ANOTHER CHARACTERIZATION OF AE(0)-SPACES

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We prove that a space $X$ is an absolute extensor for the class of all zero-dimensional spaces if and only if $X$ is an upper semi-continuous compact-valued retract of a power of the real line.

1. Introduction. Dugundji spaces were introduced by Pelczynski [5]. Later Haydon [4] proved that the class of Dugundji spaces coincides with the class of all compact absolute extensors for zero-dimensional compact spaces (briefly, AE(0)). After Haydon's paper, compact AE(0)-spaces have been extensively studied (see Ščepin's review [9]); let us note the following result of Dranishnikov [3]: a compact $X$ is an AE(0)-space if and only if for every embedding of $X$ in a Tychonoff cube $I^\tau$ there exists an upper semi-continuous compact-valued (br. usco) mapping $r$ from $I^\tau$ to $X$ such that $r(x) = \{x\}$, for each $x \in X$ (such a usco mapping will be called a usco retraction).

Chigogidze [2] extended the notion of AE(0) from the class of compact spaces to that of completely regular spaces and gave a characterization of such AE(0)-spaces.

The aim of the present paper is to give another characterization of completely regular AE(0)-spaces which is similar to the above mentioned result of Dranishnikov. We prove that $X \in$ AE(0) iff $X$ is a usco retract of $R^\tau$ for some $\tau$, where $R$ is the real line with the usual topology. Our technique is different from Dranishnikov's.

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2. Notations and terminology. All spaces considered are completely regular and all single-valued mappings are continuous. A set-valued mapping $r$ from $X$ to $Y$ is called upper semi-continuous (br. u.s.c.) if the set $r^*(U) = \{x \in X: r(x) \subseteq U\}$ is open in $X$ whenever $U$ is open in $Y$. We say that a usco mapping $r$ is minimal if every usco selection for $r$ coincides with $r$. It follows from the Kuratowski-Zorn lemma that every usco mapping has a minimal usco selection.

A mapping $f$ from $Y$ to $X$, where $Y \subseteq Z$, is called $Z$-normal if, for every continuous function $g$ on $X$, the function $g \circ f$ is continuously extendable to $Z$. A space $X$ is called an absolute extensor for zero-dimensional spaces [2], if every $Z$-normal mapping $f$ from $Y$ to $X$, where $Y \subseteq Z$...
and dim $Z = 0$, is continuously extendable to $Z$; if $f$ is continuously extendable only to a neighbourhood of $Y$ in $Z$, the space $X$ is called an absolute neighbourhood extensor for 0-dimensional space, briefly ANE(0). Here, dim stands for the dimension defined by finite functionally open covers.

A mapping $f$ from $X$ to $Y$ will be called 0-soft [2], if for every 0-dimensional space $Z$ and every two $Z$-normal mappings $g: Z_0 \to X$, $h: Z_1 \to Y$ such that $Z_0 \subset Z_1 \subset Z$ and $f \circ g = h|Z_0$, there exists a $Z$-normal mapping $k: Z_1 \to X$ such that $g = k|Z_0$ and $f \circ k = h$. In the case $Z$ is paracompact and $Z_0$ and $Z_1$ are closed subsets of $Z$, one gets Ščepin's notion [8] of a 0-soft mapping, defined earlier.

A space $X$ is said to be a multivalued absolute (resp. neighbourhood) extensor (br. $X \in \text{MA(N)}E$) if every $Z$-normal mapping $f: Z_0 \to X$ with $Z_0 \subset Z$, can be extended to a usco mapping from $Z$ (resp. from a neighbourhood of $Z_0$ in $Z$) to $X$.

A mapping $f: X \to Y$ is said to be functionally open if $f(U)$ is functionally open in $Y$ for every functionally open subset $U$ of $X$.

3. AE(0)-spaces.

**Lemma 1.** Let $X = \prod\{X_s: s \in S\}$ be a product of separable metric spaces and let $U$ be a $G_\delta$-set in $X$. Then there exists a countable set $B \subset S$ such that $p_B(U)$ is a $G_\delta$-set in $X_B$ and $G_\delta(U) = X_S \setminus B \times p_B(U)$. If $U$ is open in $X$ then $G_\delta(U)$ is functionally open too.

