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**THE ETA INVARIANT,  $\text{Pin}^c$  BORDISM, AND EQUIVARIANT  
 $\text{Spin}^c$  BORDISM FOR CYCLIC 2-GROUPS**

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THE ETA INVARIANT,  $\text{Pin}^c$  BORDISM,  
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The eta invariant and equivariant Stiefel-Whitney numbers completely detect  $Z_n$  equivariant  $\text{Spin}^c$  bordism and  $\text{Pin}^c$  bordism. The additive structure of  $\text{Pin}^c$  bordism and of equivariant  $\text{Spin}^c$  bordism for cyclic 2-groups is determined using these invariants in terms of  $K$ -theory. The analysis is used to embed the  $K$ -theory in the bordism.

**0. Introduction.** The eta invariant of Atiyah-Patodi-Singer [4] is an  $R/Z$  valued measure of the spectral asymmetry of self-adjoint partial differential operators. It defines both equivariant bordism and locally flat  $K$ -theory invariants in suitable categories. Let  $n$  be a power of 2, let  $Z_n = \{\lambda \in C: \lambda^n = 1\}$  act by scalar multiplication on the unit sphere  $S^{2k-1}$  of  $C^k$ , and let  $L^k(n) = S^{2k-1}/Z_n$  be a generalized lens space.

**THEOREM 0.1.** (a) *The eta invariant and equivariant Stiefel-Whitney numbers completely detect the  $Z_n$ - $\text{Spin}^c$  reduced bordism groups  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ .*

(b) *Let  $A_{2k}(n) = 0$ , and let  $A_{2k-1}(n)$  be the subgroup of  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  generated by all possible  $\text{Spin}^c$  structures on  $L^k(n)$ . Let  $\ker_*(\eta, n)$  be the kernel of all eta invariants. Then as additive groups,*

$$\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n) \approx \{A_*(n) \otimes Z[CP^2, CP^4, \dots]\} \oplus \ker_*(\eta, n),$$

$$\ker_m(\eta, n) \approx \bigoplus_{j < m} \text{Tor}(\Omega_j^{\text{Spin}^c}), \quad \text{and} \quad A_{2k-1}(n) \approx \tilde{K}(S^{2k+1}/Z_n).$$

**REMARK.** Equivariant bordism decomposes as a direct sum for different primes so it suffices to study prime powers. Gilkey [13] showed the eta invariant completely detects  $\tilde{\Omega}_*^U(BZ_v)$  for all  $v$  and  $\tilde{\Omega}_*^{\text{SO}}(BZ_v)$  for  $v$  odd. The arguments given there generalize to show the eta invariant completely detects  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_v)$  if  $v$  is odd. See also Wilson [20]. The coefficient ring  $\Omega_*^U$  is torsion free; all the torsion in  $\Omega_*^{\text{SO}}$  and  $\Omega_*^{\text{Spin}^c}$  is of order 2. The torsion in  $\Omega_*^{\text{Spin}^c}$  enters in an essential fashion when  $n$  is a power of 2 as we shall see. The  $K$ -theory of the lens spaces is well known. Let  $n = 2^v$ . If  $1 \leq i < n$  choose  $s$  so  $2^s \leq i < 2^{s+1}$ . If  $i > k$ , let  $t(i, k, n) = 0$ . If  $i \leq k$ , let  $t(i, k, n) = v - s + [(k - i)/2^s]$ . Then  $A_{2k-1}(n) = \bigoplus_{i=1}^{n-1} Z_{2^{t(i,k,n)}}$  by Fujii et al. [9]. One can also show  $A_*(n) = bu_*(BZ_n)$ .

The Smith homomorphism defines an isomorphism between  $\tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)$  and  $\tilde{\Omega}_{m-1}^{\text{Pin}^c}$ ; under this isomorphism, the eta invariant for odd dimensional  $Z_2\text{-Spin}^c$  manifolds corresponds to the eta invariant of even dimensional  $\text{Pin}^c$  manifolds.

**THEOREM 0.2.** (a) *The eta invariant and Stiefel-Whitney numbers completely detect the  $\text{Pin}^c$  bordism groups  $\Omega_*^{\text{Pin}^c}$ .*

(b) *Let  $B_{2k+1} = 0$  and  $B_{2k} = Z_{2^{k+1}}$  be the subgroup of  $\Omega_{2k}^{\text{Pin}^c}$  generated by real projective space  $RP^{2k}$ . Let  $\ker_*(\eta)$  be the kernel of all eta invariants. Then as additive groups*

$$\Omega_*^{\text{Pin}^c} \approx \{ B_* \otimes Z[CP^2, CP^4, \dots] \} \oplus \ker_*(\eta) \quad \text{and}$$

$$\ker_m(\eta) \approx \bigoplus_{j \leq m} \text{Tor}(\Omega_j^{\text{Spin}^c}).$$

In [5] we studied  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  and  $\Omega_*^{\text{Pin}^c}$  using the Anderson, Brown, and Peterson splitting of the spectrum  $M\text{Spin}^c$ . The methods of that paper were homotopy theoretic. In this paper, we use geometry as a bridge between the analysis and the topology and obtain explicit representatives of the generators. In particular, the splitting  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n) = \{ A_* \otimes Q_* \} \oplus \ker_*(\eta, n)$  is an analytic splitting whereas in [5] it was a purely algebraic splitting. This paper rests heavily upon the results of Anderson-Brown-Peterson [1] and of Stong [18, 19]. We refer to Giambalvo [10] for analogous results regarding (S)Pin bordism. The results of Stong show that for unitary and  $\text{Spin}^c$  bordism, all relations among characteristic numbers follow from the Atiyah-Singer index theorem. Theorem 0.1 is a generalization of these results to equivariant bordism.

Here is a brief guide to the paper. In the first section, we discuss the results concerning  $\Omega_*^{\text{Spin}^c}$  and the equivariant Stiefel-Whitney numbers we shall need. In the second section, we define the eta invariant and recall its properties. In the third section we discuss the Smith homomorphism and in the fourth section we complete the proofs of Theorems 0.1 and 0.2. We acknowledge with pleasure helpful correspondence and conversations with Professors Giambalvo, Landweber, and Peterson.

**1. The  $\text{Spin}^c$  bordism ring and equivariant Stiefel-Whitney classes.** Let  $\text{Spin}(m)$  be the universal cover group of the special orthogonal group

$\mathrm{SO}(m)$  for  $m > 2$ . Since  $\pi_1(\mathrm{SO}(m)) = \mathbb{Z}_2$ , there is a non-trivial short exact sequence  $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathrm{Spin}(m) \xrightarrow{\pi} \mathrm{SO}(m) \rightarrow 0$ ; define  $\mathrm{Spin}(m)$  in terms of Clifford algebras if  $m = 1, 2$  (see Atiyah-Bott-Shapiro [3]). Let  $\mathrm{Spin}^c(m) = \mathrm{Spin}(m) \times \mathrm{U}(1)/\mathbb{Z}_2$  by identifying  $(g, \lambda) = (-g, -\lambda)$ . Let  $\tau(g, \lambda) = \lambda^2$  define a representation  $\tau: \mathrm{Spin}^c(m) \rightarrow \mathrm{U}(1)$ , then  $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathrm{Spin}^c(m) \xrightarrow{\pi \times \tau} \mathrm{SO}(m) \times \mathrm{U}(1) \rightarrow 0$ . Similarly if  $\mathrm{O}(m)$  is the orthogonal group, let  $\mathrm{Pin}(m)$  be the non-trivial double cover for  $m > 2$ , and let  $\mathrm{Pin}^c(m) = \mathrm{Pin}(m) \times \mathrm{U}(1)/\mathbb{Z}_2$ .  $\mathrm{Spin}^c(m)$  is the connected component of the identity in  $\mathrm{Pin}^c(m)$ . The forgetful homomorphism from  $\mathrm{U}(m)$  to  $\mathrm{SO}(2m)$  lifts to  $\mathrm{Spin}^c(2m)$  and the determinant representation lifts to  $\tau$  (see Hitchin [15]).

Let  $W^* = H^*(\mathrm{BO}; \mathbb{Z}_2) = \mathbb{Z}_2[w_j]$  be the algebra of Stiefel-Whitney classes. Let  $V$  be a real vector bundle over a compact manifold.  $V$  is orientable if  $w_1(V) = 0$ .  $V$  admits a  $\mathrm{Pin}^c$  structure if  $w_2(V)$  is the mod 2 reduction of an integral class.  $V$  admits a  $\mathrm{Spin}^c$  structure if  $w_1 = 0$  and if  $w_2$  is integral—i.e.  $V$  is both oriented and  $\mathrm{Pin}^c$ .

LEMMA 1.1. *Let  $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$  be a short exact sequence of vector bundles over a compact manifold  $M$ .*

(a) *If two of the  $V_i$  admit  $\mathrm{Spin}^c$  structures, then there is a natural  $\mathrm{Spin}^c$  structure induced on the third.*

(b) *If one of the  $V_i$  admits a  $\mathrm{Spin}^c$  structure and another admits a  $\mathrm{Pin}^c$  structure, then there is a natural  $\mathrm{Pin}^c$  structure induced on the third.*

(c) *If two of the  $V_i$  admit  $\mathrm{Pin}^c$  structures and if the third is orientable, then there is a natural  $\mathrm{Spin}^c$  structure on the third.*

*Proof.* As this is an elementary calculation in characteristic classes, we omit details in the interests of brevity. This fails if we replace  $(\mathrm{S})\mathrm{Pin}^c$  by  $(\mathrm{S})\mathrm{Pin}$ .

A  $(\mathrm{S})\mathrm{Pin}^c$  structure on a compact manifold  $N$  with boundary  $M$  is a  $(\mathrm{S})\mathrm{Pin}^c$  structure on the tangent bundle  $T(N)$ . Since  $T(N)|_M = T(M) \oplus 1$ , we use Lemma 1.1 to induce a  $(\mathrm{S})\mathrm{Pin}^c$  structure on  $M$ . Let  $\Omega_m^{(\mathrm{S})\mathrm{Pin}^c}$  denote the bordism group of compact smooth  $m$ -dimensional  $(\mathrm{S})\mathrm{Pin}^c$  manifolds modulo the subgroup which bound and let  $\Omega_*^{(\mathrm{S})\mathrm{Pin}^c}$  denote the corresponding graded group. We define  $\Omega_*^{(\mathrm{S})\mathrm{O}}$  similarly. If  $M$  admits a stable unitary structure, then there is a natural  $\mathrm{Spin}^c$  structure induced on  $M$ . Let  $CP^k$  be complex projective space with the  $\mathrm{Spin}^c$  structure induced

from the holomorphic structure. The direct sum of two  $\text{Spin}^c$  bundles is a  $\text{Spin}^c$  bundle and the direct sum of a  $\text{Spin}^c$  bundle and a  $\text{Pin}^c$  bundle is a  $\text{Pin}^c$  bundle. Consequently, Cartesian product makes  $\Omega_*^{\text{Spin}^c}$  into a graded ring and makes  $\Omega_*^{\text{Pin}^c}$  into an  $\Omega_*^{\text{Spin}^c}$  module. Since the direct sum of two  $\text{Pin}^c$  bundles is not a  $\text{Pin}^c$  bundle in general, there is no natural ring structure on  $\Omega_*^{\text{Pin}^c}$ . The Stiefel-Whitney numbers are  $Z_2$  valued  $\Omega_*^0$  bordism invariants completely detecting  $\Omega_*^0$ . The forgetful functor maps  $\Omega_*^{(\text{S})\text{Pin}^c} \rightarrow \Omega_*^0$  and lets us regard the Stiefel-Whitney numbers as (S) $\text{Pin}^c$  bordism invariants.

The representation  $\tau: \text{Spin}^c(m) \rightarrow U(1)$  defines a complex line bundle over any  $\text{Spin}^c$  manifold  $M$ ; let  $c_1(M) \in H^2(M; Z)$  be the Chern class of this bundle. Let  $p_k(M) \in H^{4k}(M; Z)$  be the Pontrjagin classes. We form the ring generated by  $\{c_1, p_*\}$ . The integer and rational characteristic numbers obtained from this ring are called the Chern/Pontrjagin numbers. Let  $\Lambda^k$  be the complexified exterior representations of  $\text{SO}(m)$ ; extend the  $\Lambda^k$  to  $\text{Spin}^c(m)$  by composing with the projection  $\pi$ . Let  $R(\text{Spin}^c)$  be the free polynomial algebra  $R(\text{Spin}^c) = Z[\Lambda^k, \tau]$ ; this is not the full representation ring of  $\text{Spin}^c$ . If  $\theta \in R(\text{Spin}^c)$  and if  $M \in \Omega_{\text{even}}^{\text{Spin}^c}$ , let  $\text{index}(\theta, M) \in Z$  be the index of the  $\text{Spin}^c$  complex over  $M$  with coefficients in the virtual bundle defined by  $\theta$ . Since  $\text{index}(\theta, M)$  is expressible in terms of rational Chern-Pontrjagin numbers, it is a bordism invariant. Since  $\text{index}(\theta, M) \in Z$ ,  $\text{index}(\theta, M) = 0$  if  $M \in \text{Tor}(\Omega_*^{\text{Spin}^c})$ . We set  $\text{index}(\theta, M) = 0$  for notational convenience if  $M \in \Omega_{\text{odd}}^{\text{Spin}^c}$ . We summarize below the results we shall need concerning  $\Omega_*^{\text{Spin}^c}$  and refer to Anderson-Brown-Peterson [1] and Stong [18, 19] for details.

