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OF TWO STARSHAPED SETS IN  $R^3$**

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Set  $S$  in  $R^d$  has property  $P_k$  if and only if  $S$  is a finite union of  $d$ -polytopes and for every finite set  $F$  in  $\text{bdry} S$  there exist points  $c_1, \dots, c_k$  (depending on  $F$ ) such that each point of  $F$  is clearly visible via  $S$  from at least one  $c_i$ ,  $1 \leq i \leq k$ . The following results are established.

(1) Let  $S \subseteq R^3$ . If  $S$  satisfies property  $P_2$ , then  $S$  is a union of two starshaped sets.

(2) Let  $S \subseteq R^d$ ,  $d \geq 3$ . If  $S$  is a compact union of  $k$  starshaped sets, then there exists a sequence  $\{S_j\}$  converging to  $S$  (relative to the Hausdorff metric) such that each set  $S_j$  satisfies property  $P_k$ .

When  $d = 3$  and  $k = 2$ , the converse of (2) above holds as well, yielding a characterization theorem for compact unions of two starshaped sets in  $R^3$ .

**1. Introduction.** We begin with some definitions. Let  $S$  be a subset of  $R^d$ . Hyperplane  $H$  is said to *support  $S$  locally* at boundary point  $s$  of  $S$  if and only if  $s \in H$  and there is some neighborhood  $N$  of  $s$  such that  $N \cap S$  lies in one of the closed halfspaces determined by  $H$ . Point  $s$  in  $S$  is called a *point of local convexity* of  $S$  if and only if there is some neighborhood  $N$  of  $s$  such that  $N \cap S$  is convex. If  $S$  fails to be locally convex at  $q$  in  $S$ , then  $q$  is called a *point of local nonconvexity* (Inc point) of  $S$ . For points  $x$  and  $y$  in  $S$ , we say  $x$  *sees*  $y$  via  $S$  ( $x$  is *visible* from  $y$  via  $S$ ) if and only if the segment  $[x, y]$  lies in  $S$ . Similarly,  $x$  is *clearly visible* from  $y$  via  $S$  if and only if there is some neighborhood  $N$  of  $x$  such that  $y$  sees via  $S$  each point of  $N \cap S$ . Set  $S$  is *locally starshaped* at point  $x$  of  $S$  if and only if there is some neighborhood  $N$  of  $x$  such that  $x$  sees via  $S$  each point of  $N \cap S$ . Finally, set  $S$  is *starshaped* if and only if there is some point  $p$  in  $S$  such that  $p$  sees via  $S$  each point of  $S$ , and the set of all such points  $p$  is called the (convex) *kernel* of  $S$ .

A well-known theorem of Krasnosel'skii [3] states that if  $S$  is a nonempty compact set in  $R^d$ ,  $S$  is starshaped if and only if every  $d + 1$  points of  $S$  are visible via  $S$  from a common point. Moreover, "points of  $S$ " may be replaced by "boundary points of  $S$ " to produce a stronger result. In [1], the concept of clear visibility, together with work by Lawrence, Hare, and Kenelly [4], were used to obtain the following

Krasnosel'skii-type theorem for unions of two starshaped sets in the plane: Let  $S$  be a compact nonempty set in  $R^2$ , and assume that for each finite set  $F$  in the boundary of  $S$  there exist points  $c, d$  (depending on  $F$ ) such that each point of  $F$  is clearly visible via  $S$  from at least one of  $c, d$ . Then  $S$  is a union of two starshaped sets.

In this paper, an analogous result is proved for set  $S$  in  $R^3$ , where  $S$  satisfies the additional hypothesis of being a finite union of polytopes. Furthermore, while not every compact union  $F$  of two starshaped sets in  $R^3$  satisfies this hypothesis,  $F$  will be the limit (relative to the Hausdorff metric) for a sequence whose members do satisfy it. This in turn leads to a characterization theorem for compact unions of two starshaped sets in  $R^3$ .