**Proof.** Put $M = X \setminus G_\delta(U)$. By a result of R. Pol and E. Pol [6] there exists a countable set $B \subset S$ such that $p_B(U)$ is a $G_\delta$-set in $X_B$ and $p_B(U) \cap p_B(M) = \emptyset$. Hence $p_B^{-1}(p_B(U)) \cap M = \emptyset$. Since $p_B(G_\delta(U)) = p_B(U)$, we have $B \in k(G_\delta(U))$, so $G_\delta(U) = p_B(U) \times X_S \setminus B$. If $U$ is open in $X$ then $p_B(U)$ is functionally open in $X_B$. Thus, $G_\delta(U)$ is functionally open too.

The proof of the following (actually known) lemma is an easy exercise on the definition of a minimal usco mapping.

**Lemma 2.** Let $r$ be a minimal usco mapping from $X$ to $Y$ and let $U$ be an open set in $Y$. Then the following holds:

(i) $r(x) \subset \text{cl}(U)$ for every $x \in \text{Int}(\text{cl}(r^*(U)));$
Let $Y = \prod \{ Y_s: s \in S \}$ be a product of separable metric spaces and let $X \subset Y$. Let $r$ be a u.s.c. mapping from $Y$ to $X$. A subset $B$ of $S$ is called $r$-admissible if $B \in k(\text{cl}(r^*(U) \cap X))$ for every standard open subset $U$ of $Y$ with $B \in k(U)$. The above definition is a simple modification of the definition of $e$-admissible set, given by Shirokov [11]. The following lemma was actually proved by Shirokov [11].

**Lemma 3.** Let $Y = \prod \{ Y_s: s \in S \}$ be a product of separable metric spaces, $X \subset Y$ and let $r$ be a u.s.c. mapping from $Y$ to $X$. Then we have:

(i) for every set $B \subset S$ there is a $r$-admissible set $A$ containing $B$ and card $A = \text{card } B$;

(ii) a union of $r$-admissible subsets of $S$ is $r$-admissible too.

**Lemma 4.** Let $Y = \prod \{ Y_s: s \in S \}$ be a product of separable metric spaces, $X \subset Y$ and let $r$ be a minimal usco mapping from $Y$ to $X$. Suppose $B$ is a $r$-admissible subset of $S$. Then the following conditions are fulfilled:

(i) $B \in k(\text{cl}(r^*(\bigcup_{i=1}^n U_i \cap X)))$ for every finite family $\{U_i: i = 1, \ldots, n\}$ of standard open subsets of $Y$ with $B \in \bigcap_{i=1}^n k(U_i)$;

(ii) $p_B(r(x)) = p_B(r(y))$ whenever $p_B(x) = p_B(y)$.

**Proof.** (i) Let $U = \bigcup_{i=1}^n U_i$. By Lemma 2(ii) we have

$$\text{cl}(r^*(U \cap X)) = \text{cl}(r^{-1}(U \cap X)) = \text{cl}\left(\bigcup_{i=1}^n r^{-1}(U_i \cap X)\right)$$

$$= \bigcup_{i=1}^n \text{cl}(r^{-1}(U_i \cap X)) = \bigcup_{i=1}^n \text{cl}(r^*(U_i \cap X)).$$

Since $B$ is $r$-admissible, $B \in k(\text{cl}(r^*(U_i \cap X)))$ for each $i$. Thus, $B \in k(\text{cl}(r^*(U \cap X)))$.

(ii) Let $p_B(x) = p_B(y)$ and $p_B(r(y)) \subset p_B(V)$, where $V$ is open in $Y$. Since $r(y)$ is compact, $V$ can be considered as a finite union $\bigcup_{i=1}^n V_i$ of standard open subsets of $Y$ with $B \in \bigcap_{i=1}^n k(V_i)$. Then, by (i), we have $B \in k(\text{cl}(r^*(V \cap X)))$. Consequently, $B \in k(\text{Int}\text{(cl}(r^*(V \cap X))))$. Thus, $x \in \text{Int}\text{(cl}(r^*(V \cap X)))$ because $y \in r^*(V \cap X)$. Hence, by Lemma 2(i), $r(x) \subset \text{cl}(V \cap X)$ i.e. $p_B(r(x)) \subset \text{cl}(p_B(V))$. The last inclusion shows that $p_B(r(x)) \subset p_B(r(y))$. Analogously, $p_B(r(y)) \subset p_B(r(x))$. Therefore $p_B(r(x)) = p_B(r(y))$. 