**THEOREM 1.2.** *Let  $P_* = Z[CP^1, CP^2, CP^4, \dots, CP^{2k}, \dots]$ .*

(a)  $\Omega_*^{\text{Spin}^c}$  is a commutative ring. All the torsion in  $\Omega_*^{\text{Spin}^c}$  is of order 2. If  $M \in \Omega_*^{\text{Spin}^c}$ , then  $M = 0$  iff all the rational Chern/Pontrjagin numbers of  $M$  vanish and all the Stiefel-Whitney numbers of  $M$  vanish.

(b)  $P_*$  embeds in  $\Omega_*^{\text{Spin}^c}$  and  $\Omega_*^{\text{Spin}^c} \otimes_Z Z_2 = (P_* \otimes_Z Z_2) \oplus \text{Tor}(\Omega_*^{\text{Spin}^c})$ . If  $M \in P_*$ , then  $M \in 2^v P_*$  iff  $\text{index}(\theta, M)$  is divisible by  $2^v$  for all  $\theta \in R(\text{Spin}^c)$ .

**REMARK.** (b) is a scholium to the theorem of Stong [18, 19] that all relations among characteristic numbers in  $\Omega_*^{\text{Spin}^c}/\text{torsion}$  are given by the index theorem. Let  $\pi_Z(m) = \text{Rank}_Z(\Omega_m^{\text{Spin}^c}) = \text{Rank}_Z(P_m)$  and let  $\pi_2(m) = \text{Rank}_2(\text{Tor}(\Omega_m^{\text{Spin}^c}))$ ; since all torsion in  $\Omega_m^{\text{Spin}^c}$  is of order 2, these numbers determine the additive structure. We computed them for  $m \leq 59$  on a computer using the Anderson-Brown-Peterson [1] algorithm and list

them below as follows:

$m$	$\pi_2(m)$	$\pi_Z(m)$	$m$	$\pi_2(m)$	$\pi_Z(m)$	$m$	$\pi_2(m)$	$\pi_Z(m)$
0	0	1	20	1	19	40	26	139
1	0	0	21	0	0	41	8	0
2	0	1	22	5	19	42	59	139
3	0	0	23	0	0	43	10	0
4	0	2	24	2	30	44	44	195
5	0	0	25	0	0	45	16	0
6	0	2	26	9	30	46	90	195
7	0	0	27	0	0	47	20	0
8	0	4	28	4	45	48	72	272
9	0	0	29	1	0	49	29	0
10	1	4	30	14	45	50	138	272
11	0	0	31	1	0	51	36	0
12	0	7	32	8	67	52	116	373
13	0	0	33	2	0	53	51	0
14	1	7	34	24	67	54	207	373
15	0	0	35	2	0	55	64	0
16	0	12	36	15	97	56	183	508
17	0	0	37	4	0	57	88	0
18	3	12	38	37	97	58	311	508
19	0	0	39	5	0	59	110	0

We shall need the following technical lemma about Stiefel-Whitney numbers later in the paper. We acknowledge with gratitude helpful suggestions by Peter Landweber about the proof. Let  $\ker_*(SW) = \{M \in \Omega_*^{\text{Spin}^c} : x(M) = 0 \forall x \in W^*\}$ .

LEMMA 1.3. *Let  $Q_* = Z[CP^2, CP^4, \dots, CP^{2k}, \dots]$  so  $P_* = Q_*[CP^1]$ . Let  $M = M_1 + M_2$  for  $M_1 \in Q_*$  and  $M_2 \in \text{Tor}(\Omega_*^{\text{Spin}^c})$ . If  $M \in \ker_*(SW)$ , then  $M_1 \in 2 \cdot Q_*$  and  $M_2 = 0$ .*

*Proof.* We assemble the appropriate results from Stong's book (see [19] pages 42 and 352). Let  $\bar{M}$  and  $\bar{M}_i$  denote the corresponding elements reduced mod 2 in  $\Omega_*^{\text{Spin}^c} \otimes Z_2$ . Let  $F: \Omega_*^{\text{Spin}^c} \rightarrow \Omega_*^{\text{SO}}$ ,  $G: \Omega_*^{\text{SO}} \rightarrow \Omega_*^0$ , and  $p: \Omega_*^{\text{SO}} \rightarrow \Omega_*^{\text{SO}}/\text{torsion}$  be the natural maps. The Stiefel-Whitney numbers completely detect unoriented bordism so  $GF(M) = 0$ . Since  $CP^1$  bounds in  $\Omega_*^{\text{SO}}$  and since  $pF$  is surjective,  $pF: Q_* \otimes F_2 \rightarrow \{\Omega_*^{\text{SO}}/\text{torsion}\} \otimes Z_2$  is surjective by Theorem 1.2. Since  $\Omega_*^{\text{SO}}/\text{torsion} = Z[y_4, y_8, \dots]$  is a polynomial algebra, we see  $pF$  is an isomorphism by counting dimensions. With  $Z_2$  coefficients,  $G: \Omega_*^{\text{SO}} \otimes Z_2 = \Omega_*^{\text{SO}}/2\Omega_*^{\text{SO}} \rightarrow \Omega_*^0$  is injective. Since  $GF(\bar{M}) = 0$ ,  $F(\bar{M}) = 0$  so  $pF(\bar{M}) = pF(\bar{M}_1) = 0$ . Since  $pF$  is an isomorphism with coefficients in  $Z_2$ , this implies  $\bar{M}_1 = 0$  so  $M_1 \in 2 \cdot \Omega_m^{\text{Spin}^c}$ . Consequently  $\text{index}(\theta, M_1) \equiv 0(2) \forall \theta \in R(\text{Spin}^c)$  so  $M_1 \in 2 \cdot P_*$

and hence  $M_1 \in 2Q_*$  by Theorem 1.2. Consequently  $M_1 \in \ker_*(SW)$  so  $M_2 \in \ker_*(SW)$  so  $M_2 = 0$  which completes the proof.

Let  $BZ_n$  be the classifying space of  $Z_n$ . A  $Z_n$ -structure on  $M$  is a homotopy class of a map  $M \rightarrow BZ_n$ . This is equivalent to either a principal  $Z_n$  bundle over  $M$  or to a representation of  $\pi_1(M)$  in  $Z_n$ . If  $N$  is a compact  $\text{Spin}^c$  manifold with boundary  $M$  and if the classifying map extends over  $N$ , then  $M$  bounds. Let  $\Omega_m^{\text{Spin}^c}(BZ_n)$  be the resulting bordism group and let  $\Omega_*^{\text{Spin}^c}(BZ_n)$  be the graded direct sum. Cartesian product makes  $\Omega_*^{\text{Spin}^c}(BZ_n)$  into an  $\Omega_*^{\text{Spin}^c}$  module. The forgetful functor induces an  $\Omega_*^{\text{Spin}^c}$  module morphism  $\Omega_*^{\text{Spin}^c}(BZ_n) \rightarrow \Omega_*^{\text{Spin}^c}$ . Since any  $\text{Spin}^c$  manifold admits a trivial  $Z_n$  structure, this map is surjective and splits. Let  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  be the kernel of the forgetful functor;  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  is an  $\Omega_*^{\text{Spin}^c}$  module and we split  $\Omega_*^{\text{Spin}^c}(BZ_n) = \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n) \oplus \Omega_*^{\text{Spin}^c}$ .

Let  $W^*(BZ_n) = W^* \otimes H^*(BZ_n; Z_2)$  be the algebra of  $Z_n$  equivariant Stiefel Whitney classes. If  $x \in W^m(BZ_n)$  and if  $M \in \tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)$ , then there is a natural evaluation  $x(M) \in Z_2$  obtained by cupping the Stiefel-Whitney classes of the tangent bundle with the cohomology classes of  $H^*(BZ_n; Z_2)$  of the principal bundle and then evaluating on the fundamental class of  $M$ . This defines a pairing  $W^m(BZ_n) \otimes \Omega_m^{\text{Spin}^c}(BZ_n) \rightarrow Z_2$ . We refer to Conner-Floyd [7] for further details.

Let  $\rho_s(\lambda) = \lambda^s$  be the irreducible representations of  $Z_n$  where  $s$  is defined modulo  $n$  and let  $V_s$  be the complex line bundle corresponding to  $\rho_s$ .  $V_{n/2}$  is a real bundle. Let  $x_1 = w_1(V_{n/2}) \in H^1(BZ_n; Z_2)$  and let  $x_2 \in H^2(BZ_n; Z_2)$  be the mod 2 reduction of  $c_1(V_1)$ . Let  $a(m, n) = |\oplus_{j \geq 0} P_{m-2j-1} \otimes Z_n|$ , let  $b(m) = |\oplus_{j \geq 0} \text{Tor}(\Omega_{m-2j-1}^{\text{Spin}^c})|$ , and let  $c(m) = b(m-1) = |\oplus_{j \geq 0} \text{Tor}(\Omega_{m-2j-2}^{\text{Spin}^c})|$ .

**LEMMA 1.4.** (a)  $\tilde{H}_{2j}(BZ_n; Z) = 0$ ,  $\tilde{H}_{2j-1}(BZ_n; Z) = Z_n$ , and  $\tilde{H}_j(BZ_n; Z_2) = Z_2$  for  $j > 0$ .

(b)  $H^*(BZ_n; Z_2) = Z_2[x_1]$ . If  $n > 2$ , then  $H^*(BZ_n; Z_2) = Z_2[x_1, x_2]/\{x_1^2 = 0\}$ .

(c)  $|\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)| \leq a(m, n)b(m)c(m)$  and  $\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)$  is a finite 2-group.

(d) If  $M \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ , then

$$M \times \Omega_*^{\text{Spin}^c} = M \times P_* + M \times \text{Tor}(\Omega_*^{\text{Spin}^c}).$$

*Proof.* We refer to [6] for (a, b).  $|\oplus_j \tilde{H}_j(BZ_n; \Omega_{m-j}^{\text{Spin}^c})| = a(m, n)b(m)c(m)$  by (a) and Theorem 1.2. Since the  $E_{p,q}^2$  term of the bordism spectral sequence is  $\tilde{H}_p(BZ_n; \Omega_q^{\text{Spin}^c})$  (see [7]),  $\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)$  is a

finite 2-group and  $|\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)| \leq a(m, n)b(m)c(m)$  which proves (c); (d) now follows from Theorem 1.2 which completes the proof.

REMARK. In Lemma 4.1, we will show

$$|\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)| = a(m, n)b(m)c(m)$$

so the bordism spectral sequence degenerates.

**2. The eta invariant.** Let  $M$  be a smooth compact Riemannian manifold of dimension  $m$  without boundary and let  $D$  be a self-adjoint elliptic differential operator on  $M$ . If  $\lambda \in R$ , let  $E(D, \lambda)$  denote the eigenspace of  $D$  corresponding to the eigenvalue  $\lambda$ . Let

$$\eta(s, D) = \frac{1}{2} \left\{ \dim E(D, 0) + \sum_{\lambda \neq 0} \dim E(D, \lambda) \cdot \text{sign}(\lambda) \cdot |\lambda|^{-s} \right\}$$

be an analytic measure of the spectral asymmetry of  $D$ . The series converges to define a holomorphic function of  $s$  for  $\text{Re}(s) > > 0$ ;  $\eta(s, D)$  has a meromorphic extension to  $C$  with isolated simple poles on the real axis. The value at  $s = 0$  is regular and we let  $\eta(D) = \eta(0, D) \in R/Z$ . If  $D_t$  is a smooth 1-parameter family of such operators, although  $\eta(0, D_t)$  has integer jumps as spectral values cross the origin, the mod  $Z$  reduction is smooth in  $t$ .

There is a general procedure for constructing such operators. Let  $N$  be a possibly non-compact manifold with boundary  $M$ . Choose the metric to be product near  $M$ . Let  $Q: C^\infty(V_1) \rightarrow C^\infty(V_2)$  be an elliptic first order complex over  $N$ . Use the geodesic flow to identify a neighborhood of  $M$  in  $N$  with  $M \times [0, \epsilon)$ . Let  $n \in [0, \epsilon)$  be the distance to the boundary  $M$ . Using the symbol of  $dn$ , identify  $V_1$  with  $V_2$  on the collar and decompose  $Q = \partial/\partial n + D$  where  $D$  is a first order elliptic tangential differential operator on  $C^\infty(V_1|_M)$ . For the classical elliptic complexes,  $D$  is selfadjoint and does not depend on the particular  $N$  chosen. If  $N$  is compact, the generalized index theorem of Atiyah et al [4] is

THEOREM 2.1. *With the notation above,*

$$\text{index}(Q) = \int_N a_0(x, Q) dx - \{ \eta(0, D) + \dim \ker(D) \} / 2.$$

$a_0(x, Q)$  is a local invariant of  $Q$  and of the formal adjoint  $Q^*$  which vanishes if  $M$  is even dimensional.  $\text{Index}(Q)$  is computed with respect to suitable non-local elliptic boundary conditions.

