LOCALIZATION IN THE CLASSIFICATION OF FLAT MANIFOLDS

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Two compact flat Riemannian manifolds are called comparable if each one is a covering space of the other in such a way that the covering maps are affine and both the compositions of the covering maps increase distance locally by a constant factor. Considering comparability classes instead of affine-equivalence classes corresponds to localizing the algebra in calculations.

Introduction. This paper is concerned with compact flat Riemannian manifolds, i.e. smooth compact manifolds with a Riemannian connection for which the Levi-Civita connection is flat. These are all quotients of Euclidean space $\mathbb{R}^n$ by a group of isometries $\Gamma$ acting properly discontinuously. A continuous map between two such manifolds is called affine if it lifts to an affine map of $\mathbb{R}^n$. The rotational part of $\Gamma$, i.e. its image in $\text{GL}(\mathbb{R}^n)$, is called the holonomy group of the manifold and is always finite. Charlap [4] showed that the affine-equivalence classes of these manifolds with given holonomy group $G$ correspond bijectively with the isomorphism classes of a category $E_Z(G)$ defined in terms of the integral representations of $G$. For the purpose of calculation it is convenient to localize the integral representations to get a category $\hat{E}(G)$. This will be seen to correspond to the following geometric notion.

Definition. Two compact flat Riemannian manifolds $B_1$, $B_2$ are comparable if there exist affine covering maps $\theta_1: B_1 \to B_2$, $\theta_2: B_2 \to B_1$ such that $\theta_1 \circ \theta_2$ and $\theta_2 \circ \theta_1$ both increase distance locally by a factor $m$.

Section 1 covers the background material. In §2 we shall look at the endomorphisms of these manifolds and in §3 we shall prove the following.

Theorem A. There is a natural bijection between the isomorphism classes of $\hat{E}(G)$ and the comparability classes of compact flat Riemannian manifolds with holonomy group $G$. 
Section 4 investigates the extent to which the cohomology of the manifold is an invariant of the comparability class and §5 contains a calculation of the comparability classes when the holonomy group is the metacyclic group $D_{pq}$, $p, q$ primes.

1. Background. The main theorem on the structure of compact flat Riemannian manifolds is the following one ([1], [3], [11]).

**Theorem.** If $\Gamma$ is the fundamental group of a compact flat Riemannian manifold of dimension $n$ then $\Gamma$ fits into a short exact sequence

\[
0 \to M \to \Gamma \to G \to 1
\]

in which

(i) $G$ is finite and
(ii) $M$ is free-Abelian of rank $n$.

$G$ acts on $M$ by conjugation and
(iii) this action is faithful.

Any two embeddings of $\Gamma$ in Isom($\mathbb{R}^n$), the group of isometries of $\mathbb{R}^n$, are conjugate in Aff($\mathbb{R}^n$), the group of affine transformations of $\mathbb{R}^n$, and under such an embedding $M$ corresponds to the subgroup of pure translations and $G$ to the holonomy group.

Conversely, any abstract group which fits into an exact sequence with these properties can be realised as the fundamental group of a compact flat Riemannian manifold of dimension $n$, unique up to affine equivalence, and for each $n$ there are only finitely many such groups up to isomorphism. In fact any finite group can occur as the holonomy group $G$ of a compact flat Riemannian manifold for sufficiently large $n$ [1].

Thus the task of determining the compact flat Riemannian manifolds up to affine equivalence is equivalent to that of determining the groups $\Gamma$ which fit into a sequence $(\ast)$ and satisfy conditions (i), (ii) and (iii). First of all, recall that if $R$ is a Dedekind ring and $G$ a finite group then a finitely-generated $RG$-module which is $R$-projective is called an $RG$-lattice. In the sequence $(\ast)$ $M$ is a $ZG$-lattice and the sequence is determined by an element $\alpha \in H^2(G; M)$. We shall label the group $\Gamma$ and also the corresponding manifold (up to affine isomorphism) by $B(M, \alpha)$, or $B(G; M, \alpha)$ if we wish to stress the role of $G$. There remains the question of whether different $M$ or $\alpha$ can lead to the same $\Gamma$. This was considered in [4]: define $\alpha \in H^2(G; M)$ to be special if its restriction to each subgroup of prime order is non-zero. Then for any finite group $G$ we can define a category $E_R(G)$ to have as objects pairs $(M, \alpha)$ where $M$ is a
faithful $RG$-lattice and $a \in H^2(G; M)$ is special, and as morphisms pairs $(f, A): (M, \alpha) \rightarrow (N, \beta)$ where $A$ is an automorphism of $G$ and to specify $f$ we define another $RG$-lattice $\hat{A}(N)$ to be like $N$ as an $R$-module but with a new action of $G$, denoted $*$,