Therefore $p_B(r(x)) = p_B(r(y))$. 

(iii) $\text{cl}(r^{-1}(U)) = \text{cl}(r^*(U))$, where $r^{-1}(U) = \{ x \in X: r(x) \cap U \neq \emptyset \}$. 

Let $Y = \prod \{ Y_s: s \in S \}$ be a product of separable metric spaces and let $X \subset Y$. Let $r$ be a u.s.c. mapping from $Y$ to $X$. A subset $B$ of $S$ is called $r$-admissible if $B \in k(\text{cl}(r^*(U) \cap X))$ for every standard open subset $U$ of $Y$ with $B \in k(U)$. The above definition is a simple modification of the definition of $e$-admissible set, given by Shirokov [11]. The following lemma was actually proved by Shirokov [11].

**Lemma 3.** Let $Y = \prod \{ Y_s: s \in S \}$ be a product of separable metric spaces, $X \subset Y$ and let $r$ be a u.s.c. mapping from $Y$ to $X$. Then we have:

(i) for every set $B \subset S$ there is a $r$-admissible set $A$ containing $B$ and card $A = \text{card } B$;

(ii) a union of $r$-admissible subsets of $S$ is $r$-admissible too.
A mapping \( f: X \to Y \) is said to have a polish kernel [2], if there exists a polish (i.e. complete separable metric) space \( P \) such that \( X \) is \( C \)-embedded in \( Y \times P \) and \( f \) coincides with the restriction \( p_Y|X \), where \( p_Y: Y \times P \to Y \) is the natural projection. The following lemma is proved by Chigogidze [2].

**LEMMA 5.** Let the mapping \( f \) from \( X \) to \( Y \) have a polish kernel, where \( X \) and \( Y \) are AE(0)-spaces. Then \( f \) is 0-soft if and only if \( f \) is functionally open.

**LEMMA 6.** Let \( Y = \prod \{ Y_s : s \in S \} \) be a product of separable metric spaces and let \( r \) be a minimal usco retraction from \( Y \) to \( X \). Then for every \( r \)-admissible set \( B \subset S \) the following conditions are fulfilled:

(i) the restriction \( p_B|X \) is functionally open;

(ii) \( p_B(X) \) is a usco retract of \( Y_B \).

**Proof.** (i) First we prove that for every \( C \subset S \) the projection \( p_C \) is functionally open. Let \( U \) be a functionally open subset of \( Y \). Then, by Lemma 1, there exists a countable set \( D \subset S \) such that \( U = p^{-1}_D(p_D(U)) \). This permits us to present \( U \) as a countable union \( \bigcup_{i=1}^\infty U_i \) of standard open subsets of \( Y \) with \( D \in k(U_i) \), for each \( i \). Hence, \( p_C(U) = \bigcup_{i=1}^\infty p_C(U_i) \). Since every \( p_C(U_i) \) is a standard open subset of \( Y_C \), the set \( p_C(U) \) is a countable union of functionally open subsets of \( Y_C \). Therefore \( p_C(U) \) is functionally open.

Now, suppose \( B \) is \( r \)-admissible and \( U \) is functionally open in \( X \). Since \( G_\delta(r^\#(U)) \) is functionally open in \( Y \) (by Lemma 1), in order to prove that \( p_B|X \) is functionally open it suffices to show that \( p_B(U) = p_B(G_\delta(r^\#(U))) \cap p_B(X) \). Let \( x \in X \) and let \( p_B(x) = p_B(y) \) for some \( y \in G_\delta(r^\#(U)) \). If we assume \( r(y) \subset X \setminus U \) then \( y \in r^\#(X \setminus U) \). However \( r^\#(X \setminus U) \) is a \( G_\delta \)-set in \( Y \) because \( X \setminus U \) is a zero-set in \( X \). Hence, \( r^\#(X \setminus U) \cap r^\#(U) \neq \emptyset \), which is impossible. Thus, \( r(y) \cap U \neq \emptyset \). By Lemma 4(ii), we have \( p_B(x) = p_B(r(x)) = p_B(r(y)) \), so \( p_B(x) \in p_B(U) \). Therefore \( p_B(G_\delta(r^\#(U))) \cap p_B(X) \subset p_B(U) \). The inverse inclusion is obvious.