The following terminology will be used throughout the paper:  $\text{Conv}S$ ,  $\text{cl}S$ ,  $\text{int}S$ ,  $\text{rel int}S$ ,  $\text{bdry}S$ ,  $\text{rel bdry}S$ , and  $\text{ker}S$  will denote the convex hull, closure, interior, relative interior, boundary, relative boundary, and kernel, respectively, for set  $S$ . The distance from point  $x$  to point  $y$  will be denoted  $\text{dist}(x, y)$ . For distinct points  $x$  and  $y$ ,  $L(x, y)$  will be the line determined by  $x$  and  $y$ , while  $R(x, y)$  will be the ray emanating from  $x$  through  $y$ . For  $x \in S$ ,  $A_x$  will represent  $\{x: z \text{ is clearly visible via } S \text{ from } x\}$ . The reader is referred to Valentine [7] and to Lay [5] for a discussion of these concepts and to Nadler [6] for information on the Hausdorff metric.

## 2. The results. The following definition will be helpful.

**DEFINITION 1.** Let  $S \subseteq R^d$ . We say that  $S$  has property  $P_k$  if and only if  $S$  is a finite union of  $d$ -polytopes and for every finite set  $F \subseteq \text{bdry}S$  there exist points  $c_1, \dots, c_k$  (depending on  $F$ ) such that each point of  $F$  is clearly visible via  $S$  from at least one  $c_i$ ,  $1 \leq i \leq k$ .

Several lemmas will be needed to prove Theorem 1. The first of these is a variation of [2, Lemma 2].

**LEMMA 1.** *Let  $S \subseteq R^d$ ,  $z \in S$ , and assume that  $S$  is locally starshaped at  $z$ . If  $p \in \text{conv}A_z$  and  $p \neq z$ , then there exists some point  $p' \in [p, z)$  such that  $p' \in A_z$ .*

*Proof.* As in [2, Lemma 2], use Carathéodory's theorem to select a set of  $d + 1$  or fewer points  $p_1, \dots, p_k$  in  $A_z$  with  $p \in \text{conv}\{p_1, \dots, p_k\}$ . Say  $p = \sum\{\lambda_i p_i: 1 \leq i \leq k\}$ , where  $0 \leq \lambda_i \leq 1$  and  $\sum\{\lambda_i: 1 \leq i \leq k\} = 1$ . Observe that for any  $0 \leq \mu \leq 1$ , point  $\mu z + (1 - \mu)p$  on  $[z, p]$  is a convex combination of the points  $\mu z + (1 - \mu)p_i$ ,  $1 \leq i \leq k$ . Also  $\mu z + (1 - \mu)p_i \in [z, p_i]$ ,  $1 \leq i \leq k$ . By the definition of locally starshaped,

together with the definition of clear visibility, we may choose a spherical neighborhood  $N$  of  $z$ ,  $p \notin N$ , such that  $z$  and each  $p_i$  see via  $S$  every point of  $N \cap S$ . We may choose  $\mu_0$ ,  $0 < \mu_0 < 1$  and  $\mu_0$  sufficiently near 1 that each point  $\mu_0 z + (1 - \mu_0)p_i = p'_i$  belongs to  $N$ . Define

$$\begin{aligned} p' &= \Sigma\{\lambda_i p'_i: 1 \leq i \leq k\} \\ &= \mu_0 z + (1 - \mu_0)p \in \text{conv}\{p'_1, \dots, p'_k\} \cap (z, p) \cap N. \end{aligned}$$

We will show that  $p'$  satisfies the lemma. For  $x \in N \cap S$ ,  $[x, z] \subseteq N \cap S$ ,  $p_1$  sees  $[x, z]$  via  $S$ , and hence  $\text{conv}\{p'_1, x, z\} \subseteq N \cap S$ . By an easy induction,  $\text{conv}\{p'_k, \dots, p'_1, x, z\} \subseteq N \cap S$ . Since  $p' \in \text{conv}\{p'_k, \dots, p'_1\}$ ,  $[p', x] \subseteq S$ . We conclude that  $p'$  sees via  $S$  each point of  $N \cap S$ ,  $p' \in A_z$ , and Lemma 1 is established.