$$g^*m = A(g)m, \quad g \in G, m \in M;$$

$f$ is an $RG$-module homomorphism $M \rightarrow \hat{A}(N)$. There is an induced isomorphism $A_*: H^2(G; M) \rightarrow H^2(G; \hat{A}(N));$ we require $f_*(\alpha) = A_*(\beta)$.

**Theorem (Charlap).** The map $B$ defines a bijection between the isomorphism classes of $E_Z(G)$ and the affine-equivalence classes of compact flat Riemannian manifolds with holonomy group $G$.

This reduces the geometric problem of classifying the compact flat Riemannian manifolds with given holonomy group to an algebraic one involving $ZG$-lattices $M$ and $H^2(G; M)$. Charlap [4] gave the solution for $G = C_p$, the cyclic group of order $p$, which is one of the few cases where all the $ZG$-lattices are known.

The advantage of the comparability classes is that we only need to consider the genus of the $ZG$-lattice $M$, where two $ZG$-lattices $M, N$ are in the same genus if they are isomorphic when localized at any prime $p$ (we write $M_p \equiv N_p$) or equivalently when completed at any prime $p$ ($\hat{M}_p \equiv \hat{N}_p$). See [7]. These are much easier to handle.

**2. Construction of automorphisms.** In order to decide whether two manifolds $B(G; M, \alpha_1), B(G; M, \alpha_2)$ are affinely equivalent we have to be able to construct automorphisms of $M$. We shall use the following proposition which applies to any ring $\Lambda$.

**Proposition.** Let $0 \rightarrow L \xrightarrow{r} M \xrightarrow{s} N \rightarrow 0$ be an exact sequence of $\Lambda$-modules determined by $\phi \in \text{Ext}_\Lambda(N, L)$ and let $\alpha$ be an endomorphism of $L, \beta$ one of $N$ such that for some $\psi \in \text{Ext}_\Lambda(N, L)$, $\alpha_\psi = \psi_\beta = \phi$. Then there is an endomorphism $\gamma$ of $M$ which makes the following diagram commute.

$$
\begin{array}{ccc}
0 & \rightarrow & L \xrightarrow{r} M \xrightarrow{s} N \rightarrow 0 \\
\alpha \downarrow & & \beta \downarrow \\
0 & \rightarrow & L \xrightarrow{r} M \xrightarrow{s} N \rightarrow 0
\end{array}
$$

**Proof.** Hilton and Stammbach [9] give an interpretation of the functor Ext in terms of extensions, not only for the modules but also for the morphisms.
Let \( \psi \) determine the extension \( 0 \to L \to M' \to N \to 0 \). By definition of \( \alpha_* \) there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & L \\
\downarrow \alpha & & \downarrow \delta \\
0 & \to & M \\
\downarrow \text{id} & & \downarrow \text{id} \\
0 & \to & N \\
\end{array}
\]

in which the left-hand square is a push-out. By definition of \( \beta_* \) there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & L \\
\downarrow \text{id} & & \downarrow \epsilon \\
0 & \to & M \\
\downarrow \beta & & \downarrow \beta \\
0 & \to & N \\
\end{array}
\]

in which the right-hand square is a pull-back. Take \( \gamma = \delta \epsilon \). \( \square \)

Such an interpretation also works for \( H^2(G; M) \). In particular consider the endomorphism of \( M \) which is just multiplication by a constant \( m \). If \( m \) is prime to \( |G| \) then it does not affect \( H^2(G; M) \) so for any group \( \Gamma \) which is an extension of \( G \) by \( M \) we get a monomorphism \( \gamma \) satisfying