(ii) Let \( B \) be a \( r \)-admissible set. Define a compact-valued mapping \( r_1: Y_B \to p_B(X) \) by letting \( r_1(p_B(x)) = p_B(r(x)) \). Lemma 4(ii) implies that this definition is correct and that \( r_1(p_B(x)) = p_B(x) \) for every \( x \in X \). It remains to prove that \( r_1 \) is u.s.c. Let \( r_1(p_B(x_0)) \subset U \) for some \( x_0 \in Y \), where \( U \) is open in \( Y_B \). Then, by Lemma 4(i), we have \( B \in k(\text{cl}(r^\#(p_B^{-1}(U) \cap X))) \). Consequently, \( B \in k(V) \), where \( V = \text{Int(cl}(r^\#(p_B^{-1}(U) \cap X))) \). The set \( p_B(V) \) is a neighbourhood of \( p_B(x_0) \).
because $x_0 \in r^\#(p_B^{-1}(U) \cap X)$. Let $p_B(x) \in p_B(V)$. Then $x \in V$ and, by Lemma 2(i), $r(x) \subset \text{cl}(p_B^{-1}(U) \cap X)$; so $r_1(p_B(x)) \subset \text{cl}(U)$. Therefore, $r_1$ is u.s.c.

**Lemma 7.** Let $Y = \prod\{Y_s: s \in S\}$ be a product of separable metric spaces and let $X$ be a usco retract of $Y$. Then the following conditions are fulfilled:

(i) $X$ is C-embedded in $Y$;

(ii) there exists a set $B \subset S$ of cardinality $w(X)$ such that $p_B|X$ is a homeomorphism and $p_B(X)$ is a usco retract of $Y_B$.

**Proof.** (i) Suppose $f$ is a continuous function on $X$. Consider the family $\mathcal{L}$ of all open intervals in $R$ with rational endpoints. Using Lemma 1, for every $U \in \mathcal{L}$ choose a countable set $B(U) \subset S$ such that $B(U) \in k(G_b(r^\#(f^{-1}(U))))$, where $r$ is a minimal usco retraction from $Y$ to $X$. It follows from Lemma 3(i) that there exists a countable $r$-admissible set $C$ containing $\bigcup\{B(U): U \in \mathcal{L}\}$. One can easily see that $p_C(x) = p_C(y)$ implies $f(x) = f(y)$ for every $x, y \in X$. Since $p_C|X$ is open, there exists a continuous function $g$ on $p_C(X)$ such that $f(x) = g(p_C(x))$, for each $x \in X$. Since $p_C(X)$ is a usco retract of $Y_C$, it is closed in $Y_C$. Hence, $g$ is continuously extendable on $Y_C$; so $f$ is continuously extendable on $Y$.

(ii) Suppose $r$ is a minimal usco retraction from $Y$ to $X$. Let $\mathcal{Q}$ be a family of standard open subsets of $Y$ such that card $\mathcal{Q} = w(X)$. Put $B_1 = \bigcup\{m(U): U \in \mathcal{Q}\}$, where $m(U) = \{s \in S: p_s(U) \neq Y_s\}$. Clearly, card $B_1 = w(X)$. By Lemma 3(i), pick a $r$-admissible set $B$ containing $B_1$ and such that card $B = w(X)$. Observe that $p_B|X$ is one-to-one. Since $p_B|X$ is open (by Lemma 6(i), we conclude that $p_B|X$ is a homeomorphism. Next, by Lemma 6(ii), $p_B(X)$ is a usco retract of $Y_B$.

**Theorem 1.** For a space $X$, the following conditions are equivalent:

(i) $X \in \text{AE}(0)$;

(ii) $X \in \text{MAE}$;

(iii) $X$ is a usco retract of $R^A$, for some $A$.