**LEMMA 2.** *Let  $S$  be a closed set in  $R^d$ . Let  $P$  be a plane in  $R^d$ ,  $B$  a component of  $P \sim S$ , with  $S$  locally starshaped at  $z \in \text{bdry} B$ . Assume that line  $L$  in plane  $P$  supports  $B$  locally at  $z$  and that  $B \cap M$  is in the open halfplane  $L_1$  determined by  $L$  for an appropriate neighborhood  $M$  of  $z$ . Then  $(\text{conv} A_z) \cap P \subseteq \text{cl} L_2$ , where  $L_2$  is the opposite open halfplane determined by  $L$ .*

*Proof.* Suppose on the contrary that there is some point  $p \in (\text{conv} A_z) \cap P \cap L_1$ , to obtain a contradiction. Then  $p \neq z$ , so by Lemma 1 there exist point  $p' \in [p, z)$  and convex neighborhood  $N$  of  $z$  such that  $p'$  sees via  $S$  each point of  $N \cap S$ . For convenience of notation, assume that  $N \subseteq M \subseteq P$ .

By a simple geometric argument, we may choose a point  $b \in B \cap N$  such that  $R(p', b)$  meets  $N \cap L$  at some point  $w$ . Since  $B \cap N \subseteq B \cap M \subseteq L_1$ ,  $w \notin B$ , so  $(b, w]$  meets  $\text{bdry} B$  at a point  $c$ . We have  $c \in [b, w] \subseteq N$  and  $c \in \text{bdry} B \subseteq S$ , so  $c \in N \cap S$ . Therefore, by our choice of  $p'$ ,  $[p', c] \subseteq S$ . Hence  $b \in [p', c] \subseteq S$ , impossible since  $b \in B \subseteq P \sim S$ . We have a contradiction, our supposition is false, and  $(\text{conv} A_z) \cap P \subseteq \text{cl} L_2$ . Thus Lemma 2 is proved.

**LEMMA 3.** *Let  $S$  be a compact set in  $R^3$ , and assume that  $S$  is a finite union of polytopes. Let  $P$  be a plane in  $R^3$ , with  $b$  a bounded component of  $P \sim S$ . For  $z$  a point of local convexity of  $\text{cl} B$ ,  $z$  in edge  $e \subseteq \text{rel bdry cl} B$ , there exists a plane  $H$  such that the following are true:*

- (1)  $H \cap P$  is a line containing  $e$ .
- (2) The two open halfspaces determined by  $H$  can be denoted  $H_1$  and  $H_2$  in such a way that for  $N$  any neighborhood of  $z$  such that  $(\text{cl} B) \cap N$  is convex,  $B \cap N$  lies in  $H_1$  while  $A_z \subseteq \text{cl} H_2$ .

*Proof.* Notice that  $S$  is locally starshaped at each of its points and that  $\text{bdry} B$  is a closed polygonal curve in  $P$ . Let  $J$  be a plane,  $J \neq P$ , such that  $J$  contains edge  $e$  of  $\text{bdry} B$ . If  $N$  is any neighborhood of  $z$  such that  $(\text{cl} B) \cap N$  is convex, then  $J$  supports  $(\text{cl} B) \cap N$  at  $e$ , and  $B \cap N$  necessarily lies in one of the open halfspaces  $J_1$  determined by  $J$ . If  $A_z \subseteq \text{cl} J_2$ , then  $J$  satisfies the lemma. Otherwise,  $A_z \cap J_1 \neq \emptyset$ .

For convenience of notation, let  $P_1$  and  $P_2$  denote distinct open halfspaces in  $R^3$  determined by plane  $P$ , let  $L = P \cap J$ , and label the halfplanes in  $P$  determined by  $L$  so that  $B \cap N \subseteq L_1 \equiv J_1 \cap P$ . (See Figure 1.) Observe that  $\text{conv} A_z$  is necessarily disjoint from one of  $J_1 \cap P_1$  or  $J_1 \cap P_2$ , for otherwise  $(\text{conv} A_z) \cap J_1 \cap P \equiv (\text{conv} A_z) \cap L_1 \cap P \neq \emptyset$ , contradicting Lemma 2. Thus we may assume that  $(\text{conv} A_z) \cap J_1 \cap P_2 = \emptyset$ , and since  $(\text{conv} A_z) \cap L_1 = \emptyset$ ,  $(\text{conv} A_z) \cap J_1 \subseteq P_1$ .