Let  $M$  be an odd dimensional  $Z_n\text{-Spin}^c$  manifold. Choose a not necessarily compact even dimensional  $\text{Spin}^c$  manifold  $N$  so  $\partial N = M$ ; for

example we could take  $N = M \times [0, \infty)$ . Since  $N$  is even dimensional, the  $\text{Spin}^c$  complex is defined over  $N$ . Let  $Q$  be the operator of the  $\text{Spin}^c$  complex over  $N$  and let  $D$  be the tangential operator of the  $\text{Spin}^c$  complex over  $M$ . If  $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$ , let  $\theta(M)$  be the virtual bundle defined by  $\theta$  over  $M$  and let  $D_\theta$  be  $D$  with coefficients in  $\theta(M)$ . Let  $\eta(\theta, M) = \eta(D_\theta)$  and set  $\eta(\theta, M) = 0$  if  $M$  is even dimensional. Similarly, if  $M$  is an even dimensional  $\text{Pin}^c$  manifold and if  $\theta \in R(\text{Spin}^c)$ , let  $\bar{D}_\theta$  be the tangential operator of the  $\text{Pin}^c$  complex over  $M$  with coefficients in  $\theta(M)$  and let  $\eta(\theta, M) = \eta(\bar{D}_\theta)$ . Set  $\eta(\theta, M) = 0$  if  $M$  is odd dimensional.

LEMMA 2.2. (a) Let  $M = M_1 \times M_2$ , let  $Q$  be a first order elliptic complex over  $M_1$ , let  $R$  be a first order self-adjoint elliptic operator over  $M_2$ , and let

$$P = \begin{pmatrix} R \otimes 1 & 1 \otimes Q^* \\ 1 \otimes Q & -R \otimes 1 \end{pmatrix} \quad \text{on } M.$$

Then  $\eta(P)(M) = \text{index}(Q)(M_1) \cdot \eta(R)(M_2)$ .

(b)  $\eta: R_0(Z_n) \otimes R(\text{Spin}^c) \otimes \Omega_*^{\text{Spin}^c}(BZ_n) \rightarrow Q/Z$ .

(c)  $\eta: R(\text{Spin}^c) \otimes \Omega_*^{\text{Pin}^c} \rightarrow Q/Z$ .

*Proof.* We refer to Gilkey [11, 14] for the proof of (a). To prove (b), we use Theorem 2.1. Suppose  $N$  is a compact  $Z_n$ - $\text{Spin}^c$  manifold with boundary  $M$  and let  $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$ . We extend the bundle  $\theta(M)$  over  $N$  as follows. Let  $r(\Lambda^k) = \Lambda^k + \Lambda^{k-1}$  and  $r(\tau) = \tau$  define an  $R(Z_n)$  module ring isomorphism of  $R_0(Z_n) \otimes R(\text{Spin}^c)$ . Choose  $\Theta$  so  $r(\Theta) = \theta$ . Since  $T(N)|_M = T(M) \oplus 1$ ,  $\Theta(N)|_M = \theta(M)$  so this provides the desired extension. Let  $Q_\Theta$  be the operator of the  $\text{Spin}^c$  complex over  $N$  with coefficients in  $\Theta(N)$ . The tangential operator of  $Q_\Theta$  is  $D_\theta$ . If  $\Theta = \rho \otimes \psi$  for  $\rho \in R_0(Z_n)$  and  $\psi \in R(\text{Spin}^c)$ , then  $Q_\Theta$  is locally isomorphic to  $\dim(\rho)$  copies of  $Q_\psi$ . Since  $a_0$  is a local invariant,  $a_0(x, Q_\Theta) = \dim(\rho)a_0(x, Q_\psi) = 0$  since  $\dim(\rho) = 0$  for  $\rho$  in the augmentation ideal. In general  $Q_\Theta$  is a sum of such operators. Since  $a_0(x, -)$  is additive with respect to direct sums, the local term vanishes and Theorem 2.1 implies  $\text{index}(Q_\Theta) = -\{\eta(0, D_\theta) + \dim \ker(D_\theta)\}/2 \in Z$ . Therefore  $\eta(\theta, M) = 0$  in  $R/Z$  so eta is a bordism invariant. If the  $Z_n$  structure on  $M$  is trivial, then  $\theta(M) = 0$  so  $\eta(\theta, M) = 0$ ; consequently we will often restrict eta to  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  with no loss of information. Since  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  is a torsion group by Lemma 1.4, eta takes values in  $Q/Z$  which proves (b).

The proof of (c) is similar. The representations of  $R(\text{Spin}^c)$  extend to  $\text{Pin}^c$  so the operators in question are well defined. Let  $\bar{Q}_\Theta$  be the operator of the  $\text{Pin}^c$  complex with coefficients in the bundle defined by  $\Theta$ . Since  $M$

is even dimensional,  $a(x, \overline{Q}_\Theta) = 0$ . The remainder of the argument is the same; we refer to Gilkey [11] for a proof  $\eta(\theta, M) \in Z[1/2]/Z \subseteq Q/Z$  which proves (c).

We introduce the following notation for certain Dedekind sums which arise in evaluating the eta invariant. Let  $C(Z_n)$  denote the space of complex class functions on  $Z_n$ . The map  $\rho \rightarrow \text{Tr}(\rho)$  embeds  $R(Z_n)$  in  $C(Z_n)$  with  $R(Z_n) \otimes C = C(Z_n)$ . If  $f, g \in C(Z_n)$ , let  $(f, g) = \sum_{\lambda \in Z_n} f(\lambda)g(\lambda)/n$ . If  $\rho, \hat{\rho} \in R(Z_n)$ , then  $(\rho, \hat{\rho}) \in Z$  by the orthogonality relations. Let  $\alpha(1) = 0$  and  $\alpha(\lambda) = \lambda/(\lambda - 1)$  for  $\lambda \neq 1$ .

LEMMA 2.3. (a)  $L^k(n) \in \tilde{\Omega}_{2k-1}^{\text{Spin}^c}(BZ_n)$ . If  $\theta \in R_0(Z_n)$ , then  $\eta(\theta, L^k(n)) = (\theta, \alpha^k)$ .

(b)  $\tilde{\Omega}_0^{\text{Spin}^c}(BZ_n) = 0$ .  $\eta(\rho_s - \rho_0, L^1(n)) = -s/n$ .  $L^1(n)$  generates  $\tilde{\Omega}_1^{\text{Spin}^c}(BZ_n) = Z_n$ .

*Proof.*  $T(L^k(n)) \oplus 1$  inherits a natural unitary structure. Since  $L^k(n)$  is odd dimensional, it bounds in  $\Omega_*^U$  and hence in  $\Omega_*^{\text{Spin}^c}$ . Let  $Z_n \rightarrow S^{2k-1} \rightarrow L^k(n)$  define a  $Z_n$  structure on  $L^k(n)$  so  $L^k(n) \in \tilde{\Omega}_{2k-1}^{\text{Spin}^c}(BZ_n)$ . Atiyah-Patodi-Singer [4, see II-(2.9)] calculate the eta invariant of the tangential operator of the Signature complex in terms of Dedekind sums; the same argument proves (a).  $\tilde{\Omega}_0^{\text{Spin}^c}(BZ_n) = 0$  by Lemma 1.4. If  $\chi(1) = 0$  and  $\chi(\lambda) = 1$  for  $\lambda \neq 1$ , then

$$\begin{aligned} \eta(\rho_s - \rho_0, L^1(n)) &= (\rho_1(\rho_s - \rho_0)/(\rho_1 - \rho_0), \chi) = (\rho_s + \cdots + \rho_1, \chi) \\ &= (\rho_s + \cdots + \rho_1, \rho_0) - s/n = -s/n \text{ mod } Z \end{aligned}$$

by (a) and the orthogonality relations. Thus  $L^1(n)$  is an element of order at least  $n$  in  $\tilde{\Omega}_1^{\text{Spin}^c}(BZ_n)$ . Since  $|\tilde{\Omega}_1^{\text{Spin}^c}(BZ_n)| \leq n$  by Lemma 1.4,  $L^1(n)$  generates  $Z_n = \tilde{\Omega}_1^{\text{Spin}^c}(BZ_n)$  which completes the proof.

Define

$$\ker_*(\eta, n) = \{ M \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n) : \eta(\theta, M) = 0 \forall \theta \in R_0(Z_n) \otimes R(\text{Spin}^c) \},$$

$$\ker_*(\eta) = \{ M \in \Omega_*^{\text{Pin}^c} : \eta(\theta, M) = 0 \forall \theta \in R(\text{Spin}^c) \},$$

$$\ker_*(SW, n) = \{ M \in \Omega_*^{\text{Spin}^c}(BZ_n) : x(M) = 0 \forall x \in W^*(BZ_n) \}.$$

LEMMA 2.4. Let  $M \in L^1(n) \times \Omega_{m-1}^{\text{Spin}^c}$ .

(a)  $M = L^1(n) \times (N_1 + N_2)$  for  $N_1 \in P_{m-1}$  and  $N_2 \in \text{Tor}(\Omega_{m-1}^{\text{Spin}^c})$ .

(b)  $M \in \ker_m(\eta, n)$  iff  $N_1 \in n \cdot P_{m-1}$ .

(c)  $M \in \ker_m(SW, n)$  iff  $N_2 = 0$  and  $N_1 \in 2 \cdot Q_{m-1} + CP^1 \cdot P_{m-3}$ .

- (d)  $|L^1(n) \times \Omega_{m-1}^{\text{Spin}^c}| = |\Omega_{m-1}^{\text{Spin}^c} \otimes Z_n| = a(m, n)b(m)/a(m-2, n)b(m-2)$ .  
(e)  $L^1(n) \times \Omega_{m-1}^{\text{Spin}^c} \cap \ker_m(SW, n) \cap \ker_m(\eta, n) = 0$ .

*Proof.* If  $M \in L^1(n) \times \Omega_{m-1}^{\text{Spin}^c}$ , then  $M$  has this form by Lemma 1.4 which proves (a). Suppose  $M \in \ker_m(\eta, n)$ . If  $\dim(M)$  is even, then  $N_1 = 0$  so we suppose  $\dim(M)$  odd. If  $\psi \in R(\text{Spin}^c)$ , choose  $\Psi$  so  $r(\Psi) = \psi$ . Since  $T(L^1(n)) = 1$ ,  $\Psi(L^1(n) \times N) = \Psi(1 \oplus T(N)) = \psi(N)$ . Let  $Q$  be the operator of the  $\text{Spin}^c$  complex on  $N$  with coefficients in  $\psi$ , let  $D$  be the tangential operator of the  $\text{Spin}^c$  complex on  $L^1(n)$  with coefficients in  $\rho_1 - \rho_0$ , and let  $\hat{D}$  be the tangential operator of the  $\text{Spin}^c$  complex on  $M$  with coefficients in  $(\rho_1 - \rho_0) \otimes \Psi$ . Then  $\hat{D} = \begin{pmatrix} D \otimes Q & 1 \otimes Q^* \\ 1 \otimes Q & -D \otimes 1 \end{pmatrix}$  so  $\eta((\rho_1 - \rho_0) \otimes \Psi, M) = \eta(\rho_1 - \rho_0, L^1(n)) \cdot \text{index}(\psi, N_1 + N_2) = -\text{index}(\psi, N_1)/n$  by Lemmas 2.2 and 2.3 since  $\text{index}(\psi, N_2) = 0$ . Consequently,  $M \in \ker_m(\eta, n)$  iff  $\text{index}(\psi, N_1) \equiv 0 \pmod{n}$  iff  $N_1 \in n \cdot P_{m-1}$  by Theorem 1.2 which proves (b). The only possibly non-zero equivariant Stiefel-Whitney numbers of  $L^1(n) \times N$  are of the form  $x_1 \cdot y$  for  $y \in W_{m-1}$ . Since  $x_1(L^1(n)) = 1$ ,  $x_1 \cdot y(L^1(n) \times N) = y(N)$ . Decompose  $N_1 = X_1 + CP^1 \times X_2$  for  $X_1 \in Q_{m-1}$  and  $X_2 \in P_{m-3}$ . Since  $CP^1$  bounds a 3 ball in  $\Omega_2^{\text{SO}}$ ,  $CP^1 \times X_2 \in \ker_{m-1}(SW)$  and  $y(N) = y(X_1) + y(N_2)$ .  $M \in \ker_m(SW, n)$  iff  $X_1 + N_2 \in \ker_{m-1}(SW)$  iff  $X_1 \in 2 \cdot Q_*$  and  $N_2 = 0$  by Lemma 1.3 which proves (c). Since  $L^1(n)$  is an element of order  $n$ ,  $L^1(n) \times (N_1 + N_2) = 0$  iff  $N_1 \in n \cdot P_{m-1}$  and  $N_2 = 0$  so  $|L^1(n) \times \Omega_{m-1}^{\text{Spin}^c}| = n^{\pi_2(m-1)} \cdot 2^{\pi_2(m-1)} = |\Omega_{m-1}^{\text{Spin}^c} \otimes Z_n|$  which proves (d). (e) is a direct consequence of (b, c) which completes the proof.

Let  $s(\tau) = \tau \otimes \tau$  and  $s(\Lambda^k) = \sum_{i+j=k} \Lambda^i \otimes \Lambda^j$  define an  $R(Z_n)$  module coproduct  $s: R_0(Z_n) \otimes R(\text{Spin}^c) \rightarrow \{R_0(Z_n) \otimes R(\text{Spin}^c)\} \otimes R(\text{Spin}^c)$ . Let  $M \in \Omega_{\text{odd}}^{\text{Spin}^c}(BZ_n)$  and let  $N \in \Omega_{\text{even}}^{\text{Spin}^c}$ . If  $s(\theta) = \sum_i a_i \otimes b_i$ , then  $\theta(M \times N) = \sum_i a_i(M) \otimes b_i(N)$ . By Lemma 2.2,  $\eta(\theta, M \times N) = \sum_i \eta(a_i, M) \cdot \text{index}(b_i, N)$ . If  $N$  is a torsion class, then  $\text{index}(b_i, N) = 0$ . This proves

LEMMA 2.5.  $\tilde{\Omega}_{\text{odd}}^{\text{Spin}^c}(BZ_n) \times \text{Tor}(\Omega_{\text{even}}^{\text{Spin}^c}) \subseteq \ker_*(\eta, n)$ .

