal{S}_{\mathcal{F}}(\mathcal{O}_E, P_E^{n-1})$. Therefore $T(w)\phi$ does not vanish on U_E. Write $T(w)\phi = \phi_1 + \phi_2$, ϕ_1 being a sum of eigenfunctions supported on U_E, and ϕ_2 a sum of those supported on $P_E - P_E^2$. Say $\mathcal{I}$ acts as cI on those $\phi \in H_K^\theta$ supported on U_E and as $c'I$ on ϕ supported on $P_E - P_E^2$. Then $\mathcal{I}T(w)\phi = c\phi_1 + c'\phi_2$ and $\mathcal{I}T(w)\phi = T(w)\mathcal{I}\phi = T(w)c'\phi = c'\phi_1 + c'\phi_2$. This implies $c = c'$ if $\phi_1 \neq 0$. But if $\phi_1 = 0$, then $T(\omega)\phi$ vanishes on U_E, which it does not.

We next analyze the commuting algebra of the induced representation.

PROPOSITION 6.10. $\mathrm{Ind}_K^G H_K^\theta$ *is not irreducible if $\omega(\theta) \geq 2$.*

Proof. Consider functions $S: U(3) \to \mathrm{End}\, H_K^\theta$ such that $S(k_1 g k_2) = T(k_1)S(g)T(k_2)$ for $k_1, k_2 \in K$. S is determined by its values on

$$K \backslash G/K = \left\{ \begin{pmatrix} \pi^{-n} & & \\ & 1 & \\ & & \pi^n \end{pmatrix} = d_n \,\middle|\, n \geq 0 \right\}.$$

The eigenspaces of $Z(N)$ correspond to characters of the form $n(b) \to \chi(\varepsilon b N(x))$, $b \in F$, $x \in \mathcal{O}_E$. Let H_x be the eigenspace corresponding to $x \in \mathcal{O}_E$. Choose $\phi \in H_x$. Then $S(d_n)\phi$ is an eigenfunction for the character $n(b) \to \chi(\varepsilon b N(\pi^n x))$, so $S(d_n)\phi \in H_{\pi^n x}$. For $n(b) \to \chi(\varepsilon b N(\pi^n x))$ to be an eigencharacter, there must exist a function $\phi' \in H_K^\theta$ such that $\phi'(\pi^n x) \neq 0$. But ϕ' is supported on $\mathcal{O}_E - P_E^2$. Since $x \in \mathcal{O}_E$ and $\phi'(\pi^n x) \neq 0$, we must have $x \in U_E$ and $n = 0$ or 1, or $\nu_E(x) = 1$ and $n = 0$. But $S(d_n): H_x \to H_{\pi^n x}$, so $S(D_n) = 0$ if $n \geq 2$, and S is completely determined by $S(I)$ and $S(d_1)$. $S(I)$ is a scalar since H_K^θ is irreducible. It remains to show that we may have $S(d_1)$ be non-zero. Write $H_K^\theta = H_1 \oplus H_2$, where $H_1 = \{\phi \in H_K^\theta \mid T(d_1)\phi \in H_K^\theta\}$. Let $S(d_1) = T(d_1)R$, where R is the projection of H_K^θ on H_1. It is then easy to check that if $d_1 = k_1 d_1 k_2$, with $k_1, k_2 \in K$, we have $S(d_1) = T(k_1)S(d_1)T(k_2)$.

In order to exhibit H^θ as an induced representation for $\omega(\theta) \geq 2$, we must consider the other conjugacy class of maximal compact subgroups. Let a representative for this class be given by

$$L = \begin{pmatrix} \mathcal{O}_E & P_E^{-1} & P_E^{-1} \\ P_E & \mathcal{O}_E & \mathcal{O}_E \\ P_E & \mathcal{O}_E & \mathcal{O}_E \end{pmatrix} \cap U(3).$$

For $\omega(\theta) \geq 2$, let $H_L^\theta = \{\phi \in H_K^\theta \mid \phi$ is supported on $P_E - P_E^2$ and $\phi(y) \in \mathcal{S}_{\mathrm{C}}(\mathcal{O}_E, P_E^{n-1})$ for all $y \in E\}$.

LEMMA 6.11. *H_L^θ is invariant under the action of L.*

Proof. L is generated by B, the Iwahori subgroup, and the element

$$\begin{pmatrix} 0 & 0 & \pi^{-1} \\ 0 & 1 & 0 \\ \pi & 0 & 0 \end{pmatrix}.$$

A calculation shows that H_L^θ is invariant by both B and this element.

LEMMA 6.12. *H_L^θ is an irreducible L-module.*

Proof. The proof that showed H_K^θ was an irreducible K-module applies here, since the only elements $k \in K$ which were used to show H_K^θ was irreducible are also on L, namely, the elements $u(c)$, $n(b)$, and $m(a)$.

PROPOSITION 6.13. *For $\omega(\theta) \geq 2$, $\mathrm{Ind}_L^G H_L^\theta$ is irreducible and is equivalent to H^θ.*

Proof. We have $G = \bigcup_{n \geq 0} Ld_n L$, where

$$d_n = \begin{pmatrix} \pi^{-n} & & \\ & 1 & \\ & & \pi^n \end{pmatrix}.$$

Using the argument in Proposition 6.10, we see that any function S is determined in this case by its value at the identity, since any function in H_L^θ is supported on the single shell $P_E - P_E^2$. Since H_L^θ is irreducible, $S(I)$ is a scalar and the induced representation is thus irreducible. By Frobenius reciprocity, it is equivalent to H^θ.