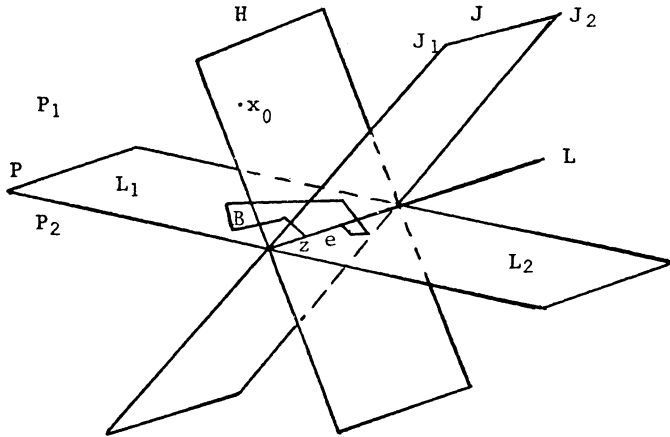


FIGURE 1

Examine the points of  $A_z \cap J_1 \subseteq P_1$ . For  $x \in A_z \cap J_1$ ,  $x$  sees via  $S$  a nondegenerate segment  $s_z$  at  $z$  contained in edge  $e$ , thus generating a planar set  $T_x \equiv \text{conv}(s_x \cup \{x\})$ . Since none of the  $T_x$  sets lie in  $P$ , each determines with  $\text{cl} L_1$  an angle of positive measure  $m(x)$ . Define  $m \equiv \text{glb}\{m(x) : x \in A_z \cap J_1\}$ . Since  $S$  is a finite union of polytopes, the  $T_x$  sets lie in a finite union of polytopes, each meeting edge  $e$  in a nondegenerate segment at  $z$ , each contained in  $P_1 \cup L$ . This forces  $m$  to be positive. Using a standard argument, select sequence  $\{x_i\}$  in  $A_z \cap J_1$  so that  $\{m(x_i)\}$  converges to  $m$ . Some subsequence of  $\{x_i\}$  also converges, say to  $x_0$ . Moreover, the angle determined by  $\text{conv}(e \cup \{x_0\})$  and  $\text{cl} L_1$  has measure  $m$ , and  $x_0 \in (\text{cl} A_z) \cap J_1 \subseteq P_1$ . Let  $H$  be the plane determined by  $\text{conv}(e \cup \{x_0\})$ . Of course  $H \cap P = L$ . Furthermore, for an

appropriate labeling of halfspaces determined by  $H$ ,  $L_1 \subseteq H_1$  so  $B \cap N \subseteq H_1$ .

It remains to show that  $A_z \subseteq \text{cl } H_2$ . Suppose on the contrary that  $y \in A_z \cap H_1$ . If  $y \in P_1$ , then the angle  $m$  chosen above would not be minimal. If  $y \in P$ , then  $y \in A_z \cap P \cap L_1$ , contradicting Lemma 2. If  $y \in P_2$ , then since  $y \in P_2 \cap H_1$  and  $x_0 \in P_1 \cap H$ ,  $[y, x_0]$  would meet  $P \cap H_1 = L_1$ . Moreover, since  $x_0 \in \text{cl } A_z$ , there would be a point  $x'_0 \in A_z$  sufficiently near  $x_0$  that  $[y, x'_0]$  would meet  $P \cap H_1 = L_1$  also, say at point  $w$ . Then  $w \in (\text{conv } A_z) \cap P \cap L_1$ , again contradicting Lemma 2. We conclude that  $A_z \cap H_1 = \emptyset$ , and  $A_z \subseteq \text{cl } H_2$ , finishing the proof of Lemma 3.

The final lemma follows immediately from [4, Theorem 1].

**LEMMA 4 (Lawrence, Hare, Kenelly Lemma).** *Let  $S$  be a closed set in  $R^d$ . Assume that every finite set  $F$  in  $\text{bdry } S$  may be partitioned into two sets  $F_1$  and  $F_2$  such that each point of  $F_i$  is clearly visible from a common point of  $S$ . Then  $\text{bdry } S$  may be partitioned into two sets  $S_1$  and  $S_2$  such that for every finite set  $F$  in  $\text{bdry } S$ , each point of  $F \cap S_i$  is clearly visible from a common point of  $S$ ,  $i = 1, 2$ .*

We are ready to prove the following theorem.