Embed  $L^{k-1}(n)$  into  $L^k(n)$  using the first  $k-1$  coordinates. The complex normal bundle of the embedding is given by the representation  $\rho_1$  so that complexification of the real normal bundle corresponds to  $\rho_1 + \rho_{-1}$ . Let  $t(\tau) = \rho_1 \otimes \tau$  and  $t(\Lambda^k) = \Lambda^{k-2} + (\rho_1 + \rho_{-1}) \otimes \Lambda^{k-1} + \Lambda^k$ . Extend  $t$  to an  $R(Z_n)$  module algebra isomorphism of  $R_0(Z_n) \otimes R(\text{Spin}^c)$ . If  $s$  is as above, then  $(t \otimes 1) \cdot s = s \cdot t$ . If  $\theta \in R_0(Z_n) \otimes$

$R(\text{Spin}^c)$  and  $N \in \Omega_*^{\text{Spin}^c}$ , then

$$\theta(L^k(n)) \times N|_{L^{k-1}(n) \times N} = t(\theta)(L^{k-1}(n) \times N).$$

Let  $\beta = (\rho_1 - \rho_0)/\rho_1 \in R_0(Z_n)$ . We will use the following Lemma to discuss the Smith homomorphism later.

**LEMMA 2.6.** (a) *If  $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$  and if  $N \in \Omega_*^{\text{Spin}^c}$ , then  $\eta(\theta \cdot \beta, L^k(n) \times N) = \eta(t(\theta), L^{k-1}(n) \times N)$ .*

(b) *If  $\sum_k L^k(n) \times N_k \in \ker_*(\eta, n)$ , then  $\sum_k L^{k-1}(n) \times N_k \in \ker_*(\eta, n)$ .*

*Proof.* If  $s(\theta) = \sum_i a_i \otimes b_i$ , then  $s(t(\theta)) = (t \otimes 1)s(\theta) = \sum_i t(a_i) \otimes b_i$ , so

$$\eta(\theta \cdot \beta, L^k(n) \times N) = \sum_i \eta(a_i \cdot \beta, L^k(n)) \cdot \text{index}(b_i, N),$$

$$\eta(t(\theta), L^{k-1}(n) \times N) = \sum_i \eta(t(a_i), L^{k-1}(n)) \cdot \text{index}(b_i, N).$$

$\tau(L^k(n)) = \rho_k(L^k(n))$  is given by the representation theory. The  $\Lambda^k$  are the complexified exterior representations. Since  $(T(L^k(n)) \oplus 1) \otimes C$  corresponds to  $k \cdot \rho_1 + k \cdot \rho_{-1}$ ,  $\Lambda^k + \Lambda^{k-1}$  corresponds to the  $2k$ th elementary symmetric function in the  $k \cdot \rho_1$  and  $k \cdot \rho_{-1}$ . We solve this relation to express  $\Lambda^k$  in terms of the (virtual) representation theory. Choose  $\phi_i \in R_0(Z_n)$  so that  $\phi_i(L^k(n)) = a_i(L^k(n))$ . Since  $a_i(L^k(n))|_{L^{k-1}(n)} = t(a_i)(L^{k-1}(n))$ ,  $\phi_i(L^{k-1}(n)) = t(a_i)(L^{k-1}(n))$ . Since  $\beta \cdot \alpha^k = \alpha^{k-1}$ ,

$$\begin{aligned} \eta(a_i \cdot \beta, L^k(n)) &= \eta(\phi_i \cdot \beta, L^k(n)) = (\phi_i \cdot \beta, \alpha^k) = (\phi_i, \alpha^{k-1}) \\ &= \eta(\phi_i, L^{k-1}(n)) = \eta(t(a_i), L^{k-1}(n)) \end{aligned}$$

by Lemmas 2.2 and 2.3 which proves (a); (b) follows from (a) since  $t$  is surjective.

**3. The Smith homomorphism.** The classifying space  $BZ_n$  is the limit  $\text{Lim}_{k \rightarrow \infty} L^k(n)$  with respect to the inclusions defined previously. If  $M \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ , let  $f: M \rightarrow L^k(n)$  be the classifying map for  $k$  large. Make  $f$  transverse to  $L^{k-1}(n)$  and let  $\Delta(M) = f^{-1}(L^{k-1}(n))$ . The normal bundle of  $L^{k-1}(n)$  in  $L^k(n)$  is given by the representation  $\rho_1$  and has a natural  $\text{Spin}^c$  structure. We use Lemma 1.1 to give  $\Delta(M)$  a  $\text{Spin}^c$  structure.  $\Delta$  defines an  $\Omega_*^{\text{Spin}^c}$  module morphism in bordism.

If  $n = 2$ , we split  $\Delta = \delta_2 \cdot \delta_1$ . Let  $RP^k = S^k/Z_2$  be real projective space and let  $L$  be the non-trivial real line bundle over  $RP^k$ . We also use the notation  $L_{RP^k}$  to emphasize the base space is  $RP^k$ . Identify the

0-section with  $RP^k$ . If  $x = 1 = w_1(L)$  generates  $H^1(RP^k; Z_2) = Z_2$ , then

$$w(k \cdot L) = (1 + x_1)^k = 1 + k \cdot x_1 + k(k-1)/2 \cdot x_1^2 + \cdots.$$

$x_1^2$  is the reduction mod 2 of  $c_1(L \otimes C)$ . The stable tangent bundle  $T(RP^k) \oplus 1 = (k+1) \cdot L$  so by Lemma 1.1

$$RP^k \text{ admits a } \left\{ \begin{array}{ll} \text{Pin}^c \text{ structure and } \tau(RP^k) = L \otimes C & \text{if } k \equiv 0(4) \\ \text{Spin}^c \text{ structure and } \tau(RP^k) = L \otimes C & \text{if } k \equiv 1(4) \\ \text{Pin structure and } \tau(RP^k) = 1 & \text{if } k \equiv 2(4) \\ \text{Spin structure and } \tau(RP^k) = 1 & \text{if } k \equiv 3(4) \end{array} \right\}$$

Embed  $S^{k-1}$  equivariantly in  $S^k$  using the first  $k$  coordinates to induce an embedding of  $RP^{k-1}$  in  $RP^k$ . The classifying space  $BZ_2 = \text{Lim}_{k \rightarrow \infty} RP^k$ . Let  $\infty$  be the image of the north pole  $(0, 0, \dots, 0, 1)$  of  $S^k$  in  $RP^k$ . There is a diffeomorphism between  $(RP^k - \infty, RP^{k-1})$  and  $(L_{RP^{k-1}}, RP^{k-1})$  so the normal bundle of  $RP^{k-1}$  in  $RP^k$  is  $L_{RP^{k-1}}$ .  $RP^k - RP^{k-1}$  is a contractable neighborhood of  $\infty$  where  $L$  is trivial.

If  $M \in \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)$ , let  $f: M \rightarrow RP^k$  be the classifying map for  $k$  large. Make  $f$  transverse to  $RP^{k-1}$  and let  $\delta_1(M) = f^{-1}(RP^{k-1})$ . The normal bundle of  $\delta_1(M)$  in  $M$  is  $L_M = f^*(L)$ .  $L_M$  has a  $\text{Pin}^c$  structure so  $\delta_1(M)$  inherits a natural  $\text{Pin}^c$  structure by Lemma 1.1.  $\delta_1: \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2) \rightarrow \Omega_{m-1}^{\text{Pin}^c}$  and  $\delta_1(RP^{2k+1}) = RP^{2k}$ .  $L_M|_{\delta_1(M)}$  is the orientation line bundle of  $\delta_1(M)$ . Similarly if  $M \in \Omega_{m-1}^{\text{Pin}^c}$ , let  $f: M \rightarrow RP^{k-1}$  classify the orientation line bundle  $L_M$ . Make  $f$  transverse to  $RP^{k-2}$  and let  $\delta_2(M) = f^{-1}(RP^{k-2})$ . Since the normal bundle of  $\delta_2(M)$  in  $M$  is  $L_M|_{\delta_2(M)}$ ,  $\delta_2(M)$  has a natural  $\text{Spin}^c$  structure by Lemma 1.1.  $\delta_2: \Omega_{m-1}^{\text{Pin}^c} \rightarrow \Omega_{m-2}^{\text{Spin}^c}(BZ_2)$  and  $\delta_2(RP^{2k}) = RP^{2k-1}$ .  $\delta_1$  and  $\delta_2$  are  $\Omega_*^{\text{Spin}^c}$  morphisms and  $\Delta = \delta_2 \cdot \delta_1$  if  $n = 2$ .

LEMMA 3.1 (*The Smith homomorphism*). (a)  $\delta_1: \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2) \rightarrow \Omega_{m-1}^{\text{Pin}^c}$  is an isomorphism.

(b) Let  $\nu: \Omega_*^{\text{Spin}^c} \rightarrow \Omega_*^{\text{Pin}^c}$  be the forgetful functor. Then  $\delta_2$  defines a short exact sequence

$$0 \rightarrow \nu(\Omega_{m-1}^{\text{Spin}^c}) \rightarrow \Omega_{m-1}^{\text{Pin}^c} \xrightarrow{\delta_2} \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c}) \rightarrow 0.$$

(c) If  $\Delta = \delta_2 \cdot \delta_1$ , then  $\Delta$  defines a short exact sequence

$$0 \rightarrow RP^1 \times \Omega_{m-1}^{\text{Spin}^c} \rightarrow \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2) \xrightarrow{\Delta} \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c}) \rightarrow 0.$$

*Proof.* We adopt the following notational conventions. If  $V$  is a vector bundle, let  $S(V)$  and  $D(V)$  be the unit sphere and disk bundles. Let  $L_M$  be a real line bundle over some manifold  $M$ . Let  $O_a$  be a simple

cover of  $M$  and let  $w_a$  be sections to the  $Z_2$  bundle  $S(L_M)$  over  $O_a$ . Let  $w_a = g_{ab}w_b$  for  $g_{ab} = \pm 1$  over  $O_a \cap O_b$ . Let  $S(L_M \oplus 1)$  be a circle bundle over  $M$ . Let  $\theta_a \in [0, 2\pi]$  be local angular parameters where we identify  $\theta_a$  with  $(\sin(\theta_a)w_a, \cos(\theta_a))$  in  $S(L_M \oplus 1)$ ;  $\theta_a = g_{ab}\theta_b$  on  $O_a \cap O_b$ . Parametrize the associated projective bundle  $RP(L_M \oplus 1)$  by letting  $\theta_a \in [0, \pi]$ . We construct the inverse to  $\delta_1$  using some ideas of Korschorke and Stong [16, 19]. Let  $M \in \Omega_{m-1}^{\text{Pin}^c}$  and let  $L_M$  be the orientation line bundle. Let  $Z_2 \rightarrow S(L_M \oplus 1) \rightarrow RP(L_M \oplus 1)$  define a  $Z_2$  structure on  $RP(L_M \oplus 1)$ . Since  $T(L_M \oplus 1) = T(M) \oplus L_M \oplus 1$ , the orientation bundle of the manifold  $L_M \oplus 1$  is  $L_M \otimes L_M = 1$  so  $L_M \oplus 1$  is orientable. Consequently  $L_M \oplus 1$  has a  $\text{Spin}^c$  structure by Lemma 1.1. Give

$$S(L_M \oplus 1) = \partial D(L_M \oplus 1)$$

the bounding  $\text{Spin}^c$  structure. The map  $\theta_a \rightarrow 2\theta_a$  is a diffeomorphism from  $RP(L_M \oplus 1)$  to  $S(L_M \oplus 1)$  which we use to define a  $\text{Spin}^c$  structure on  $RP(L_M \oplus 1)$ . Let  $\alpha_1(M) = RP(L_M \oplus 1)$  define an  $\Omega_{\star}^{\text{Spin}^c}$  module morphism from  $\Omega_{m-1}^{\text{Pin}^c}$  to  $\tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)$ . We compute  $\delta_1 \cdot \alpha_1$  as follows. Let  $f: M \rightarrow RP^{k-1}$  be the classifying map for the line bundle  $L_M$ .  $f$  induces a  $Z_2$  equivariant map  $f: S(L_M) \rightarrow S^{k-1}$ . Extend  $f: L_M \rightarrow R^k$  to be fiber linear. Let  $F(x, t) = (f(x), t)$  be a  $Z_2$  equivariant map from  $S(L_M \oplus 1)$  to  $S^k$ . This descends to a map  $F: RP(L_M \oplus 1) \rightarrow RP^k$  which is the classifying map for the  $Z_2$  structure.  $F$  is transverse to  $RP^{k-1}$  and  $F^{-1}(RP^{k-1}) = M$  corresponds to the embedding of  $M$  as  $(0, -1)$  in  $S(L_M \oplus 1) \subseteq L_M \oplus 1$ . Since this is homotopic to the embedding of  $M$  as the zero section of  $L_M \oplus 1$ , the induced  $\text{Pin}^c$  structure agrees with the original  $\text{Pin}^c$  structure on  $M$  so  $\delta_1 \cdot \alpha_1(M) = M$ . This shows  $\delta_1$  is surjective. We will show  $\alpha_1 \cdot \delta_1 = \text{id}$  by showing  $\delta_1$  is an isomorphism presently.