**Proof.** (i) $\rightarrow$ (ii) Let $f: H \to X$ be a $Z$-normal mapping, where $H \subset Z$. Consider the absolute $aZ$ of $Z$ and the natural projection $g: aZ \to Z$. Put $Y = g^{-1}(H)$. Observe that $f \circ g$ is $aZ$-normal. Since dim $aZ = 0$ and $X \in \text{AE}(0)$, there exists an extension $h: aZ \to X$ of $f \circ g$. Then the usco mapping $r: Z \to X$, defined by $r(z) = h(g^{-1}(z))$, is an extension of $f$. Thus, $X \in \text{MAE}$. 
(ii) \(\rightarrow\) (iii) Denote by \(C(X)\) the family of all continuous functions on \(X\). Consider \(X\) as a \(C\)-embedded subset of \(R^{C(X)}\). Hence, there exists a usco retraction from \(R^{C(X)}\) to \(X\).

(iii) \(\rightarrow\) (i) Let \(\mathcal{H}\) be the class of all spaces \(Y\) with the following property: \(Y\) is a usco retract of \(R^A\), for some \(A\). We will prove (by transfinite induction) that every element of \(\mathcal{H}\) is an AE(0)-space. Let \(X \in \mathcal{H}\) and \(w(X) = \mathcal{S}_0\). In this case, by Lemma 7(ii), \(X\) is a usco retract of \(R^{\omega_0}\). Hence, \(X\) is a polish space and, by a result of Chigogidze [2], \(X \in \text{AE}(0)\). Assume that \(\tau > \mathcal{S}_0\) and that for every \(X \in \mathcal{H}\) with \(w(X) < \tau\) we have \(X \in \text{AE}(0)\). Consider a space \(X \in \mathcal{H}\) with \(w(X) = \tau\). By Lemma 7(ii), \(X\) is a usco retract of \(R^\tau = \prod\{R_\alpha: \alpha < \omega(\tau)\}\), where \(\omega(\tau)\) is the initial ordinal of cardinality \(\tau\). Let \(r\) be a minimal usco retraction from \(R^\tau\) to \(X\). By Lemma 3(i), for every \(\alpha < \omega(\tau)\) there exists a countable \(r\)-admissible set \(B_\alpha\) containing \(\alpha\). Next, denote \(A(\alpha) = \bigcup\{B_\beta: \beta < \alpha\}\), \(q_\alpha = p_{A(\alpha)}\vert X\) and \(X_\alpha = q_\alpha(X)\) for each \(\alpha < \omega(\tau)\). If \(\alpha > \beta\) we put \(p_\beta^\alpha = q_\beta \circ q_\alpha^{-1}\). Thus, we actually construct a continuous inverse system \(S = \{X_\alpha, q_\beta^\alpha, \beta < \alpha < \Omega(\tau)\}\), in the sense of Ščepin [8], such that \(X = \lim S\). According to Lemmas 3(ii) and 6, we have that, for every \(\alpha < \omega(\tau)\), \(X_\alpha \in \mathcal{H}\) and \(q_\alpha\) is functionally open. Hence, \(q_\alpha^{\alpha+1}\) is functionally open. But \(w(X_\alpha) < \tau\), so \(X_\alpha \in \text{AE}(0)\) for each \(\alpha < \omega(\tau)\). Finally, Lemma 7(i) implies that \(q_\alpha^{\alpha+1}\) has a polish kernel. Therefore, it follows from Lemma 5 that \(q_\alpha^{\alpha+1}\) is 0-soft for every \(\alpha < \omega(\tau)\). So, all spaces \(X_\alpha\) and all mappings \(q_\alpha^{\alpha+1}\) are AE(0) and 0-soft, respectively. Therefore, \(X \in \text{AE}(0)\).

**Lemma 8.** Let \(r\) be a usco mapping from \(M\) to a compact space \(X\) and let \(M\) be a dense subset of \(Y\). Then \(r\) can be extended to a usco mapping from \(Y\) to \(X\).