**THEOREM 1.** *Let  $S \subseteq R^3$ . If  $S$  satisfies property  $P_2$ , then  $S$  is a union of two starshaped sets.*

*Proof.* Using Lemma 4, select a partition  $S_1, S_2$  for  $\text{bdry } S$  such that for every finite set  $F$  in  $\text{bdry } S$ , each point of  $F \cap S_i$  is clearly visible via  $S$  from a common point. For  $i = 1, 2$ , define  $\mathcal{T}_i = \{\text{cl } A_z : z \in S_i\}$ . Then each  $\mathcal{T}_i$  is a collection of compact subsets of  $S$ . Moreover, by our choice of  $S_1$  and  $S_2$ , each  $\mathcal{T}_i$  has the finite intersection property. Hence  $\bigcap \{T : T \text{ in } \mathcal{T}_i\} \neq \emptyset$ , and we may select points  $c$  and  $d$  with  $c \in \bigcap \{T : T \text{ in } \mathcal{T}_1\}$  and  $d \in \bigcap \{T : T \text{ in } \mathcal{T}_2\}$ . Observe that for  $z \in \text{bdry } S = S_1 \cup S_2$ , one of  $c$  or  $d$ , say  $c$ , belongs to  $\text{cl } A_z$ . Then  $[c, z] \subseteq S$ . We conclude that each boundary point of  $S$  sees via  $S$  either  $c$  or  $d$ .

We will show that each point of  $S$  sees via  $S$  either  $c$  or  $d$ . Portions of the argument will resemble the proof of [1, Theorem 1]. Let  $x \in S$  and suppose on the contrary that neither  $c$  nor  $d$  sees  $x$ , to reach a contradiction. Certainly  $x \notin \{c, d\}$ , and by a previous observation,  $x \in \text{int } S$ . As in [1, Theorem 1], choose the segment at  $x$  in  $S \cap L(c, x)$  having maximal length, and let  $p$  and  $q$  denote its endpoints, with the order of



Define line  $L'$  and associated point  $t$  as follows: Clearly  $L(c, v) \cap J = \emptyset$ . In case  $L(c, v) \cap K \neq \emptyset$ , let  $L_1$  denote the open halfplane of  $P$  determined by  $L(c, v)$  and containing  $J$ . Let  $L'$  be the line from  $c$  supporting  $\text{conv}K$  at a point of  $L_1$ . Using our previous observation,  $L' \cap (\text{bdry conv}K)$  contains some point  $t$  of  $\text{bdry}K$  such that  $[c, t] \subseteq S$ . In case  $L(c, v) \cap K = \emptyset$ , rotate  $L(c, v)$  about  $c$  toward  $d$  until  $\text{bdry}K$  is met. Let  $L'$  be the corresponding rotated line. Again using our observation, there is some  $t \in L' \cap (\text{bdry conv}K) \cap (\text{bdry}K)$  with  $[c, t] \subseteq S$ . Of course, in each case  $t$  may be chosen to be the furthest point from  $c$  having the required property. Moreover,  $[c, t] \cap J = \emptyset$ , and we may label the open halfplanes of  $P$  determined by  $L'$  so that  $J \subseteq L'_1$ . Then  $K \cup \{d\}$  lies in the opposite halfplane  $L'_2$ .

Since  $S$  is a finite union of polytopes,  $\text{bdry}K$  is necessarily a simple closed polygonal curve in plane  $P$ . By our choice of  $t$ , clearly  $t$  is a point of local convexity of  $\text{cl}K$ . Also,  $t$  must be a vertex of  $\text{bdry}K$ , so  $\text{bdry}K$  contains two edges  $e_1$  and  $e_2$  at  $t$ . Moreover, for an appropriate labeling of these edges,  $e_1 \subseteq \text{cl}L'_2$ ,  $e_2 \subseteq L'_2 \cup \{t\}$ , and for any neighborhood  $N$  of  $t$  with  $(\text{cl}K) \cap N$  convex,  $K \cap N$  and  $c$  lie in the same open halfplane of  $P$  determined by  $L(e_2)$ .