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow m & & \downarrow \gamma \\
0 & \to & \Gamma \\
\downarrow \text{id} & & \downarrow \text{id} \\
0 & \to & G \\
\end{array}
\]

If \( M \) is a faithful \( ZG \)-lattice then \( \Gamma \) can be embedded in \( \text{Aff}(\mathbb{R}^n) \) and \( \text{im} \gamma \) must be conjugate to \( \Gamma \) in \( \text{Aff}(\mathbb{R}^n) \) so if \( \Gamma \) is torsion-free we get an affine map from the corresponding flat manifold to itself which evidently increases distances by a factor \( m \).

**Definition.** An endomorphism of a flat manifold which increases all distances by a factor \( m \) is called expanding of degree \( m \).

**Theorem.** A flat manifold has an expanding endomorphism of degree \( m \) if and only if \( m \) is prime to the order of the holonomy group.

**Proof.** We have shown the existence above. On the other hand any expanding map of degree \( m \) leads to a diagram

\[
\begin{array}{ccc}
0 & \to & M \\
\downarrow m \phi & & \downarrow \gamma \\
0 & \to & \Gamma \\
\downarrow A & & \downarrow A \\
0 & \to & G \\
\end{array}
\]
with \( A \) an automorphism of \( G \) and \( \phi \) an automorphism of \( M \). The extension is determined by \( \alpha \in H^2(G; M) \) and the diagram implies \( m\phi_*\alpha A_*^{-1} = \alpha \), but \( \alpha \) is special so for any prime \( p \) dividing \( |G| \) let \( C \) be a cyclic subgroup of order \( p \); \( \text{res}^G_C \alpha \neq 0 \) so \( m\phi_* \text{res}^{-1}(C) \alpha \neq 0 \) so \( p \) does not divide \( m \), i.e. \( m \) is prime to \( |G| \).

**REMARK.** Epstein and Shub \([8]\) construct expanding endomorphisms of degree \( k|G| + 1, k \in \mathbb{Z}, k \geq 0 \).

3. Comparability. Recall that we defined two compact flat Riemannian manifolds \( B_1, B_2 \) to be comparable if there exist affine covering maps \( \theta_1: B_1 \rightarrow B_2, \theta_2: B_2 \rightarrow B_1 \) such that \( \theta_1 \circ \theta_2 \) and \( \theta_2 \circ \theta_1 \) are both expanding maps.

If \( B_1 = B(G_1; M_1, \alpha_1), B_2 = B(G_2; M_2, \alpha_2) \) and \( B_1 \) and \( B_2 \) are comparable then we have a diagram

\[
\begin{array}{ccc}
0 & \rightarrow & M_1 \rightarrow \pi_1(B_1) \rightarrow G_1 \rightarrow 1 \\
\phi_1 \downarrow & & \downarrow A_1 \downarrow \\
0 & \rightarrow & M_2 \rightarrow \pi_1(B_2) \rightarrow G_2 \rightarrow 1 \\
\phi_2 \downarrow & & \downarrow A_2 \downarrow \\
0 & \rightarrow & M_1 \rightarrow \pi_1(B_1) \rightarrow G_1 \rightarrow 1.
\end{array}
\]

\( \phi_2 \circ \phi_1 \) is multiplication by an integer \( m \), prime to \( p \), and \( A_2 \circ A_1 \) is an automorphism of \( G_1 \). Also \( A_1 \circ A_2 \) must be an automorphism of \( G_2 \) so \( A_1 \) and \( A_2 \) are isomorphisms.

\[
\phi_1 \circ \alpha_1 = \alpha_2 A_2^*, \quad \phi_2 \circ \alpha_2 = \alpha_1 A_2^*
\]

so

\[
m\alpha_1 = \alpha_1 A_2^* A_1^*.
\]

For any prime \( p \) dividing \( |G| \) let \( C \) be a cyclic subgroup of order \( p \), \( \text{res}^G_C \alpha_1 \neq 0 \) so \( m \text{res}^G_{(A_2A_1)}^{-1}C \alpha_1 \neq 0 \) so \( p \) does not divide \( m \), i.e. \( m \) is prime to \( |G| \).