**Proof.** For every \(y \in Y\) denote by \(U(y)\) the local base at \(y\) in \(Y\). Then the usco mapping \(r_1\), defined by \(r_1(y) = \cap\{\text{cl}(r(U \cap M)): U \subseteq U(y)\}\), is the required extension.

**Lemma 9.** Suppose \(Z = \prod\{Z_s: s \in S\}\) is a product of separable metric spaces and \(Y\) is closed in \(Z\). Let \(r\) be a minimal usco mapping from \(Z\) to \(Y\) and let \(X\) be a subset of \(Y\) such that \(r(x) = \{x\}\) for every \(x \in X\). Then the following holds:

(i) \(r(x) = \{x\}\) for every \(x \in G_\delta(X)\);

(ii) \(r(G_\delta(M)) \subseteq G_\delta(H)\) for every \(H \subseteq Y\) and every \(M \subseteq r^\#(H)\).
Proof. (i) Suppose \( r(x_0) \neq x_0 \) for some \( x_0 \in G_\delta(x) \). Take a point \( y \in r(x_0) \setminus \{x_0\} \) and a countable \( r \)-admissible set \( B \subset S \) such that \( p_B^{-1}(p_B(x_0)) \cap X \neq \emptyset \). Choose \( x \in p_B^{-1}(p_B(x_0)) \cap X \). Lemma 4(ii) implies \( p_B(x) = p_B(r(x_0)) \). This is impossible because \( x_0, y \in r(x_0) \) and \( p_B(x_0) \neq p_B(y) \). Hence, \( r(x) = \{x\} \) for every \( x \in G_\delta(X) \).

(ii) Assume \( H \subset Y \) and \( M \subset r^+(H) \). Let \( r(x_0) \setminus G_\delta(H) \neq \emptyset \) for some \( x_0 \in G_\delta(M) \). Take a point \( y \in r(x_0) \setminus G_\delta(H) \) and a countable \( r \)-admissible set \( B \subset S \) such that \( p_B(y) \notin p_B(H) \). Next choose a point \( x \in p_B^{-1}(p_B(x_0)) \cap M \). Then, by Lemma 4(ii), we have \( p_B(r(x)) = p_B(r(x_0)) \). But \( r(x) \subset H \); so \( p_B(r(x_0)) \subset p_B(H) \). This contradicts \( p_B(y) \notin p_B(H) \). Therefore, \( r(G_\delta(M)) \subset G_\delta(H) \).

**Theorem 2.** For a space \( X \), the following conditions are equivalent:

(i) \( X \in \text{ANE}(0) \);

(ii) \( X \in \text{MANE} \);

(iii) \( X \) is open in its Hewitt-realcompactification \( vX \) and \( vX \in \text{AE}(0) \).

**Proof.** (i) \( \rightarrow \) (ii) This implication can be proved as the implication (i) \( \rightarrow \) (ii) of Theorem 1.

(ii) \( \rightarrow \) (iii) Consider \( X \) as a \( C \)-embedded subset of \( R^A \), where \( A \) is the family of all continuous functions on \( X \). Clearly, \( vX = \text{cl}(X) \). Since \( X \in \text{MANE} \) there exists a usco retraction \( r_1 \) from an open subset \( U \) of \( R^A \) to \( X \). It is easily seen that \( U \cap vX = X \) i.e. \( X \) is open in \( vX \). Identifying \( R \) with \( (0,1) \), we consider \( R^A \) as a dense subset of \( I^A \), where \( I = \{0,1\} \). Put \( Y = \text{cl}_{I^A}(X) \). By Lemma 8, there exists a usco extension \( r_2: \text{Int}_{I^A} \text{cl}_{I^A}(U) \rightarrow Y \) of \( r_1 \). Let \( r_3 \) be a usco mapping from \( I^A \) to \( Y \) defined by letting \( r_3(y) = r_2(y) \), for \( y \in \text{Int}_{I^A} \text{cl}_{I^A}(U) \), and \( r_3(y) = Y \), otherwise. Denote by \( r \) a minimal usco selection for \( r_3 \). Since each point \( z \in I^A \setminus R^A \) is contained in a \( G_\delta \)-subset \( H(z) \) of \( I^A \) with \( H(z) \cap R^A = \emptyset \), the \( G_\delta \)-closure \( G_\delta(X) \) of \( X \) in \( I^A \) coincides with \( vX \). So, by Lemma 9, \( r \) is a usco retraction from \( G_\delta(U) \) to \( vX \). Here, \( G_\delta(U) \) is the \( G_\delta \)-closure of \( U \) in \( R^A \). It follows from Lemma 1 that there exists a countable set \( B \subset A \) such that \( G_\delta(U) = p_B(U) \times R^A \setminus B \). The space \( p_B(U) \), being a polish space, is an \( \text{AE}(0) \). Hence, \( G_\delta(U) \in \text{AE}(0) \) as a product of \( \text{AE}(0) \)-spaces. Thus, \( vX \) is a usco retract of an \( \text{AE}(0) \)-space. Therefore, by Theorem 1, \( vX \in \text{AE}(0) \).