Since  $\delta_1(RP^1) = RP^0$ ,  $\delta_1(RP^1 \times \Omega_{m-1}^{\text{Spin}^c}) = \nu(\Omega_{m-1}^{\text{Spin}^c})$ . Conversely, if  $N \in \Omega_{m-1}^{\text{Spin}^c}$ , then the orientation line bundle  $L_N$  is trivial so  $\alpha_1(N) = RP^1 \times N$ . Thus  $\delta_1$  and  $\alpha_1$  provide isomorphisms between  $RP^1 \times \Omega_{m-1}^{\text{Spin}^c}$  and  $\nu(\Omega_{m-1}^{\text{Spin}^c})$ . Since  $\delta_1$  is surjective and since  $\tilde{\Omega}_{\star}^{\text{Spin}^c}(BZ_2)$  is a 2-group by Lemma 1.4,  $\Omega_{\star}^{\text{Pin}^c}$  is a torsion group so

$$\delta_2(\Omega_{m-1}^{\text{Pin}^c}) \subseteq \text{Tor}(\Omega_{m-2}^{\text{Spin}^c}(BZ_2)) = \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c}).$$

Conversely, let  $N \in \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c})$ . Then  $2 \cdot N = 0$  in  $\Omega_{m-2}^{\text{Spin}^c}$ . Let  $L_N$  be the real line bundle over  $N$  corresponding to the  $Z_2$  structure so  $Z_2 \rightarrow S(L_N) \rightarrow N$ . The Stiefel-Whitney numbers and Chern/Pontrjagin numbers are multiplicative under finite coverings so  $S(L_N) = 2 \cdot N = 0$  in  $\Omega_{m-2}^{\text{Spin}^c}$  by Theorem 1.2. Let  $U$  be a  $m - 1$  dimensional compact  $\text{Spin}^c$

manifold with boundary  $S(L_N)$ . Use the outward normal to orient  $D(L_N) - N$ . Let  $M = D(L_N) \cup U$  along the common boundary  $S(L_N)$  with the natural  $\text{Pin}^c$  structure. If  $L_M$  is the orientation bundle of  $M$ , then  $L_M$  is the pull back of  $L_N$  over  $D(L_N)$  and  $L_M$  is trivial over  $M - N$ . Let  $f: N \rightarrow RP^{k-1}$  be the classifying map of  $L_N$ . Extend  $f$  as a fiber linear map  $f: (D(L_N), N) \rightarrow (L_{RP^{k-1}}, RP^{k-1})$ . Since  $RP^k - RP^{k-1}$  is contractible, we can change  $f$  so  $f^{-1}(RP^{k-1}) = N$  and  $f(x) = \infty$  near the boundary  $S(L)$ . Extend  $f$  to  $M$  so  $f(U) = \infty$ . Since  $f^{-1}(RP^{k-1}) = N$  with the given  $\text{Spin}^c$  structure,  $\delta_2(M) = N$ . This shows  $\text{image}(\delta_2) = \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c})$ .

If  $N \in \nu(\Omega_{m-1}^{\text{Spin}^c})$ , then  $L_N = 1$  so we may take  $f(N) = \infty$  and  $\delta_2(N) = 0$ . Therefore  $\nu(\Omega_{m-1}^{\text{Spin}^c}) \subseteq \ker(\delta_2)$ . Since  $|\nu(\Omega_{m-1}^{\text{Spin}^c})| = |RP^1 \times \Omega_{m-1}^{\text{Spin}^c}| = |\Omega_{m-1}^{\text{Spin}^c} \otimes Z_2|$  by Lemma 2.4, we may estimate

$$\begin{aligned} |\tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)| &\geq |\Omega_{m-1}^{\text{Pin}^c}| \geq |\tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2)| \cdot |\text{Tor}(\Omega_{m-2}^{\text{Spin}^c})| \cdot |\nu(\Omega_{m-1}^{\text{Spin}^c})| \\ &\geq \Pi_{j \geq 0} |\text{Tor}(\Omega_{m-2j-2}^{\text{Spin}^c})| \cdot |\Omega_{m+1-2j}^{\text{Spin}^c} \otimes Z_2| = a(m, 2)b(m)c(m). \end{aligned}$$

By Lemma 1.4,  $|\tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)| \leq a(m, 2)b(m)c(m)$  so all the inequalities are equalities. This shows  $\delta_1$  is an isomorphism and  $\ker(\delta_2) = \nu(\Omega_{m-1}^{\text{Spin}^c})$ . This proves (a,b); we combine (a) and (b) to prove (c).

Since  $\ker(\delta_2) = \delta_1(RP^1 \times \Omega_{m-1}^{\text{Spin}^c})$  consists of elements of order 2,  $2\delta_2^{-1}$  is well defined. Let  $M \in \text{image}(\delta_2)$  and let  $L_M$  be the real line bundle over  $M$  corresponding to the  $Z_2$  structure. Since  $T(L_M \oplus 1) = T(M) \oplus L_M \oplus 1$ ,  $D(L_M \oplus 1)$  has a natural  $\text{Pin}^c$  structure and  $S(L_M \oplus 1)$  inherits a  $\text{Pin}^c$  structure that bounds. This is not the structure we wish to use. Define a complex line bundle  $H$  over  $S(L_M \oplus 1)$  with transition functions  $h_{ab} = e^{i(\theta_a - \theta_b)/2}$ ;  $H^2 = 1$ . Inequivalent  $\text{Pin}^c$  structures are parametrized by complex line bundles; we twist the bounding  $\text{Pin}^c$  structure on  $S(L_M \oplus 1)$  by  $H$  to define a new  $\text{Pin}^c$  structure on  $S(L_M \oplus 1)$ ; we denote this by  $\alpha_2(M)$  and the map  $M \rightarrow \alpha_2(M)$  extends as an  $\Omega_*^{\text{Spin}^c}$  module morphism in bordism.

A more geometric description of this structure can be given as follows. Let  $S_{\pm} = \{(z, t) \in S(L_M \oplus 1) : t \geq 0 \text{ or } t \leq 0\}$  be the closed upper/lower hemispheres. Let  $D_{\pm}$  be two copies of the unit disk bundle  $D(L_M)$ . Use the diffeomorphism  $d_{\pm} \rightarrow (\pm d_{\pm}, \pm(1 - |d_{\pm}|^2)^{1/2})$  to identify  $D_{\pm}$  with  $S_{\pm}$  where the zero sections  $M_{\pm}$  go to the north and south poles respectively. This decomposes  $S(L_M \oplus 1) = D_+ \cup D_-$  where the glueing is the antipodal map  $-1$  on the boundary  $S(L_M)$ . The antipodal map on  $S(L_M \oplus 1)$  becomes the shift  $\chi$  which interchanges  $D_+$  and  $D_-$ . Give  $D_{\pm} - M_{\pm}$  opposite orientations to define a  $\text{Spin}^c$  structure on  $S(L_M \oplus 1) - M_+ - M_-$  and a  $\text{Pin}^c$  structure on  $S(L_M \oplus 1)$ .  $\chi$  is a  $\text{Pin}^c$  involution reversing the orientation of  $S(L_M) - M_+ - M_-$ .

LEMMA 3.2. *If  $M \in \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2)$ , then  $2\delta_2^{-1}(M) = \alpha_2(M)$ .*

*Proof.* Let  $U$  be an  $m - 1$  dimensional  $\text{Spin}^c$  manifold with boundary  $\partial U = S(L_M)$ . Let  $U_{\pm}$  be two copies of  $U$ . Identify  $\partial U_{\pm} \times (0, 1]$  with  $S_{\pm}(L_M) \times (0, 1]$  in  $D_{\pm}(L_M) - M_{\pm}$ . Glue  $\partial U_{\pm} \times [.2, .8]$  to  $D_{\pm}(L_M) \times \{1\}$  in  $S(L_M \oplus 1) \times \{1\}$  to construct the bordism

$$S(L_M \oplus 1) \times [0, 1] \cup U_+ \times [.2, .8] \cup U_- \times [.2, .8].$$

This admits a  $\text{Pin}^c$  structure and is orientable off the core  $M_{\pm} \times [0, 1]$  using the natural orientation for  $U_+ \times [.2, .8]$  and the reversed orientation for  $U_- \times [.2, .8]$ . The boundary of this bordism consists of four pieces. The first is  $\alpha_2(M) = S(L_M \oplus 1) \times \{0\}$ . The second and third pieces are  $D_{\pm, 2}(L_M) \cup U_{\pm} \times \{.2\}$  which yields  $2\delta_2^{-1}(M)$  as was shown in the proof of Lemma 3.1. The final piece is an error term

$$\begin{aligned} E &= (S(L_M \oplus 1) - \text{int}(D_{+, .8}(L_M)) - \text{int}(D_{-, .8}(L_M))) \\ &\cup U_+ \times \{.8\} \cup U_- \times \{.8\}. \end{aligned}$$

Since  $E \cap M_{\pm} \times [0, 1] = \emptyset$ ,  $E$  is orientable. Let the shift  $\chi$  induce an action of  $Z_2$  on  $E$  and let  $\hat{E} = E/Z_2$  with the induced  $\text{Pin}^c$  structure. Let  $\hat{L}$  be the orientation line bundle of  $\hat{E}$ . Since  $\chi$  is orientation reversing,  $S(\hat{L}) = E$ .  $T(\hat{L}) = \hat{L} \oplus T(\hat{E})$  so  $\hat{T}(\hat{L})$  is orientable and has a  $\text{Spin}^c$  structure. Since  $E$  is the boundary of  $D(\hat{L})$ , it is zero in  $\Omega_{\star}^{\text{Pin}^c}$ . This completes the proof.

We now study the behavior of the eta invariant with respect to the  $\delta_i$ . Let  $M \in \Omega_m^{\text{Spin}^c}(BZ_2)$  and let  $N \in \Omega_{m-1}^{\text{Pin}^c}$ . Give  $N$  the  $Z_2$  structure corresponding to the orientation class so  $\rho_1(N) = L_N$ . This extends  $\eta: R(Z_2) \otimes R(\text{Spin}^c) \otimes \Omega_{\star}^{\text{Pin}^c} \rightarrow Q/Z$ . It is an easy exercise using Clifford algebras to show  $\tau(\delta_1(M)) = \tau(M)|_{\delta_1(M)}$  and  $\tau(\delta_2(N)) = \tau(N)|_{\delta_2(N)} \otimes L_N$ ; we omit details in the interests of brevity. Let  $s_1(\Lambda^k) = 1 \otimes \Lambda^k + \rho_1 \otimes \Lambda^{k-1}$ ,  $s_1(\tau) = \tau$ ,  $s_2(\Lambda^k) = 1 \otimes \Lambda^k + \rho_1 \otimes \Lambda^{k-1}$ , and  $s_2(\tau) = \rho_1 \otimes \tau$  define  $R(Z_2)$  module algebra isomorphisms  $s_i$  of  $R(Z_2) \otimes R(\text{Spin}^c)$ . If  $\theta \in R(Z_2) \otimes R(\text{Spin}^c)$ , then  $\theta(M)|_{\delta_1(M)} = s_1(\theta)(\delta_1(M))$  and  $\theta(N)|_{\delta_2(N)} = s_2(\theta)(\delta_2(N))$ .

LEMMA 3.3. *Let  $\psi \in R(Z_2) \otimes R(\text{Spin}^c)$ ,  $M_1 \in \tilde{\Omega}_{\text{odd}}^{\text{Spin}^c}(BZ_2)$ , and  $M_2 \in \Omega_{\text{even}}^{\text{Pin}^c}$ .*

(a)  $\eta((\rho_0 - \rho_1) \otimes \psi, M_1) = \eta(s_1(\psi), \delta_1(M_1))$ .  $M_1 \in \ker_{\star}(\eta, 2)$  iff  $\delta_1(M_1) \in \ker_{\star}(\eta)$ .

(b)  $2\eta(\psi, M_2) = \eta((\rho_0 - \rho_1) \cdot s_2(\psi), \delta_2(M_2))$ . If  $2 \cdot M_2 \in \ker_*(\eta)$ , then  $\delta_2(M_2) \in \ker_*(\eta, 2) \oplus \text{Tor}(\Omega_*^{\text{Spin}^c})$ .

(c) If  $2M_1 \in \ker_*(\eta, 2)$ , then  $\Delta(M_1) \in \ker_*(\eta, 2) \oplus \text{Tor}(\Omega_*^{\text{Spin}^c})$ .