Thus if we localize at any prime \( p \) in \( |G| \) then, in the category \( E_{\mathbb{Z}_p}(G) \), \((\phi_2\phi_1,p,A_2 A_1)\) is an isomorphism as is \((\phi_1\phi_2,p,A_1 A_2)\), so \((\phi_1,p,A_1)\) is mono and epi and hence \( \phi_1,p \) is an isomorphism of \( M_{1,p} \) with \( M_{2,p} \) if we identify \( G_1 \) with \( G_2 \) using \( A_1 \). \( M_1 \) and \( M_2 \) must be in the same genus under this identification and \( \phi_1 \) induces an isomorphism

\[
\phi_1^*: H^2(G_1; M_1) \rightarrow H^2(G_1; M_2) \quad \text{with} \quad \phi_1^* \alpha_1 = \alpha_2.
\]

**DEFINITION.** Suppose \( B_1 = B(G_1; M_1, \alpha_1), B_2 = B(G_2; M_2, \alpha_2) \). We shall say that \( B_1 \) and \( B_2 \) are algebraically comparable if and only if for some identification \( G_1 \equiv G_2 = G \), say, \( M_1 \) is in the same genus as \( M_2 \) and for each prime \( p \) dividing \( |G| \) there is an isomorphism \( \psi_p: \hat{M}_{1,p} \rightarrow \hat{M}_{2,p} \) such that if \( \alpha_{i,p} \) is the image of \( \alpha_i \) in \( H^2(G; \hat{M}_{i,p}) \) then \( \psi_p \alpha_{1,p} = \alpha_{2,p} \).
We shall need the following lemma which is based on one of Roiter [cf. 7, p. 645], see also [5].

**Lemma A.** Let $M$ and $N$ be $\mathbb{Z}G$-lattices in the same genus, so for each $p$ dividing $|G|$ there are isomorphisms $\psi_p : \hat{M}_p \rightarrow \hat{N}_p$. Then there is a homomorphism $\phi : M \rightarrow N$ such that $\phi_\ast = \psi_\ast : H^\bullet(G; \hat{M}_p) \rightarrow H^\bullet(G; \hat{N}_p)$ and there is an exact sequence

$$0 \rightarrow M \overset{\phi}{\rightarrow} N \rightarrow T \rightarrow 0$$

with $T$ finite of order prime to $|G|$.

**Proof.** $\hat{Z}_p \otimes \text{Hom}_{\mathbb{Z}G}(M, N) \cong \text{Hom}_{\mathbb{Z}G}(\hat{M}_p, \hat{N}_p)$. $\text{Hom}_{\mathbb{Z}G}(M, N)$ is densely embedded in $\text{Hom}_{\mathbb{Z}G}(\hat{M}_p, \hat{N}_p)$ under the $p$-adic topology so for each $p$ dividing $|G|$ we may choose $u_p \in \text{Hom}(M, N)$ with $\hat{u}_p \equiv \psi_p \mod p^r$ where $r$ is large enough that $p^r$ does not divide $|G|$. By the Chinese Remainder Theorem choose integers $\alpha_p$ such that

$$\alpha_p \equiv 1 \mod p^r, \quad \alpha_p \equiv 0 \mod p^{r'}, \quad p' \neq p$$

where $p'$ varies the over all primes dividing $|G|$ except $p$, and $p^{r'} \nmid |G|$. Then $\phi = \sum_{p \mid |G|} \alpha_p u_p$ has the correct effect on cohomology since $\phi_p \equiv u_p \mod p^r$ and $H^\bullet(G; -)_p$ is annihilated by $p^r$. Similarly we can get $\theta \in \text{Hom}(N, M)$ and

$$\hat{\theta}_p \hat{\phi}_p \equiv 1 \mod p^r, \quad \hat{\phi}_p \hat{\theta}_p = 1 \mod p^r.$$ 

By Nakayama's lemma, $\hat{\phi}_p \hat{\theta}_p$ is onto $\hat{M}_p$ so $\hat{\theta}_p \hat{\phi}_p$ must be an automorphism of $\hat{M}_p$ and in particular $\ker \hat{\phi}_p = 0$. Hence $(\ker \phi)_p = 0$ and since $\ker \phi$ is torsion-free, $\ker \phi = 0$. Let $T = \text{coker} \phi$, so

$$0 \rightarrow M \overset{\phi}{\rightarrow} N \rightarrow T \rightarrow 0$$

Since $\hat{\phi}_p$ is an isomorphism, $\hat{T}_p = 0$, so $T$ is finite and $|T|$ is prime to $|G|$.