(iii) \( \rightarrow \) (i) This implication is obvious.
**Corollary 1.** Let $X \in \text{A(N)E}(0)$ and let $F$ be a $G_{\delta}$-subset of $X$. Then the $G_{\delta}$-closure of $F$ in $X$ is also an A(N)E(0)-space.

**Proof.** Let $X \in \text{ANE}(0)$. Since $\nu X \in \text{AE}(0)$ there is a minimal usco retraction $r$ from $R^A$ to $\nu X$ for some $A$. The set $F$ is $G_{\delta}$ in $\nu X$ because $X$ is open in $\nu X$. Hence, $r^\#(F)$ is a $G_{\delta}$-subset of $R^A$. By Lemma 1, $G_{\delta}(r^\#(F))$ is a product of polish spaces, so $G_{\delta}(r^\#(F)) \in \text{AE}(0)$. Next, Lemma 9 implies that the $G_{\delta}$-closure $G_{\delta}(F)$ of $F$ in $\nu X$ is a usco retract of $G_{\delta}(r^\#(F))$. Thus, $G_{\delta}(F)$ is also an AE(0)-space. But $G_{\delta}(F) \cap X$ is open and dense in $G_{\delta}(F)$. Consequently $G_{\delta}(F) \cap X \in \text{ANE}(0)$. However, $G_{\delta}(F) \cap X$ is the $G_{\delta}$-closure of $F$ in $X$.

By the same arguments one can prove that the $G_{\delta}$-closure of $F$ in $X$ is an AE(0)-space if $X \in \text{AE}(0)$.

**Theorem 3.** Let $X$ be a pinnate in the sense of Arhangel’skii [1] ANE(0)-space. Then $\nu X$ is Lindelöf and Čech-complete.

**Proof.** First we will prove that $X$ is Čech-complete. Consider the Stone-Čech compactification $\beta X$ of $X$. Denote by $Z$ the space obtained from $\beta X$ by means of making the points of $\beta X \setminus X$ isolated. We observe that $X$ is a closed C-embedded subset of $Z$. Since $X \in \text{ANE}(0)$, there is a usco retraction from $U$ to $X$, where $U$ is an open set in $Z$ containing $X$. Now, to prove that $X$ is Čech-complete one can use the arguments of Przymusinski [7, the proof of Lemma 2].

Next, let $r_1$ be a usco mapping from $R^A$ to $\nu X$ for some $A$. Consider $R^A$ as a dense subset of $I^A$ by identifying $R$ with $(0,1)$, and put $Y = \text{cl}_{I^A}(\nu X)$. By Lemma 8, $r_1$ is extendable to a usco mapping $r$ from $I^A$ to $Y$. Wlog, we assume that $r$ is minimal. Put $H = r^\#(X)$. $H$ is a $G_{\delta}$-subset of $I^A$ because $X$ is Čech-complete. Since $G_{\delta}(X) = \nu X$, it follows from Lemma 9 that $r$ is a usco retraction from $G_{\delta}(H)$ to $\nu X$. So, $\nu X$ is closed in $G_{\delta}(H)$. But, by Lemma 1, $G_{\delta}(H)$ is a Lindelöf $G_{\delta}$-subset of $I^A$. Therefore, $\nu X$ is Lindelöf and Čech-complete.