*Proof.* Since  $\alpha_1 \cdot \delta_1 = \text{id}$ , we may assume  $M_1 = RP(L_N \oplus 1)$ . The parity involved plays an important role so we suppose first  $s_1(\psi) \in R(\text{Spin}^c)$ . Let  $X = \psi(M_1)$  so  $X|_N = s_1(\psi)(N)$ . We study the eta invariant on  $RP(L_N \oplus 1)$  by working equivariantly over  $S(L_N \oplus 1)$ . Let  $\bar{D}: C^\infty(V) \rightarrow C^\infty(V)$  be the tangential operator of the Pin<sup>c</sup> complex over  $N$  with coefficients in  $X$ . Let  $\beta_a$  denote normalized Clifford multiplication by the local orientations  $\omega_a$ ;  $\beta_a$  defines an automorphism of  $V$  which anti-commutes with  $\bar{D}$ , see Gilkey [11, 14]. The tangential operator of the Spin<sup>c</sup> complex with coefficients in  $X$  over  $S(L_N \oplus 1)$  is  $D = \bar{D} + \beta_a \cdot \partial/\partial\theta_a: C^\infty(V) \rightarrow C^\infty(V)$ . Let  $\{\lambda_k, f_k\}$  be a spectral resolution of  $\bar{D}$ . Since  $\cos(\sin)$  are even(odd) functions,  $\cos(n\theta_a) \cdot f_k$  and  $\sin(n\theta_a)\beta_a f_k$  are well defined sections to  $V$  over  $S(L_N \oplus 1)$ . If  $n$  is even(odd), these define sections to  $V(V \otimes L_{M_1})$  over  $RP(L_N \oplus 1)$ . These functions form a complete orthogonal system for  $L^2(V)$  over  $S(L_N \oplus 1)$  if we omit the  $\sin(n\theta)$  terms for  $n = 0$ . If  $n > 0$ , then

$$\begin{aligned} D(a \cos(n\theta)f_k + b \sin(n\theta)\beta f_k) \\ = (\lambda_k a - nb)\cos(n\theta)f_k + (-na - \lambda_k b)\sin(n\theta)\beta f_k \end{aligned}$$

so  $D$  is given by the matrix  $\begin{pmatrix} \lambda_k & -n \\ n & -\lambda_k \end{pmatrix}$  on this subspace. This matrix has two unequal non-zero eigenvalues  $\pm(\lambda_k^2 + n^2)^{1/2}$  which cancel in pairs and contribute nothing to the eta invariant. If  $n = 0$ , then  $f_k$  is an eigenvector corresponding to the eigenvalue  $\lambda_k$ . Consequently  $\eta(D) = \eta(\bar{D})$  and  $\eta(D_{\rho_1}) = 0$ . This shows  $\eta(\bar{D}) = \eta(\rho_0 - \rho_1, D)$  and establishes the formula in this case. If  $s_1(\psi) \in \rho_1 \cdot R(\text{Spin}^c)$ , then the parities involved are reversed owing to the twisting of the line bundle  $L_N$ . We use Gilkey [11] and the case previously considered to compute

$$\begin{aligned} \eta(s_1(\psi), N) &= -\eta(L_N \cdot s_1(\psi), N) = -\eta(s_1(\rho_1 \cdot \psi), N) \\ &= -\eta((\rho_0 - \rho_1) \cdot \rho_1 \cdot \psi, M) = \eta((\rho_0 - \rho_1) \cdot \psi, M) \end{aligned}$$

which establishes the formula in general. If  $N \in \ker_*(\eta)$ , then  $\eta(\theta, N) = 0 \forall \theta \in R(Z_2) \otimes R(\text{Spin}^c)$  since  $\eta(\rho_1 \otimes \theta, N) = -\eta(\theta, N)$ . Consequently  $M \in \ker_*(\eta, 2)$  by the formula. Conversely,  $1 \otimes R(\text{Spin}^c) \subseteq \text{image}(s_1)$  so  $M \in \ker_*(\eta, 2)$  implies  $N \in \ker_*(\eta)$ . This proves (a).

We use Lemma 3.2 to prove (b). If  $N = \delta_2(M_2)$ , then  $2M_2 = S(L_N \oplus 1)$ . Let  $X = \psi(S(L_N \oplus 1))$  so  $X|_N = s_2(\psi)(N)$ . Let  $D$  be the tangential operator of the Spin<sup>c</sup> complex over  $N$  with coefficients in  $X$  and let  $D_1$  be  $D$  with coefficients in  $L_N$ . The pull back of  $L_N$  is the orientation bundle of  $S(L_N \oplus 1)$ . Normalized Clifford multiplication by

local sections  $\omega_a$  of  $L_N$  define isomorphisms  $\beta_a: 1 \rightarrow L_N$  and  $\beta_a: L_N \rightarrow 1$  which intertwine  $D$  and  $D_1$ . Let

$$\bar{D} = \begin{pmatrix} D \otimes 1 & 1 \otimes \beta \partial / \partial \theta \\ 1 \otimes \beta \partial / \partial \theta & -D_1 \otimes 1 \end{pmatrix}$$

be the tangential operator of the  $\text{Pin}^c$  complex over  $S(L_N \oplus 1)$  with coefficients in  $s_2(\psi)$ . If  $\{f_k, \lambda_k\}$  and  $\{g_k, \mu_k\}$  is a spectral resolution of  $D$  and  $D_1$ , then the collection  $\{(\cos(n\theta)f_k, 0), (0, \beta \sin(n\theta)f_k), (\beta \sin(n\theta)g_k, 0), (0, \cos(n\theta)g_k)\}$  is a complete orthogonal system for  $L^2(V \oplus (V \otimes L_N))$  over  $S(L_N \oplus 1)$  if we omit the terms in  $\sin(n\theta)$  for  $n = 0$ . If  $n \neq 0$ ,

$$\begin{aligned} & \bar{D}(a \cos(n\theta)f_k \oplus b \sin(n\theta)\beta f_k) \\ &= (a\lambda_k - nb)\cos(n\theta)f_k \oplus (-na - \lambda_k b)\sin(n\theta)\beta f_k \\ & \bar{D}(a \sin(n\theta)\beta g_k \oplus b \cos(n\theta)g_k) \\ &= (a\mu_k - nb)\sin(n\theta)\beta f_k \oplus (-nb - \mu_k b)\cos(n\theta)g_k \end{aligned}$$

so  $\bar{D}$  is given by a matrix  $\begin{pmatrix} \lambda_k & -n \\ -n & -\lambda_k \end{pmatrix}$  or  $\begin{pmatrix} \mu_k & -n \\ -n & -\mu_k \end{pmatrix}$ . These matrices have two unequal non-zero eigenvalues  $\pm(\lambda_k^2 + n^2)^{1/2}$  or  $\pm(\mu_k^2 + n^2)^{1/2}$  which cancel in pairs and make no contribution to the  $\eta$  invariant. If  $n = 0$ , only the eigenvalues of  $D$  and  $-D_1$  contribute. Since  $2M_2 = S(L_N \oplus 1)$ ,

$$\begin{aligned} 2\eta(\psi, M_2) &= \eta(\bar{D})(S(L_N \oplus 1)) = \eta(D) - \eta(D_1) \\ &= \eta((\rho_0 - \rho_1) \cdot s_2(\psi), N). \end{aligned}$$

If  $2 \cdot M_2 \in \ker_*(\eta, 2)$ , then  $\eta(\theta, N) = 0 \forall \theta$  since  $s_2$  is an isomorphism. Let  $N = N_1 + N_2$  for  $N_1 \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_2)$  and  $N_2 \in \text{Tor}(\Omega_*^{\text{Spin}^c})$ . Since  $\eta(\theta, N_2) = 0$ ,  $N_1 \in \ker_*(\eta, 2)$  which completes the proof of (b). We combine (a, b) to prove (c).

It is easier to study the behavior of the Stiefel-Whitney numbers with respect to the  $\delta_i$ . Let  $M \in \Omega_m^0(BZ_2)$  and let  $f: M \rightarrow RP^k$  be the classifying map. Make  $f$  transverse to  $RP^{k-1}$  and let  $\delta(M) = f^{-1}(RP^{k-1})$ . Let  $s(w_k) = w_k \otimes 1 + w_{k-1} \otimes x_1$  define an  $H^*(BZ_2; Z_2)$  module algebra isomorphism of  $W^*(BZ_2)$ . Since  $T(M)|_{\delta(M)} = T(\delta(M)) \oplus L_M$ , the cohomology classes  $x(M)|_{\delta(M)}$  and  $s(x)(\delta(M))$  agree.

**LEMMA 3.4.** (a) *If  $M \in \Omega_m^0(BZ_2)$  and if  $x \in W^{m-1}(BZ_2)$ , then  $(x_1(L_M) \cdot x)(M) = s(x)(\delta(M))$ .*

(b)  *$M \in \ker_m(SW, 2)$  iff  $\delta_1(M) \in \ker_{m-1}(SW)$ .*

(c) *If  $N \in \Omega_{m-1}^{\text{Pin}^c} \cap \ker_{m-1}(SW)$ , then  $\delta_2(N) \in \ker_{m-2}(SW, 2) \subseteq \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2)$ .*

(d) *If  $M \in \ker_m(SW, n)$ , then  $\Delta(M) \in \ker_{m-2}(SW, n) \subseteq \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_n)$ .*

*Proof.* Let  $i: \delta_1(M) \rightarrow M$  be the inclusion and let  $\theta = x(L_M) \in H^{m-1}(M; Z_2)$ . Since  $[M]$  is the Poincaré dual of  $w_1(L_M)$ ,  $i_*[\delta_1(M)] = w_1(L_M) \cap [M]$  and hence

$$\begin{aligned} \theta w_1(L_M)(M) &= \langle \theta \cup w_1(L_M), [M] \rangle = \langle \theta, w_1(L_M) \cap [M] \rangle \\ &= \langle \theta, i_*[\Delta(M)] \rangle = \langle i^*(\theta), [\Delta(M)] \rangle. \end{aligned}$$

This proves (a) since  $i^*(\theta) = s(x)(\delta(M))$ . To prove (b, c), let  $M \in \tilde{\Omega}_m^{\text{Spin}^c}(BZ_2)$  and let  $N = \delta_1(M)$ . Suppose  $M \in \ker_m(SW, n)$ . If  $x \in W^{m-1}$ , choose  $y \in W^{m-1}(BZ_2)$  so  $s(y) = x$ . Then  $0 = (x_1 \cdot y)(M) = x(N)$  so  $N \in \ker_{m-1}(SW)$ . Suppose  $N \in \ker_{m-1}(SW)$ . Let the orientation line bundle give a  $Z_2$  structure to  $N$ . The equivariant Stiefel-Whitney numbers of  $N$  can be computed in terms of the ordinary Stiefel-Whitney numbers of  $N$  so all the equivariant Stiefel-Whitney numbers of  $N$  vanish. If  $x \in W^m$ , then  $x(M) = 0$  since  $M = 0$  in  $\Omega_*^{\text{Spin}^c}$ . We therefore suppose  $x = x_1 \cdot y$  for  $y \in W^{m-1}(BZ_2)$ . Then  $0 = y(N) = (x_1 \cdot y)(M)$  so  $M \in \ker_m(SW, n)$  which proves (b). If  $y \in W^{m-2}(BZ_2)$ , choose  $x$  so  $s(x) = y$ . Then  $y(\delta_2(N)) = (x_1 \cdot x)(N) = 0$ . Decompose  $\delta_2(N) = X_1 + X_2$  for  $X_1 \in \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2)$  and  $X_2 \in \text{Tor}(\Omega_{m-2}^{\text{Spin}^c})$ . Since  $X_1$  bounds,  $X_2 \in \ker_{m-2}(SW)$  so  $X_2 = 0$ . This shows  $\delta_2(N) \in \ker_{m-2}(SW, 2)$  and proves (c). The proof of (d) is the same as that of (c) and is omitted in the interests of brevity. This completes the proof of the Lemma.

**4.  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  and  $\Omega_*^{\text{Pin}^c}$ .** Let  $\pi_2(m): \Omega_m^{\text{Spin}^c}(BZ_2) \rightarrow \text{Tor}(\Omega_m^{\text{Spin}^c})$  be the natural projection and let  $\pi = \bigoplus_{k>0} \pi_2(m-2k)\Delta^k$ . Let  $R_*(n)$  be the  $\Omega_*^{\text{Spin}^c}$  submodule of  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  generated by the  $L^k(n)$  and let  $E(m) = \bigoplus_{k \geq 0} \text{Tor}(\Omega_{m-2k-2}^{\text{Spin}^c})$ .

LEMMA 4.1. (a)  $0 \rightarrow R_m(n) \rightarrow \tilde{\Omega}_m^{\text{Spin}^c}(BZ_n) \xrightarrow{\pi} E(m) \rightarrow 0$  is exact.

(b)  $0 \rightarrow L^1(n) \times \Omega_{m-1}^{\text{Spin}^c} \rightarrow R_m(n) \xrightarrow{\Delta} R_{m-2}(n) \rightarrow 0$  is exact.

(c)  $|R_m(n)| = a(m, n)b(m)$  and  $|\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)| = a(m, n)b(m)c(m)$ .

(d)  $\ker_m(SW, n) \cap \ker_m(\eta, n) = 0$ .

REMARK. (d) completes the proof of Theorem 0.1(a); we apply  $\delta_1$  to (d) and use the results of the third section to complete the proof of Theorem 0.2(a).