Now if we let $m = |T|$, $mN \subset \phi(M)$, so we can define $\phi' : N \rightarrow M$ by $\phi'(n) = \phi^{-1} (mn)$, $n \in N$. On cohomology $\phi'_\ast$ is iso and $\phi'_\ast \phi_\ast = m$ is iso so $\phi'_\ast$ is iso. By the proposition of §2 and the discussion afterwards, $\phi$ and $\phi'$ lead to covering maps $\theta : B_1 \rightarrow B_2$, $\theta' : B_2 \rightarrow B_1$ which show that $B_1$ and $B_2$ are comparable. We have shown:

**Theorem B.** Algebraic comparability is equivalent to comparability for flat manifolds.
It is now clear how to define the category \( \hat{\mathcal{E}}(G) \) in order to make Theorem A valid. The objects are the same as those of \( E_Z(G) \) but the morphisms must be those occurring in the definition of algebraically comparable. That is, a morphism \((M, \alpha) \to (N, \beta)\) is now a pair \(\{(f_p), A\}\) where \(A\) is an automorphism of \(G\) and for each \(p\) which divides \(G\) there is a \(\mathbb{Z}G\)-module automorphism \(f_p: M_p \to A(N_p)\) such that \(f_p(\alpha_p) = A(\beta_p)\).

**REMARKS.** (a) Since there are exist non-isomorphic \(\mathbb{Z}G\)-lattices in the same genus, for example when \(G\) is cyclic of order 23, the construction yields examples of manifolds which cover each other in a non-trivial way. Charlap [4] has examples which become affinely equivalent after taking the product with a circle.

(b) It is perhaps interesting to see what happens if we weaken the condition in the definition of comparable that the compositions should be expanding. A lot of the integral representation theory is lost: for example let \(B(G; M, \alpha)\) be a flat manifold and let \(S, T\) be two \(\mathbb{Z}G\)-lattices which are isomorphic as \(\mathbb{Q}G\)-spaces, but not even in the same genus. Then \(B(G; M \oplus S, \alpha \oplus 0)\) and \(B(G; M \oplus T, \alpha \oplus 0)\) both cover each other (since \(S\) is isomorphic to a submodule of \(T\) of finite index and vice versa).

4. The cohomology of a flat manifold. It is interesting to see how the cohomology of a flat manifold \(B(G; M, \alpha)\) depends on \(M\) and \(\alpha\). We shall calculate \(H^*\#B\) using the spectral sequence of the extension \(0 \to M \to \pi_1(B) \to G \to 1\), (since \(B\) is an Eilenberg-MacLane space \(K(\pi_1(B), 1))\). For coefficients in a ring \(R\),

\[
E_2^{ij}(R) = H^i(G; H^j(M; R)) = H^i(G; \Lambda^j H^1(M; R)).
\]

**PROPOSITION.** The Betti numbers of \(B(G; M, \alpha)\) depend only on \(M\).

**Proof.** \(E_2^{ij}(\mathbb{Q}) = 0\) unless \(i = 0\).

\[
H_j(B; \mathbb{Q}) \cong E_2^{0j}(\mathbb{Q}) \cong (\Lambda^j (M \otimes \mathbb{Q}))^G. \quad \square
\]

**PROPOSITION.** The \(E_2\) terms of the spectral sequence (with any coefficients) depend only on the genus of \(M\).

**Proof.** The torsion-free part is taken care of as above. As for the torsion, \(H^1(M; R) \cong \text{Hom}(M, R)\) and \(H^i(G; N)_p \cong H^i(G; N_p)\) and \(\text{Hom}(M, R) \otimes \mathbb{Z}_p \cong \text{Hom}(M_p, R \otimes \mathbb{Z}_p)\). \(\square\)
Theorem. If two flat manifolds $B_1$, $B_2$ are comparable then the cohomology groups $H^*(B_1; \mathbb{Z})$, $H^*(B_2; \mathbb{Z})$ are isomorphic. If $R$ is a subring of $\mathbb{Q}$ with all primes not in a finite set inverted or $R$ is a field then there is an affine map $f: B_1 \to B_2$ which induces an isomorphism of rings $f^*: H^*(B_2; R) \cong H^*(B_1; R)$.