Using Lemma 3, select a plane  $H$  such that  $H \cap P$  is a line containing  $e_2$ ,  $K \cap N \subseteq H_1$ , and  $A_t \subseteq \text{cl}H_2$ . Similarly, select plane  $M$  for  $e_1$  so that  $K \cap N \subseteq M_1$  and  $A_t \subseteq \text{cl}M_2$ . Recall that by our choice of  $c$  and  $d$ , at least one of these points lies in  $\text{cl}A_t \subseteq \text{cl}H_2 \cap \text{cl}M_2$ . Since  $c$  and  $K \cap N$  are in the same open halfplane of  $P$  determined by  $L(e_2)$ ,  $c \in H_1$ . This forces  $d$  to belong to  $\text{cl}H_2 \cap \text{cl}M_2 \cap P$ . However, clearly  $\text{cl}H_2 \cap \text{cl}M_2 \cap P \subseteq \text{cl}L'_1$ , while  $d \in L'_2$ . We have a contradiction, our supposition is false, and every point of  $S$  must see via  $S$  either  $c$  or  $d$ . Hence  $S$  is a union of two starshaped sets, and Theorem 1 is established.

**THEOREM 2.** *For  $k \geq 1$  and  $d \geq 1$ , let  $\mathcal{F}(k, d)$  denote the family of all compact unions of  $k$  (or fewer) starshaped sets in  $R^d$ ,  $\mathcal{C}(k, d)$  the subfamily of  $\mathcal{F}(k, d)$  whose members are finite unions of  $d$ -polytopes. Then  $\mathcal{C}(k, d)$  is dense in  $\mathcal{F}(k, d)$ , relative to the Hausdorff metric. Moreover,  $\mathcal{F}(k, d)$  is closed, relative to the Hausdorff metric.*

*Proof.* In the proof,  $h$  will denote the Hausdorff metric on compact subsets of  $R^d$ . That is, if  $(A)_\delta = \{x: \text{dist}(x, A) < \delta\}$ , then for  $A$  and  $B$  compact in  $R^d$ ,  $h(A, B) = \inf\{\delta: A \subseteq (B)_\delta \text{ and } B \subseteq (A)_\delta, \delta > 0\}$ .

To see that  $\mathcal{C}(k, d)$  is dense in  $\mathcal{F}(k, d)$ , let  $S \in \mathcal{F}(k, d)$ . For an arbitrary  $\delta > 0$ , we must find some  $C$  in  $\mathcal{C}(k, d)$  for which  $h(S, C) < \delta$ . Assume that each point of  $S$  is visible via  $S$  from one of  $s_1, \dots, s_k$ . Form

an open cover for  $S$ , using interiors of  $d$ -simplices whose diameters are at most  $\delta/2$ . Using the compactness of  $S$ , reduce to a finite subcover, say  $\{\text{int } P_j: 1 \leq j \leq m\}$ , where  $P_j$  is a  $d$ -simplex. For  $1 \leq i \leq k$ , define  $C_i = \bigcup \{\text{conv}(s_i \cup P_j): s_i \text{ sees via } S \text{ some point of } P_j, 1 \leq j \leq m\}$ . Certainly set  $C \equiv C_1 \cup \dots \cup C_k$  is a union of  $k$  starshaped sets as well as a finite union of  $d$ -polytopes. Thus  $C \in \mathcal{C}(k, d)$ .

Clearly  $S \subseteq C$ , so  $S \subseteq (C)_\delta$ . To see that  $C \subseteq (S)_\delta$ , let  $x \in C \sim S$ . Then  $x \in \text{conv}(s_i \cup P_j)$  for some  $i$  and  $j$ . Moreover, for an appropriate  $i$  and  $j$ , there is some  $y' \in P_j \cap S$  with  $[s_i, y'] \subseteq S$ . If  $x, s_i, y'$  are collinear, then since  $x \notin S$ ,  $x$  must belong to  $P_j$ , and  $\text{dist}(x, y') \leq \delta/2$ . Thus  $x \in (S)_\delta$ . If  $x, s_i, y'$  are not collinear, assume  $x \in [s_i, y]$  where  $y \in P_j$ , and let  $x'$  be the point of  $[s_i, y']$  such that  $[x, x']$  and  $[y, y']$  are parallel. Then  $x' \in S$  and  $\text{dist}(x, x') \leq \text{dist}(y, y') \leq \delta/2$ . Again  $x \in (S)_\delta$ . We conclude that  $C \subseteq (S)_\delta$ ,  $h(S, C) < \delta$ , and  $\mathcal{C}(k, d)$  is indeed dense in  $\mathcal{F}(k, d)$ .