We will next consider the case when E/F is ramified. Let θ be a character of N^1 with conductor n which does not appear in $(U(1), U(1))$. H^θ is thus supercuspidal.

Let $H_L^\theta = \{\phi \in \mathscr{S}_{\mathscr{F}}(\mathcal{O}_E, P_E^{2n-1}) \mid \phi(x) \in \mathscr{S}_{\mathbf{C}}(\mathcal{O}_E, P_E^{2n-1})\ \forall x \in E\}$.

LEMMA 6.14. *H_L^θ is an L-invariant subspace of H^θ.*

Proof. A calculation shows H_L^θ is left invariant by all generators of L.

LEMMA 6.15. *If $\phi \in H_L^\theta$, then ϕ vanishes on P_E.*

Proof. Choose $y \in P_E$, $t \in N^1$. Then $t \in T_{n-1} = N_{2n-2} \Rightarrow ty - y \in P_E^{2n-1}$, so $\phi(ty) = \phi(y)$ and $\tilde{W}_1(t)(\phi(y)) = \theta(t)\phi(y)$. Let $f = \phi(y)$. Let $\mathscr{F} = \oplus \mathscr{F}_{\theta_i}$ be the decomposition into the one-dimensional eigenspaces corresponding to the characters of N^1 appearing in $\tilde{W}_1|_T$. Let $H_1 = \oplus\{\mathscr{F}_{\theta_i} \mid \omega(\theta_i) < n\}$ and $H_2 = \oplus\{\mathscr{F}_{\theta_i} \mid \omega(\theta_i) = n\}$. Now we claim that $f = \phi(y)$ is in H_2. Write $f = \alpha + \beta$, $\alpha \in H_1$, $\beta \in H_2$. It $t \in T_{n-1}$, then $\tilde{W}_1(t)\alpha = \Sigma \theta_i(t)\alpha_i = \Sigma \alpha_i = \alpha$ since all $\theta_i(t) = 1$. Also, $\tilde{W}_1(t)f = \theta(t)f = \theta(t)\alpha + \theta(t)\beta$ for $t \in T_{n-1}$. If $\alpha \neq 0$, we must have $\theta(t) = 1\ \forall t \in T_{n-1}$, a contradiction, since $\omega(\theta) = n$. Therefore $\alpha = 0$ and $f \in H_2$. Let H_2 have basis $\{f_j\}$, $f_j \in F_{\theta_j}$. If $f = \Sigma a_j f_j$, then $\tilde{W}_1(t)f = \Sigma a_j \theta_j(t) f_j$. If $t \in T_{n-1}$, then $\theta(t)f = \tilde{W}_1(t)f = \Sigma a_j \theta_j(t) f_j$. Also, $\theta(t)f = \Sigma a_j \theta_j(t) f_j$ so $\theta(t) = \theta_j(t)\ \forall t \in T_{n-1}$ for all j such that $a_j \neq 0$. But θ_j appears in $\tilde{W}_1|_T$ and θ does not appear, and we know from [M] that whether or not a character of conductor n appears depends only on its restriction to T_{n-1}. Thus θ appears $\Leftrightarrow \theta_j$ appears, which is a contradiction, unless $f = 0$. Therefore, ϕ vanishes on P_E.

PROPOSITION 6.16. *H_L^θ is an irreducible L-module.*

Proof. The proof proceeds as in the unramified case, although this case is easier, as the functions in H_L^θ are supported on U_E rather than on $\mathcal{O}_E - P_E^2$.

PROPOSITION 6.17. $\mathrm{Ind}_L^G H_L^\theta$ *is irreducible and is equivalent to* H^θ.

Proof. Proceed as in the unramified case.

REFERENCES

[A] C. Asmuth, *Weil representations of symplectic p-adic groups*, Amer. J. Math., **101** (1979), 885–908.

[GPS] S. Gelbart and I. I. Piatetskii-Shapiro, *Automorphic Forms and L-Functions For the Unitary Group*, *Lie Group Representations* II, Lecture Notes in Mathematics, Vol. 1041, Springer, New York, 1983, pp. 141–184.

[Ho] R. Howe, *L^2 duality for stable reductive dual pairs*, preprint.

[K] D. Keys, *Principal series representations of special unitary groups over local fields*, Compositio Math., **51** (1984), 115–130.

[M] C. Moen, *The dual pair $(U(1), U(1))$ over a p-adic field*, preprint.

[Ma] D. Manderscheid, *Supercuspidal duality for the two-fold cover of* SL_2 *and the split* 0_3, Amer. J. Math., **107** (1985), 1305–1324.

[PS] I. I. Piatetskiĭ-Shapiro, *On the Saito-Kurokawa lifting*, Invent. Math., **71** (1983), 309–338.

[Ra] S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math., **104** (1982), 469–515.

Received April 2, 1985 and in revised form February 20, 1986. Research partially supported by a Naval Academy Research Council grant.

UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MD 21402-5000

Pacific Journal of Mathematics
Vol. 127, No. 1 January, 1987