**Corollary 2.** Every pinnate AE(0)-space is Lindelöf and Čech-complete.

An embedding $j$ of $X$ in $Y$ is said to be $d$-regular [11] (br. a $d$-embedding) if for every open subset $U$ of $j(X)$ there exists an open subset $e(U)$ of $Y$ such that the following conditions are fulfilled:

1. $e(\emptyset) = \emptyset$;
2. $e(U) \cap j(X) = U$;
3. $e(U) \cap e(V) = e(U \cap V)$;
Shirokov [11] proved that $X$ is a Dugundji space if and only if every embedding of $X$ in a Tychonoff cube is a $d$-embedding. We give a similar characterization of Čech-complete AE(0)-spaces.

**Theorem 4.** For a Čech-complete space $X$ the following conditions are equivalent:

(i) $\nu X$ is a Čech-complete Lindelöf AE(0)-space;
(ii) every $C$-embedding of $X$ in any space is a $d$-embedding;
(iii) $X$ is a $d$-embedded subset of $R^A$, for some $A$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose $X$ is a $C$-embedded subset of a space $Y$. Then there exists a mapping $h: Y \to R^{C(X)}$ such that $h|X$ is a homeomorphism and $\text{cl}_{R^{C(X)}}(h(X)) = \nu X$. Let $r$ be a usco retraction from $R^{C(X)}$ to $\nu X$. For every open set $U$ in $X$, we let $e(U) = h^{-1}(r^*(V(U)))$, where $V(U) = \bigcup\{W: W$ is open in $\nu X$ and $W \cap h(X) = h(U)\}$. It is easily seen that this operator satisfies the above three conditions. Thus, $X$ is $d$-embedded in $Y$.

(ii) $\Rightarrow$ (iii) This implication is obvious.

(iii) $\Rightarrow$ (i) Let $X$ be a $d$-embedded subset of $R^A$ for some $A$. So, there exists a $d$-regular operator $e$ from the topology of $X$ to the topology of $R^A$. Consider $R^A$ as a dense subset of $I^A$ and put $Y = \text{cl}_{I^A}(X)$. Define a usco mapping $r_1$ from $R^A$ to $Y$ by letting $r_1(x) = \cap\{\text{cl}_{I^A}(U): x \in e(U)\}$, for $x \in \bigcup\{e(U): U$ is open in $X\}$, and $r_1(x) = Y$, otherwise. Clearly, $r_1(x) = \{x\}$ for every $x \in X$. Next, by Lemma 8, $r_1$ is extendable to a usco mapping $r$ from $I^A$ to $Y$. We assume that $r$ is minimal. Since $X$ is Čech-complete, the set $H = r^*(X) = G_\delta$ in $I^A$. Lemma 9 implies that $r$ is a usco retraction from $G_\delta(H)$ to $G_\delta(X)$. By Lemma 1, $G_\delta(H)$ is a Lindelöf Čech-complete AE(0)-space. Therefore, $G_\delta(X)$ being a usco retract of $G_\delta(H)$, is a Lindelöf Čech-complete AE(0)-space too. It remains to prove that $G_\delta(X)$ is the Hewitt-realcompactification of $X$. It is known [2] that every AE(0)-space is perfectly $k$-normal in the space of Ščepin [10] and that every $G_\delta$-dense subset of a perfectly $k$-normal space $Z$ is $C$-embedded in $Z$ [12]. Hence, $X$ is $C$-embedded in $G_\delta(X)$. Therefore, $G_\delta(X)$ is the Hewitt-realcompactification of $X$.

**Corollary 3.** For a Čech-complete realcompact space $X$ the following conditions are equivalent:

(i) $X$ is a Lindelöf AE(0)-space;
(ii) every $C$-embedding of $X$ in any space is a $d$-embedding;
(iii) $X$ is a $d$-embedded subset of $R^A$, for some $A$. 
Let us note that the completeness in Theorem and Corollary 3 is essential. Indeed, every non-complete subspace of $R^{\aleph_0}$ is $d$-embedded in $R^{\aleph_0}$ but is not an AE(0)-space.

We have been unable to decide the following problems: Is every Lindelöf AE(0)-space Čech-complete? Is every normal AE(0)-space Lindelöf?

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