Finally, to see that  $\mathcal{F}(k, d)$  is closed, let  $\{S_i\}$  be a sequence in  $\mathcal{F}(k, d)$  converging to the compact set  $S_0$ , to show that  $S_0 \in \mathcal{F}(k, d)$  also. For convenience of notation, for  $i \geq 1$ , let  $S_i$  be a union of  $k$  starshaped sets whose compact kernels are  $A_{i1}, A_{i2}, \dots, A_{ik}$ , respectively. Then by standard results concerning the Hausdorff metric [6],  $\{A_{i1}: i \geq 1\}$  has a subsequence  $\{A'_{i1}\}$  converging to some compact convex set  $A_1$ . Pass to the associated subsequence  $\{S'_i\}$  of  $\{S_i\}$ , and repeat the argument for corresponding kernels  $\{A'_{i2}\}$ . By an obvious induction, in  $k$  steps we obtain subsequences  $\{A^{(k)}_{i1}\}, \{A^{(k)}_{i2}\}, \dots, \{A^{(k)}_{ik}\}$  converging to compact convex sets  $A_1, \dots, A_k$ , respectively. It is a routine matter to show that  $S_0$  is a union of  $k$  or fewer compact starshaped sets having kernels  $A_1, \dots, A_k$ .

**THEOREM 3.** *Let  $S$  be a compact union of  $k$  starshaped sets in  $R^d$ ,  $k \geq 1$ ,  $d \geq 3$ . Then there is a sequence  $\{S_j\}$  converging to  $S$  (relative to the Hausdorff metric) such that each  $S_j$  satisfies property  $P_k$ . That is, using the notation of Theorem 2, sets having property  $P_k$  are dense in  $\mathcal{F}(k, d)$ .*

*Proof.* As in the proof of Theorem 2,  $h$  will denote the Hausdorff metric on compact subsets of  $R^d$ . For any  $\delta > 0$ , we must find some  $C$  having property  $P_k$  for which  $h(S, C) < \delta$ .

Assume that each point of  $S$  is visible via  $S$  from one of the distinct points  $s_1, \dots, s_k$ . Form an open cover for  $S$  using spheres of radius  $\delta/4$ , centered at points of  $S$ . Reduce to a finite subcover, and choose the center of each sphere. Say these centers are the points  $t_1, \dots, t_m$ . Partition

$\{t_1, \dots, t_m\}$  into  $k$  subsets  $V_1, \dots, V_k$  such that the following is true: If  $t \in V_i$ , then  $s_i$  is a point of  $\{s_1, \dots, s_k\}$  closest to  $t$  with  $[s_i, t] \subseteq S$ . Define  $T_i = \cup\{[s_i, t]: t \in V_i\}$ . Observe that  $s_i \notin T_j$  for  $i \neq j$ : Otherwise,  $s_i \in (s_j, t)$  for some  $t \in V_j$ ,  $[s_i, t] \subseteq (s_j, t] \subseteq S$ , and  $s_i$  would be closer to  $t$  than  $s_j$  is to  $t$ , impossible by the definition of  $V_j$ .

In case the sets  $T_1, \dots, T_k$  are pairwise disjoint, let  $T'_i = T_i$ ,  $1 \leq i \leq k$ , and define  $T$  to be their union. Otherwise, suppose  $T_1$  meets  $T_2 \cup \dots \cup T_k$ . Then for some point in  $V_1$ , call it  $t_1$  (for convenience of notation),  $(s_1, t_1]$  meets  $T_2 \cup \dots \cup T_k$ . Using the facts that each  $T_i$  set is a finite union of edges at  $s_i$ ,  $s_1 \notin T_2 \cup \dots \cup T_k$ , and  $d \geq 3$ , it is not hard to show that there exists an edge  $[s_1, t'_1]$  not collinear with  $[s_1, t_1]$  such that  $[s_1, t'_1]$  is disjoint from  $T_2 \cup \dots \cup T_k$  and  $\text{dist}(t_1, t'_1) < \delta/4$ . Thus  $h([s_1, t_1], [s_1, t'_1]) < \delta/4$ , also. Repeating the procedure for each edge of  $T_1$ , in finitely many steps we obtain a new set  $T'_1$  starshaped at  $s_1$  such that  $T'_1$  is disjoint from  $T_2 \cup \dots \cup T_k$  and  $h(T_1, T'_1) < \delta/4$ .