Proof. Suppose $R \subset \mathbb{Q}$ and all primes not in $|B|$ are invertible in $R$. We shall show that $\theta^*: H^*(B_2; R) \to H^*(B_1; R)$ is an isomorphism when $\theta: B_1 \to B_2$ is a covering map constructed in the proof that algebraically comparable implies comparable. Let $\phi: M_1 \to M_2$ be the induced map on $\mathbb{Z}G$-lattices; it is sufficient to show that this gives an isomorphism $\phi^*: H^1(M_2; R) \to H^1(M_1; R)$ (since $H^i(M_j; R) \cong \Lambda^i H^1(M_j; R)$). But, by construction, $\text{coker} \phi$ is finite of order prime to $|G|$ and so from the exact sequence

$$0 \to \text{Hom}(\text{coker} \phi, R) \to \text{Hom}(M_2, R) \to \text{Hom}(M_1, R) \to \text{Ext}(\text{coker} \phi, R) \to 0$$

we see that this is the case ($H^1(M; R) \cong \text{Hom}(M, R)$). For other $R$ we must examine the proof of Lemma A to see that $\text{coker} \phi$ can be made coprime to an additional finite set of primes.

The only torsion that can occur in $H^*(B; -)$ is at primes in $|G|$ since the only $E_2^{**}$ terms which are not annihilated by $|G|$ are the $E_2^{0*}$ and there are no differentials with image in $E^{0*}$. Thus we can deduce the result for $R = \mathbb{Z}$ from the case with the primes not in $|G|$ inverted. □

Remarks. (a) A version of this theorem for $G$ of prime order was proved by Charlap and Vasquez [5], who also calculate the groups $H^*(B; \mathbb{Z})$ in this case.

(b) It is not known whether the rings $H^*(B; \mathbb{Z})$ must be isomorphic.

5. Metacyclic groups. Let $D_{pq}$ be a metacyclic group of order $pq$, with $p, q$ distinct primes and $q$ dividing $p - 1$:

$$1 \to C_p \to D_{pq} \to C_q \to 1,$$

$$D_{pq} = \langle x, y | x^p = y^q = 1, yxy^{-1} = x^r \rangle,$$

where $r$ is a primitive $q$th root of 1 mod $p$. We shall find the comparability classes of flat manifolds with holonomy group isomorphic to $D_{pq}$. The genera of the indecomposable $D_{pq}$-lattices were determined by Pu [10] and are given in [7, pp. 747–751], whose description we follow here.
Let \( R = \mathbb{Z}[\xi] \), where \( \xi \) is a primitive \( p \)th root of 1. Define an automorphism \( \sigma \) of \( R \) over \( \mathbb{Z} \) by \( \sigma(\xi) = \xi^r \) and let \( D_{pq} \) act on \( R \) by
\[
xr = \xi^r, \quad yr = \sigma(R), \quad r \in R.
\]
Thus \( R \) is a \( \mathbb{Z}G \)-lattice and so is the ideal \( P = (1 - \xi)R \) or any power of it, \( P^i \). There are also the indecomposable lattices for the factor \( C_q \), in particular \( \mathbb{Z}, \mathbb{Z}H \) and its augmentation ideal \( S \), where \( H = \langle y \rangle \).

If we localize at \( q \) the indecomposable \( \mathbb{Z}_qD_{pq} \)-lattices are as follows (we shall calculate the cohomology groups later, but include them here for convenience).