*Proof.* If  $n = 2$ ,  $\text{image}(\Delta) = \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c})$  by Lemma 3.1 so  $\pi$  is surjective. The inclusion  $Z_2 \rightarrow Z_n$  induces an  $\Omega_*^{\text{Spin}^c}$  module morphism called induction  $i: \Omega_*^{\text{Spin}^c}(BZ_2) \rightarrow \Omega_*^{\text{Spin}^c}(BZ_n)$ . Let  $p: S^{2k-1}/Z_2 \rightarrow S^{2k-1}/Z_n$  be the natural covering projection;  $p$  is compatible with the inclusions defining  $BZ_2$  and  $BZ_n$ . If  $M \in \Omega_*^{\text{Spin}^c}(BZ_2)$  and if  $f: M \rightarrow RP^{2k-1}$  is the classifying map, then  $pf$  is the classifying map for  $i(M)$  so

$i \cdot \Delta = \Delta \cdot i$ . Since  $i \cdot \pi_2(m) = \pi_2(m) \cdot i$ ,  $i \cdot \pi = \pi \cdot i$  and  $\pi$  is surjective for all  $n$ . Since  $\Delta(L^k(n)) = L^{k-1}(n)$  and since  $\Delta$  is an  $\Omega_{\star}^{\text{Spin}^c}$  module morphism,  $\Delta: R_m(n) \rightarrow R_{m-2}(n) \rightarrow 0$ . Since  $R_{m-2}(n) \subseteq \tilde{\Omega}_{m-2}^{\text{Spin}^c}(BZ_n)$ ,  $R_m(n) \subseteq \ker(\pi)$ . Since  $\Delta(L^1(n)) = 0$ ,  $L^1(n) \times \Omega_{m-1}^{\text{Spin}^c} \subseteq \ker(\Delta)$ . We use Lemma 2.4 to estimate:

$$\begin{aligned} |R_m(n)| &= \prod_{j \geq 0} |\ker(\Delta) \cap R_{m-2j}(n)| \\ &\geq \prod_{j \geq 0} |L^1(n) \times \Omega_{m-2j-1}^{\text{Spin}^c}| = a(m, n)b(m), \\ |\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)| &= |\ker(\pi)| \cdot |E(m)| \\ &\geq |R_m(n)| \cdot |E(m)| \geq a(m, n)b(m)c(m). \end{aligned}$$

Since  $|\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)| \leq a(m, n)b(m)c(m)$  by Lemma 1.4, all the inequalities must have been equalities. Thus  $\ker(\Delta) \cap R_m(n) = L^1(n) \times \Omega_{m-1}^{\text{Spin}^c}$  and  $\ker(\pi) = R_{\star}(n)$ . This completes the proof of (a, b, c).

We prove (d) by induction on  $m$ ; it is immediate if  $m < 0$ . Let  $M \in \ker_m(SW, n) \cap \ker_m(\eta, n)$ . By Lemma 3.4,  $\Delta^k(M) \in \ker_{m-2k}(SW, n) \subseteq \tilde{\Omega}_{m-2k}^{\text{Spin}^c}(BZ_n)$ . Thus  $\pi(M) = 0$  and  $M \in R_m(n)$ . We therefore apply Lemma 2.6 to see  $\Delta(M) \in \ker(\eta, n)$  so  $\Delta(M) = 0$  by induction. Since  $M \in \ker(\Delta) \cap R_m(n)$ ,  $M \in L^1(n) \times \Omega_{m-1}^{\text{Spin}^c}$ . We use Lemma 2.4 to conclude  $M = 0$  which completes the proof.

Let  $\vec{q} = (q_1, \dots, q_k)$  be a collection of odd integers. Let  $\rho(\vec{q}) = \text{diag}(\lambda^{q_1}, \dots, \lambda^{q_k})$  be a fixed point free representation of  $Z_n$  in  $U(k)$ . Let  $L(n; \vec{q}) = S^{2k-1}/\rho(\vec{q})(Z_n)$  be a generalized lens space.  $L(n; \vec{q}) \in \tilde{\Omega}_{2k-1}^{\text{Spin}^c}(BZ_n)$  and  $L^k(n) = L(n; 1, \dots, 1)$ . Let

$$M_p(k+1) = L(n; -1, \dots, -1, 1, \dots, 1)$$

for  $q_1 = \dots = q_p = -1$  and  $q_{p+1} = \dots = q_{k+1} = 1$ .

LEMMA 4.2.  $\sum_{p=0}^k \binom{p}{k} M_p(k+1) = L^1(n) \times (CP^1)^k$  in  $\tilde{\Omega}_{2k+1}^{\text{Spin}^c}(BZ_n)$ .

*Proof* We use the circle trick of Conner-Floyd [7]. If  $\lambda \in Z_n$ , let  $T_1(\lambda)z = \lambda z$  and let  $T_2(\lambda)z = z$  for  $z \in S^1$ ,  $z \in D^2$ , or  $z \in C \cup \infty = CP^1$ . Let

$$N_1 = S^1 \times (CP^1)^k / (T_1 \times T_2^k)(Z_n) \quad \text{and}$$

$$N_2 = S^1 \times (CP^1)^k / (T_1 \times T_1^k)(Z_n).$$

Since  $T_2$  is trivial,  $N_1 = L^1(n) \times (CP^1)^k$ . Let  $f(z, w) = (z, zw)$  intertwine these two actions so  $N_1 = N_2$  in  $\tilde{\Omega}_{2k+1}^{\text{Spin}^c}(BZ_2)$ . The action  $T_1 \times T_1^k$  on  $D_2 \times (CP^1)^k$  has  $2^k$  isolated fixed points at  $0 \times \{0, \infty\} \times \dots \times \{0, \infty\}$ .

Cut out small  $Z_n$  equivariant spheres about each fixed point to construct a manifold on which  $Z_n$  acts freely. The quotient is an equivariant bordism between  $N_2$  and the sum of  $2^k$  lens spaces. At  $0 \in D_2$  or  $0 \in CP^1$ , the action is  $\rho_1$ . At  $\infty \in CP^1$ , the action is  $\rho_{-1}$  owing to the change of coordinates  $z = 1/w$ . There are  $\binom{k}{p}$  fixed points corresponding to  $p \cdot \infty$  and  $(k + 1 - p) \cdot 0$  so we get  $\binom{k}{p}$  copies of  $M_p(k + 1)$  which completes the proof.

Let  $M \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ . Inequivalent  $\text{Spin}^c$  structures on  $M$  are parametrized by complex line bundles over  $M$ . Let  $c(s) \cdot M$  be  $M$  with the same  $Z_n$  structure and with the  $\text{Spin}^c$  structure twisted by the complex line bundle which corresponds to the representation  $\rho_s$ . Since  $M = 0$  in  $\Omega_*^{\text{Spin}^c}$ , the Stiefel-Whitney numbers and rational Chern/Pontrjagin numbers of  $M$  vanish. Since  $M = c(s) \cdot M$  in  $\Omega_*^0$ , the Stiefel-Whitney numbers of  $c(s) \cdot M$  vanish. Since the line bundle defined by  $\rho_s$  is flat, the Chern/Pontrjagin numbers are unchanged so  $c(s) \cdot M = 0$  in  $\Omega_*^{\text{Spin}^c}$  by Theorem 1.2. The map  $M \rightarrow c(s) \cdot M$  defines an  $\Omega_*^{\text{Spin}^c}$  module isomorphism of  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$ . We extend  $c(s)$  as the identity on  $\Omega_*^{\text{Spin}^c}$ . Identify the group algebra  $Z[Z_n]$  with  $R(Z_n)$ . Since  $c(s)c(t)$  corresponds to twisting by  $\rho_s \rho_t = \rho_{s+t}$ ,  $c(s)c(t) = c(s+t)$ . We define a representation of  $R(Z_n)$  on  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  by  $c(\sum_s n_s \rho_s) = \sum_s n_s c(s)$ . Dually, let  $c(s) \cdot \Lambda^k = \Lambda^k$  and  $c(s) \cdot \tau = \rho_{2s} \cdot \tau$  define a representation of  $R(Z_n)$  on  $R_0(Z_n) \otimes R(\text{Spin}^c)$ .

LEMMA 4.3. Let  $M \in \tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  and let  $\rho \in R_0(Z_n)$ . (a)  $\Delta(c(\rho) \cdot M) = c(\rho) \cdot \Delta(M)$ .

(b) If  $\theta \in R_0(Z_n) \otimes R(\text{Spin}^c)$ , then

$$\eta(\theta, c(s) \cdot M) = \eta(\rho_s \cdot (c(s) \cdot \theta), M).$$

(c)  $\rho \in R_0(Z_n)^k$  iff  $\eta(\rho \cdot \bar{\rho}, L^k(n)) = 0 \forall \bar{\rho} \in R_0(Z_n)$ .

(d)  $c(\rho) \cdot L^k(n) = 0$  iff  $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$ .

REMARK. It is in (d) that the analysis is used at last to embed the  $K$ -theory groups into the bordism groups.

*Proof.* The classifying map is not changed if we change the  $\text{Spin}^c$  structure so  $\Delta(c(s) \cdot M) = \Delta(M)$  in  $\Omega_*^{\text{SO}}(BZ_n)$ . If we twist the  $\text{Spin}^c$  structure on  $M$  by a line bundle  $L$ , then the  $\text{Spin}^c$  structure on  $\Delta(M)$  is twisted by  $L|_{\Delta(M)}$ . Consequently  $c(s) \cdot \Delta(M) = \Delta(c(s) \cdot M)$  which proves (a). Let  $N = c(s) \cdot M$ . Since only the  $\text{Spin}^c$  structure is changed,  $\rho(N) = \rho(M)$  and  $\Lambda^k(N) = \Lambda^k(M)$ . Since  $\tau$  involves a square,  $\tau(N) = \rho_{2s}(M) \otimes \tau(M)$ . Therefore  $c(s) \cdot \theta(M) = \theta(N)$ . Let  $D(N)$  and  $D(M)$  be the tangential operators of the  $\text{Spin}^c$  complex over  $N$  and  $M$ . Since the  $\text{Spin}^c$

structure on  $N$  is twisted by  $\rho_s$ ,  $D(N)_\theta = D(M)_{\rho_s \cdot c(\theta)}$  which completes the proof of (b). We refer to Gilkey [12] for the proof of (c).

We prove (d) as follows. Let  $\rho \in R(Z_n)$ , let  $\beta = (\rho_1 - \rho_0)/\rho_1 \in R_0(Z_n)$ , and let  $\bar{\rho} \in R_0(Z_n)$ . Since  $\beta \cdot \alpha^{k+1} = \alpha^k$  for  $k > 0$ , we use (a) and Lemma 2.3 to compute

$$0 = \eta(\bar{\rho}, c(\rho) \cdot L^k(n)) = \eta(\bar{\rho} \cdot \rho, L^k(n)) = \eta(\bar{\rho} \cdot (\rho \cdot \beta), L^{k+1}(n)).$$

This implies  $\rho \cdot \beta \in R_0(Z_n)^{k+1}$  by (c). Since  $\beta R(Z_n) = R_0(Z_n)$ ,  $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$ . Conversely, let  $M = c(\rho) \cdot L^k(n)$  and assume  $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$ . We prove  $M = 0$  by showing  $M \in \ker_{2k-1}(\eta, n) \cap \ker_{2k-1}(SW, n)$ . Let  $\delta = \sum_s \rho_s$  be the regular representation. Since  $\beta \cdot \rho \in R_0(Z_n)^{k+1} = \beta \cdot R_0(Z_n)^k$ , choose  $\sigma \in R_0(Z_n)^k$  so  $\beta \cdot \rho = \beta \cdot \sigma$ . Since the only zero divisors of  $\beta$  are multiples of  $\delta$  and since  $\beta \cdot (\rho - \sigma) = 0$ ,  $\rho = \sigma + a \cdot \delta$ . If  $w \in W^{2k-1}(n)$ , then  $w(M) = \dim(\rho) \cdot w(L^k(n))$ . Since  $\dim(\rho) = a \cdot \dim(\delta) = a \cdot n$  is even,  $M \in \ker_{2k-1}(SW, n)$ . Let  $\Psi = R_0(Z_n) \otimes Z[\Lambda^k]$  so  $R_0(Z_n) \otimes R(\text{Spin}^c) = \Psi[\tau]$ . Suppose  $\theta = \psi \cdot \tau^w$  for  $\psi \in \Psi$ . Choose a representation  $\phi \in R_0(Z_n)$  so  $\phi(L^k(n))$  and  $\theta(L^k(n))$  define the same locally flat bundles. Let  $m(2w+1)$  be the algebra morphism of  $R(Z_n)$  defined by  $m(2w+1)(\rho_s) = \rho_{(2w+1)s}$ . Since  $2w+1$  is odd, it is coprime to  $n$  and  $m(2w+1)$  defines a ring isomorphism of  $R(Z_n)$  preserving  $R_0(Z_n)^j \forall j$ . Consequently  $m(2w+1)(\rho) \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$  so  $\beta \cdot m(2w+1)(\rho) \in R_0(Z_n)^{k+1}$ . Let  $\rho = \sum_s n_s \rho_s$ . We use (b) and Lemma 2.3 to compute:

$$\begin{aligned} \eta(\psi \cdot \tau^w, c(\rho)M) &= \sum_s n_s \eta(\psi \cdot \rho_{(2w+1)s}, L^k(n)) \\ &= \sum_s \eta(\psi \cdot n_s \rho_{(2w+1)s}, L^k(n)) - \eta(\psi \cdot m(2w+1)(\rho), L^k(n)) \\ &= \eta(\phi \cdot m(2w+1)(\rho) \cdot \beta, L^{k+1}(n)) \end{aligned}$$

which vanishes by (c). This shows  $M \in \ker_{2k-1}(\eta, n)$  and completes the proof.

Let  $A_{2k}(n) = 0$  and let  $A_{2k-1}(n)$  be the subgroup of  $\tilde{\Omega}_{2k-1}(BZ_n)$  generated by all possible  $\text{Spin}^c$  structures on  $L^k(n)$ .