Continuing the process for  $T_2, \dots, T_k$ , by an obvious induction we obtain pairwise disjoint starshaped sets  $T'_1, T'_2, \dots, T'_k$  with  $h(T_i, T'_i) < \delta/4$ ,  $1 \leq i \leq k$ . Define  $T = T'_1 \cup \dots \cup T'_k$ . Standard arguments reveal that

$$h(S, T_1 \cup \dots \cup T_k) < \frac{\delta}{4}, \quad h(T_1 \cup \dots \cup T_k, T) < \frac{\delta}{4},$$

and hence  $h(S, T) < \delta/2$ .

Finally, we extend the sets  $T'_1, \dots, T'_k$  to finite unions of  $d$ -polytopes. define  $m = \min\{h(T'_i, T'_j): i \neq j\}$ . Using techniques from Theorem 2, select set  $C \equiv C_1 \cup \dots \cup C_k$  in  $\mathcal{C}(k, d)$  with  $h(T_i, C_i) < \min\{\delta/2, m/2\}$  and with  $s_i \in \ker C_i$ ,  $1 \leq i \leq k$ . Since  $h(T_i, C_i) < m/2$ , certainly the  $C_i$  sets must be pairwise disjoint. Therefore, each boundary point of  $C$  is clearly visible from some  $s_i$ ,  $1 \leq i \leq k$ , and  $C$  has property  $P_k$ . Moreover,

$$h(S, C) \leq h(S, T) + h(T, C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Theorem 3 is established.

It is interesting to observe that while Theorem 3 holds when  $d \geq 3$ , it fails in the plane, as the following easy example reveals.

**EXAMPLE 1.** Let  $S$  be the set in Figure 3. Then  $S$  is a union of two starshaped sets with kernels  $\{c\}$ ,  $\{d\}$ , respectively. However, sets sufficiently close to  $S$  fail to satisfy the clear visibility condition required for property  $P_2$ .

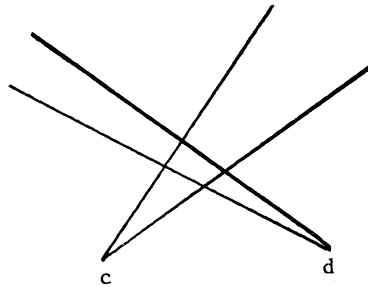


FIGURE 3

Finally, the characterization theorem for unions of two starshaped sets in  $R^3$  is an easy consequence of our previous results.

**COROLLARY 1.** *Let  $S \subseteq R^3$ . Then  $S$  is a compact union of two starshaped sets if and only if there is a sequence  $\{S_j\}$  converging to  $S$  (relative to the Hausdorff metric) such that each set  $S_j$  satisfies property  $P_2$ .*

*Proof.* The necessity follows immediately from Theorem 3. For the sufficiency, Theorem 1 implies that each set  $S_j$  is a compact union of two starshaped sets in  $R^3$ . By Theorem 2, their limit  $S$  is a compact union of two starshaped sets as well.

## REFERENCES

- [1] Marilyn Breen, *Clear visibility and unions of two starshaped sets in the plane*, Pacific J. Math., **115** (1984), 267–275.
- [2] ———, *Points of local nonconvexity, clear visibility, and starshaped sets in  $R^d$* , J. Geometry, **21** (1983), 43–52.
- [3] M. A. Krasnosel'skii, *Sur un critère pour qu'un domaine soit étoilé*, Math. Sb., **19** (61) (1946), 309–310.
- [4] J. F. Lawrence, W. R. Hare, Jr. and John W. Kenelly, *Finite unions of convex sets*, Proc. Amer. Math. Soc., **34** (1972), 225–228.
- [5] Steven R. Lay, *Convex Sets and Their Applications*, John Wiley, New York, 1982.
- [6] S. Nadler, *Hyperspaces of Sets*, Marcel Dekker, Inc., New York, 1978.
- [7] F. A. Valentine, *Convex Sets*, McGraw-Hill, New York, 1964.

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