<table>
<thead>
<tr>
<th>( \mathbb{Z}<em>qD</em>{pq} )-lattice M</th>
<th>( H^2(D_{pq}; M) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_q )</td>
<td>( \mathbb{Z}/q )</td>
</tr>
<tr>
<td>( S_q )</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbb{Z}_qH )</td>
<td>0</td>
</tr>
<tr>
<td>( R_q )</td>
<td>0</td>
</tr>
</tbody>
</table>

Alternatively if we complete at \( p \) the indecomposable lattices are

<table>
<thead>
<tr>
<th>( \hat{\mathbb{Z}}<em>pD</em>{pq} )-lattice M</th>
<th>( H^2(D_{pq}; M) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\mathbb{Z}}_i ) 0 ≤ ( i ) ≤ (q - 1)</td>
<td>( \mathbb{Z}/p, ) ( i = 1, )</td>
</tr>
<tr>
<td>( \hat{\mathbb{Z}}_i ) 0 ≤ ( i ) ≤ (q - 1)</td>
<td>0, otherwise</td>
</tr>
<tr>
<td>( \hat{\mathbb{Z}}_i^G ) 0 ≤ ( i ) ≤ (q - 1)</td>
<td>0</td>
</tr>
</tbody>
</table>

\( \hat{\mathbb{Z}}_i \) is a copy of \( \hat{\mathbb{Z}}_p \) on which \( x \) acts as 1 and \( y \) acts as \( \theta^i \), where \( \theta \) is the primitive \( q \)th root of 1 in \( \hat{\mathbb{Z}}_p \) with \( \theta \equiv r \mod(p\mathbb{Z}_p) \). \( \hat{\mathbb{Z}}_i^G \) is induced from \( \hat{\mathbb{Z}}_i \) restricted to \( H \) and is also the unique non-split extension of \( \hat{\mathbb{Z}}_i \) by \( \hat{\mathbb{Z}}_{i+1} \)

(†)

\[
0 \to \hat{\mathbb{Z}}_{i+1} \to \hat{\mathbb{Z}}_i^G \to \hat{\mathbb{Z}}_i \to 0.
\]

**Lemma.** (See [2].) Let \( N \) be a normal subgroup of \( G \) with \( |N| \) prime to \( |G/N| \). Then

\[
H'(G; M) \cong H'(G/N; M^N) \oplus H'(N; M)^{G/N}.
\]

**Proof.** Use the spectral sequence for \( 1 \to N \to G \to G/N \to 1 \).

**Remark.** If the action of \( N \) on \( M \) is trivial, so \( H^2(N; M) \cong H^1(N; \mathbb{Q} \otimes M/M) \cong \text{Hom}(N, \mathbb{Q} \otimes M/M) \), then the action of \( G \) on \( H^2(N; M) \) is given by

\[
(sf)(x) = gf(xg^{-1}), \quad f \in \text{Hom}(N, \mathbb{Q} \otimes M/M), \quad x \in N, \quad g \in G
\]

(see [2]).
The cohomology can now be calculated using the formula for the cohomology of a cyclic group. Observe that if $M$ is $\mathbb{Z}_q D_{pq}$-lattice then $H^2(D_{pq}; M)$ has only $q$-torsion so $H^2(D_{pq}; M) \cong H^2(D_{pq}/C_p; M^{C_p})$. Hence the results for $\mathbb{Z}_q$, $S_q$ and $\mathbb{Z}_q H$. Also, $R^C_q = 0$.

Over $\hat{\mathbb{Z}}_p$, $H^2(D_{pq}; M) \cong H^2(C_p; M)^H$.

$$H^2(C_p; P^i) = 0.$$ 

$H^2(D_{pq}; \hat{\mathbb{Z}}_i^G) \cong H^2(H; \hat{\mathbb{Z}}_i) \cong 0$ since $\hat{\mathbb{Z}}_i^G$ is induced from $\mathbb{Z}_i$.

$H^2(C_p; \hat{\mathbb{Z}}_i) \cong \text{Hom}(C_p, (\hat{\mathbb{Z}}_p \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_i) / \hat{\mathbb{Z}}_i) \cong \text{Hom}(C_p, \hat{\mathbb{Z}}_i / p \hat{\mathbb{Z}}_i)$.