**LEMMA 4.4.** (a)  $A_{2k-1}(n) = R_0(Z_n)/R_0(Z_n)^{k+1} = \tilde{K}(S^{2k+1}/Z_n)$  is a finite group of order  $n^k$ .

(b)  $A_{2k-1}(2) = Z_{2^k}$  is generated by  $RP^{2k-1}$ .

(c)  $L^1(n) \times (CP^1)^k \in A_{2k+1}(n)$ .

*Proof.* Since any complex line bundles over  $L^k(n)$  correspond to one of the  $\rho_s$ ,  $A_{2k-1}(n) = c(R(Z_n)) \cdot L^k(n)$ . If  $\rho \in R(Z_n)$ , let  $f(\rho) = \rho \cdot \beta \in R_0(Z_n)/R_0(Z_n)^{k+1}$  so  $f(\rho) = 0$  iff  $\rho \cdot R_0(Z_n) \subseteq R_0(Z_n)^{k+1}$ . By

Lemma 4.3, the map  $c(\rho)(M) \rightarrow f(\rho)$  is well defined and provides an isomorphism between  $A_{2k-1}(n)$  and  $R_0(Z_n)/R_0(Z_n)^{k+1}$ . This group has order  $n^k$  and is isomorphic to  $\tilde{K}(S^{2k+1}/Z_n)$  (see Atiyah [2]). Since the two  $\text{Spin}^c$  structures on  $RP^{2k-1}$  are just  $\pm RP^{2k-1}$ ,  $A_{2k-1}(2)$  is a group of order  $2^k$  generated by  $RP^{2k-1}$  which proves (b). If  $M_p(k+1) = L(n; -1, \dots, -1, 1, \dots, 1)$ , then  $M_p(k+1) = (-1)^p L^{k+1}(n)$  in  $\Omega_{2k+1}^{\text{SO}}(BZ_n)$  since we have reversed the orientation of  $p$  complex coordinates. Consequently  $M_p(k+1)$  and  $\pm L^{k+1}(n)$  only differ by the choice of  $\text{Spin}^c$  structure so  $M_p(k+1) \in A_{2k+1}(n)$ . By Lemma 4.2,  $L^1(n) \times (CP^1)^k$  is a linear combination of the  $M_p(k+1)$  which completes the proof of the Lemma.

**REMARK.** Using the description of the  $\text{Spin}^c$  structure associated to the unitary structure given in Hitchin [15] we see  $M_p(k) = (-1)^p c(p) L^k(n)$  so the  $\{M_p(k)\}$  generate  $A_{2k-1}$  for  $0 \leq p < n$ ;  $\sum_p n_p M_p(k) = 0$  iff  $\sum_p (-1)^p n_p (\rho_p - \rho_{p+1}) \in R_0(Z_n)^{k+1}$ .

Let  $S_*(n)$  be the  $Q_*$  submodule of  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  generated by the  $A_*(n)$  and let  $T_*(n)$  be the  $\text{Tor}(\Omega_*^{\text{Spin}^c})$  submodule of  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n)$  generated by the  $L^k(n)$ .

**LEMMA 4.5.** (a)  $R_m(n) = S_m(n) \oplus T_m(n)$ ,  $S_*(n) = A_*(n) \otimes Q_*$ ,  $T_*(n) = \bigoplus_{j \geq 0} \text{Tor}(\Omega_{m-2j-1}^{\text{Spin}^c})$ .

(b) If  $M \in R_m(2) \cap \ker_m(SW, 2)$ , then  $M \in 2 \cdot R_m(2)$ .

(c)  $\pi: \tilde{\Omega}_m^{\text{Spin}^c}(BZ_n) \rightarrow E(m) \rightarrow 0$  splits. The splitting  $\bar{E}(m) = \pi^{-1}E(m)$  can be chosen so  $\bar{E}(m) \subseteq \ker_m(\eta, 2)$  and so  $\Delta: \bar{E}(m) \rightarrow \bar{E}(m-2) \oplus \text{Tor}(\Omega_{m-2}^{\text{Spin}^c})$ .

(d)  $S_m(n) \cap \ker_m(\eta, n) = 0$ .

**REMARK.** The structure of  $A_*$  is given in Lemma 4.3. By Lemma 2.5,  $T_m(n) \subseteq \ker_m(\eta, n)$ . By construction  $\bar{E}(m) \subseteq \ker_m(\eta, n)$ . Thus by (d),  $\tilde{\Omega}_*^{\text{Spin}^c}(BZ_n) = A_*(n) \otimes Q_* \oplus \ker_*(\eta, n)$  and  $\ker_m(\eta, n) = T_m(n) \oplus \bar{E}(m) = \bigoplus_{j > 0} \text{Tor}(\Omega_{m-j}^{\text{Spin}^c})$ . This completes the proof of Theorem 0.1(b). If  $B_{m-1} = \delta_1 A_m(2)$ , then  $B_{2k} = Z_{2k+1}$  is generated by  $RP^{2k}$  by Lemma 4.3. We apply  $\delta_1$  to Theorem 0.1(b) and use the results of the third section to prove Theorem 0.2(b).

*Proof.*  $\Delta: T_m(n) \rightarrow T_{m-2}(n) \rightarrow 0$ . Since  $\Delta$  and  $c$  commute,  $\Delta: S_m(n) \rightarrow S_{m-2}(n) \rightarrow 0$ . Therefore  $S_m + T_m(n) \subseteq \ker(\pi) = R_m(n)$  by Lemma 4.1. By Lemma 4.2  $L^1(n) \times (CP^1)^k \subseteq S_*(n)$ . Since  $P_* = Q_*[CP^1]$ ,  $L^1(n) \times \Omega_*^{\text{Spin}^c} \subseteq S_*(n) + T_*(n)$  by Lemma 2.4. Consequently  $0 \rightarrow L^1(n) \times \Omega_{m-1}^{\text{Spin}^c} \rightarrow S_m(n) + T_m(n) \rightarrow S_{m-2}(n) + T_{m-2}(n) \rightarrow 0$  so that  $|S_m(n) + T_m(n)| = a(m, n)b(m)$  and  $S_m(n) + T_m(n) = R_m(n)$ . Since  $|A_{2k}(n)| = 1$

and  $|A_{2k-1}(n)| = n^k$ ,  $|\{A_*(n) \otimes Q_*\}_m| = a(m, n)$  so  $|S_m(n)| \leq a(m, n)$ . The map  $(N_k) \rightarrow \sum_k L^k(n) \times N_k$  defines a surjection  $\bigoplus_{j \geq 0} \text{Tor}(\Omega_{m-2j-1}^{\text{Spin}^c}) \rightarrow T_m(n)$  so  $|T_m(n)| \leq b(m)$  and  $|S_m(n) + T_m(n)| \leq a(m, n)b(m)$ . Since the inequality is an equality, there are no additional relations and this completes the proof of (a).

Let  $M \in R_*(2) \cap \ker_*(SW, 2)$ . We use (a) to decompose  $M = \sum_k RP^{2k+1} \times N(k)$  for  $N(k) \in Q_{m-2k-1} + \text{Tor}(\Omega_{m-2k-1})$ . If  $N(k) \in \ker_{m-2k-1}(SW) \forall k$ , then  $N(k) \in 2Q_{m-2k-1}$  and  $M \in 2R_m(n)$  by Lemma 1.3 which will prove (b). If this is false, choose  $j$  maximal so  $N(j) \notin \ker(SW)$ . Let  $x \in W^{m-2j-1}$  so  $x(N(j)) \neq 0$  and let  $y = x_1^{2j+1} \cdot x$ . If  $k < j$ , then  $y(RP^{2k+1} \times N(k)) = 0$  since the  $x_1^{2j+1}$  term vanishes. If  $k > j$ , then  $y(RP^{2k+1} \times N(k)) = 0$  since  $N(k) = 0$  in  $\Omega_*^0$ . Consequently  $y(M) = y(RP^{2j+1} \times N(j)) = x(N(j)) \neq 0$ . This contradiction proves (b).

Let  $i$  be induction and let  $r: R_0(Z_n) \rightarrow R_0(Z_2)$  be restriction. If  $M \in \Omega_*^{\text{Spin}^c}(BZ_2)$  and if  $\psi \in R_0(Z_n) \otimes R(\text{Spin}^c)$ , then the bundles  $\psi(i(M))$  and  $r(\psi)(M)$  agree so  $\eta(\psi, i(M)) = \eta(r(\psi), M)$  and  $i: \ker_*(\eta, 2) \rightarrow \ker_*(\eta, n)$ . Consequently it suffices to prove (c) in the special case  $n = 2$ . Let  $X \in E(m)$  and let  $0 \oplus X \in E(m+2)$ . Choose  $Y \in \tilde{Q}_{m+2}^{\text{Spin}^c}(BZ_2)$  so  $\pi(Y) = 0 \oplus X$ . Since  $2X = 0$ ,  $\pi(2Y) = 0$  so  $2Y \in R_{m+2}(2)$ . Since the Stiefel-Whitney numbers of  $2Y$  vanish,  $\exists Z \in R_{m+2}(2)$  so  $2Y = 2Z$ . Let  $M = \Delta(Y - Z)$  so  $2M = 0$  and  $\pi(M) = X$ . Since  $\delta_1(2(Y - Z)) = 0$ ,  $M \in \ker_m(\eta, 2)$  by Lemma 3.3 and the map  $X \rightarrow M$  provides a suitable splitting. If we fix  $m$  large, we can choose  $\bar{E}(m - 2k) = (1 - \pi_2)\Delta^k \bar{E}(m)$  for  $k = 0, 1, \dots$ . Since  $\tilde{\Omega}_m^{\text{Spin}^c}(BZ_n)$  is finite, we use the pigeon hole principal to choose  $\bar{E}(m)$  consistently for all  $m$  with this property. This completes the proof of (c).

Let  $M \in S_m(n) \cap \ker_m(\eta, n)$ ; we show  $M = 0$  by induction; this is trivial for  $m < 0$ . We apply  $\Delta$  and Lemma 2.6 to see  $\Delta(M) \in S_{m-2}(n) \cap \ker_{m-2}(\eta, n)$  so  $\Delta(M) = 0$ . Consequently  $M = L^1(n) \times (N_1 + N_2)$  for  $N_1 \in P_{m-1}$  and  $N_2 \in \text{Tor}(\Omega_{m-1}^{\text{Spin}^c})$ . We apply Lemma 2.4 to see  $N_1 \in n \cdot P_{m-1}$  so  $M = L^1(n) \times N_2$ . Consequently  $M \in S_m(n) \cap T_m(n) = 0$ . This completes the proof of all the assertions in the paper.

## REFERENCES

- [1] D. W. Anderson, E. H. Brown and F. P. Peterson, *Spin cobordism*, Bull. Amer. Math. Soc., **72** (1966), 257–260.
- [2] M. F. Atiyah, *K-theory*, W. A. Benjamin (1967).
- [3] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford Modules*, Topology, **3** Supp 1 (1964), 3–38.

- [4] M. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian Geometry I*, Math. Proc. Camb. Phil. Soc., **77** (1975), 43–69. II Math. Proc. Camb. Phil. Soc., **78** (1975), 405–432. III Math. Proc. Camb. Phil. Soc., **79** (1976), 71–99.
- [5] A. Bahri and P. Gilkey,  *$\text{Pin}^c$  bordism and equivariant  $\text{Spin}^c$  bordism of cyclic 2-groups*, (to appear Proc. Amer. Math. Soc.).
- [6] H. Cartan and S. Eilenberg, *Homological Algebra*, (1956), Princeton
- [7] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer Verlag 1964.
- [8] J. F. Davis and R. J. Milgram, *A survey of the Spherical Space form Problem*, Mathematical Reports, **2** (1984), 223–283. (Harwood Academic Publishers).
- [9] K. Fujii, T. Kobayashi, K. Shimomura, and M. Sugawara, *KO-groups of lens spaces modulo powers of two*, Hiroshima Math. J., **8** (1978), 469–489. Indiana Univ. Math. J., **27** (1978), 889–918.
- [10] V. Giambalvo, *Pin and Pin' cobordism*, Proc. Amer. Math. Soc., **39** (1973), 395–401.
- [11] P. Gilkey, *The eta invariant for even dimensional  $\text{Pin}^c$  manifolds*, Advances in Math., **58** (1985), 243–284.
- [12] ———, *The eta invariant and the K-theory of spherical space forms*, Inventiones Math., **76** (1984), 421–453.
- [13] ———, *The eta invariant and equivariant unitary bordism for spherical space form groups*, (to appear).
- [14] ———, *Invariance Theory, The Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish Press (1985).
- [15] N. Hitchin, *Harmonic spinors*, Advances in Math., **14** (1974), 1–55.
- [16] U. Korschörke, *Concordance and bordism of line fields*, Inventiones Math., **24** (1974), 241–268.
- [17] F. P. Peterson, *Lectures on cobordism theory*, Lecture notes in mathematics, Kyoto University (1968).
- [18] R. E. Stong, *Relations among characteristic numbers I*, Topology, **4** (1965), 267–286. II Topology, **5** (1966), 133–148.
- [19] ———, *Notes on Cobordism*, Princeton University Press (1968).
- [20] G. Wilson, *K-theory invariants for unitary G-bordism*, Quart J. Math., **24** (1973), 499–526.
- [21] J. A. Wolf, *Spaces of Constant Curvature*, Publish or Perish Press, 5th ed (1984).

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