Let $f \in \text{Hom}(C_p, \hat{\mathbb{Z}}_i / p \hat{\mathbb{Z}}_i)$; then if $f = yf$,

$$f(x) = yf(y^{-1}x) = \theta^i f(x^s)$$

where $sr \equiv 1 \mod p$

$$= s\theta^i f(x),$$

so $i = 1$.

Representatives of the $2^q + 2^{q-1} + q + 2$ indecomposable genera of the $D_{pq}$-lattices are as follows.

<table>
<thead>
<tr>
<th>lattice $M$</th>
<th>$M_q$</th>
<th>$\hat{M}_p$</th>
<th>$H^2(D_{pq}; M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p^r$, $0 \leq i \leq q - 1$</td>
<td>$R_q$</td>
<td>$\hat{P}_i$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_q$</td>
<td>$\hat{\mathbb{Z}}_0$</td>
<td>$\mathbb{Z}/q$</td>
</tr>
<tr>
<td>$S$</td>
<td>$S_q$</td>
<td>$\sum_{i=1}^{q-1} \hat{\mathbb{Z}}_i$</td>
<td>$\mathbb{Z}/p$</td>
</tr>
<tr>
<td>$H$</td>
<td>$\mathbb{Z}_q H$</td>
<td>$\sum_{i=0}^{q-1} \hat{\mathbb{Z}}_i$</td>
<td>$\mathbb{Z}/p$</td>
</tr>
<tr>
<td>$X_T$</td>
<td>$S_q + R_q^{[T]}$</td>
<td>$\sum_{i \in T} \hat{\mathbb{Z}}<em>i^{G}$ + $\sum</em>{i \notin T \cup {1}} \hat{\mathbb{Z}}_i^{G}$</td>
<td>{ $\mathbb{Z}/p$ if $2 \not\in T (mod q)$ \ $0$ otherwise $}</td>
</tr>
<tr>
<td>$Y_T$</td>
<td>$\mathbb{Z}_q H + R_q^{[T]}$</td>
<td>$\sum_{i \in T} \hat{\mathbb{Z}}<em>i^{G}$ + $\sum</em>{i \notin T} \hat{\mathbb{Z}}_i^{G}$</td>
<td>{ $\mathbb{Z}/p$ if $2 \not\in T (mod q)$ \ $0$ otherwise $}</td>
</tr>
<tr>
<td>$V$</td>
<td>$R_q + \mathbb{Z}_q$</td>
<td>$\hat{\mathbb{Z}}_0^{G}$</td>
<td>$\mathbb{Z}/q$</td>
</tr>
</tbody>
</table>

where $T$ is any non-empty subset of $\{0, 1, \ldots, q - 1\}$ except that to form $X_T$ we cannot have $1 \in T$.

The cohomology groups are as given since $H^*(G; M)_p \cong H^*(G; M_p) \cong H^*(G; \hat{M}_p)$ for any prime $p$.

$H^2(D_{pq}; M)$ has a special point if and only if it has both $p$-torsion and $q$-torsion since $\text{cor}^G_H \text{res}^G_H = |G:H|x$. In any case the comparability class depends only on the genus since all the $p$-torsion in $H^2(D_{pq}; M)$ comes from $\hat{\mathbb{Z}}_p$, and all the $q$-torsion from the irreducible $\mathbb{Z}_q$ and it is easy to construct an automorphism of these which takes any non-zero element of $H^2$ to any other.
However the automorphisms of the group permute the $\hat{\mathbb{Z}}_i$, $i \neq 0$, and hence also the $P^i$, $i \neq 1$, in order to preserve sequence $(\dagger)$. This permutes the sets $T$ by changing the elements not equal to 1 according to $\alpha \mapsto (\alpha - 1)^r + 1 \mod q$ for some $r$ depending on the automorphism, and any $r$ prime to $q$ is possible. Note that it acts in the same way on each copy of $T$.

**Theorem C.** The comparability classes of flat manifolds with holonomy group isomorphic to $D_{pq}$ are in one-to-one correspondence with the equivalence classes of the genera of faithful special $D_{pq}$-lattices under the relation on $T$ described above.

**References**


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