A CHARACTERIZATION THEOREM FOR COMPACT UNIONS OF TWO STARSHAPED SETS IN $\mathbb{R}^3$

Marilyn Breen
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Set $S$ in $R^d$ has property $P_k$ if and only if $S$ is a finite union of $d$-polytopes and for every finite set $F$ in $bdryS$ there exist points $c_1, \ldots, c_k$ (depending on $F$) such that each point of $F$ is clearly visible via $S$ from at least one $c_i$, $1 \leq i \leq k$. The following results are established.

1. Introduction. We begin with some definitions. Let $S$ be a subset of $R^d$. Hyperplane $H$ is said to support $S$ locally at boundary point $s$ of $S$ if and only if $s \in H$ and there is some neighborhood $N$ of $s$ such that $N \cap S$ lies in one of the closed halfspaces determined by $H$. Point $s$ in $S$ is called a point of local convexity of $S$ if and only if there is some neighborhood $N$ of $s$ such that $N \cap S$ is convex. If $S$ fails to be locally convex at $q$ in $S$, then $q$ is called a point of local nonconvexity (lnc point) of $S$. For points $x$ and $y$ in $S$, we say $x$ sees $y$ via $S$ ($x$ is visible from $y$ via $S$) if and only if the segment $[x, y]$ lies in $S$. Similarly, $x$ is clearly visible from $y$ via $S$ if and only if there is some neighborhood $N$ of $x$ such that $y$ sees via $S$ each point of $N \cap S$. Set $S$ is locally starshaped at point $x$ of $S$ if and only if there is some neighborhood $N$ of $x$ such that $x$ sees via $S$ each point of $N \cap S$. Finally, set $S$ is starshaped if and only if there is some point $p$ in $S$ such that $p$ sees via $S$ each point of $S$, and the set of all such points $p$ is called the (convex) kernel of $S$.

A well-known theorem of Krasnosel'skii [3] states that if $S$ is a nonempty compact set in $R^d$, $S$ is starshaped if and only if every $d + 1$ points of $S$ are visible via $S$ from a common point. Moreover, "points of $S$" may be replaced by "boundary points of $S" to produce a stronger result. In [1], the concept of clear visibility, together with work by Lawrence, Hare, and Kenelly [4], were used to obtain the following
Krasnosel'skii-type theorem for unions of two starshaped sets in the plane: Let $S$ be a compact nonempty set in $R^2$, and assume that for each finite set $F$ in the boundary of $S$ there exist points $c, d$ (depending on $F$) such that each point of $F$ is clearly visible via $S$ from at least one of $c, d$. Then $S$ is a union of two starshaped sets.

In this paper, an analogous result is proved for set $S$ in $R^3$, where $S$ satisfies the additional hypothesis of being a finite union of polytopes. Furthermore, while not every compact union $F$ of two starshaped sets in $R^3$ satisfies this hypothesis, $F$ will be the limit (relative to the Hausdorff metric) for a sequence whose members do satisfy it. This in turn leads to a characterization theorem for compact unions of two starshaped sets in $R^3$.

The following terminology will be used throughout the paper: $\text{Conv} S$, $\text{cl} S$, $\text{int} S$, $\text{rel int} S$, $\text{bdry} S$, $\text{rel bdry} S$, and $\text{ker} S$ will denote the convex hull, closure, interior, relative interior, boundary, relative boundary, and kernel, respectively, for set $S$. The distance from point $x$ to point $y$ will be denoted $\text{dist}(x, y)$. For distinct points $x$ and $y$, $L(x, y)$ will be the line determined by $x$ and $y$, while $R(x, y)$ will be the ray emanating from $x$ through $y$. For $x \in S$, $A_z$ will represent $\{x: z$ is clearly visible via $S$ from $x\}$. The reader is referred to Valentine [7] and to Lay [5] for a discussion of these concepts and to Nadler [6] for information on the Hausdorff metric.

2. The results. The following definition will be helpful.

**Definition 1.** Let $S \subseteq R^d$. We say that $S$ has property $P_k$ if and only if $S$ is a finite union of $d$-polytopes and for every finite set $F \subseteq \text{bdry} S$ there exist points $c_1, \ldots, c_k$ (depending on $F$) such that each point of $F$ is clearly visible via $S$ from at least one $c_i, 1 \leq i \leq k$.

Several lemmas will be needed to prove Theorem 1. The first of these is a variation of [2, Lemma 2].

**Lemma 1.** Let $S \subseteq R^d$, $z \in S$, and assume that $S$ is locally starshaped at $z$. If $p \in \text{conv} A_z$ and $p \neq z$, then there exists some point $p' \in [p, z)$ such that $p' \in A_z$.

**Proof.** As in [2, Lemma 2], use Carathéodory's theorem to select a set of $d + 1$ or fewer points $p_1, \ldots, p_k$ in $A_z$ with $p \in \text{conv}\{p_1, \ldots, p_k\}$. Say $p = \sum \lambda_i p_i: 1 \leq i \leq k$, where $0 \leq \lambda_i \leq 1$ and $\sum \lambda_i: 1 \leq i \leq k = 1$. Observe that for any $0 \leq \mu \leq 1$, point $\mu z + (1 - \mu)p$ on $[z, p]$ is a convex combination of the points $\mu z + (1 - \mu)p_i, 1 \leq i \leq k$. Also $\mu z + (1 - \mu)p_i \in [z, p_i], 1 \leq i \leq k$. By the definition of locally starshaped,
together with the definition of clear visibility, we may choose a spherical neighborhood $N$ of $z$, $p \notin N$, such that $z$ and each $p_i$ see via $S$ every point of $N \cap S$. We may choose $\mu_0$, $0 < \mu_0 < 1$ and $\mu_0$ sufficiently near 1 that each point $\mu_0 z + (1 - \mu_0)p = p'_i$ belongs to $N$. Define

$$p' = \sum \{ \lambda_ip_i' : 1 \leq i \leq k \}$$

$$= \mu_0 z + (1 - \mu_0)p \in \text{conv}\{ p'_1, \ldots, p'_k \} \cap (z, p) \cap N.$$ 

We will show that $p'$ satisfies the lemma. For $x \in N \cap S$, $[x, z] \subseteq N \cap S$, $p_1$ sees $[x, z]$ via $S$, and hence $\text{conv}\{ p'_1, x, z \} \subseteq N \cap S$. By an easy induction, $\text{conv}\{ p'_k, \ldots, p'_1, x, z \} \subseteq N \cap S$. Since $p' \in \text{conv}\{ p'_k, \ldots, p'_1 \}$, $[p', x] \subseteq S$. We conclude that $p'$ sees via $S$ each point of $N \cap S$, $p' \in A_z$, and Lemma 1 is established.

**Lemma 2.** Let $S$ be a closed set in $\mathbb{R}^d$. Let $P$ be a plane in $\mathbb{R}^d$, $B$ a component of $P \sim S$, with $S$ locally star shaped at $z \in \text{bdry}B$. Assume that line $L$ in plane $P$ supports $B$ locally at $z$ and that $B \cap M$ is in the open halfplane $L_1$ determined by $L$ for an appropriate neighborhood $M$ of $z$. Then $(\text{conv} A_z) \cap P \subseteq \text{cl} L_2$, where $L_2$ is the opposite open halfplane determined by $L$.

**Proof.** Suppose on the contrary that there is some point $p \in (\text{conv} A_z) \cap P \cap L_1$, to obtain a contradiction. Then $p \neq z$, so by Lemma 1 there exist point $p' \in [p, z)$ and convex neighborhood $N$ of $z$ such that $p'$ sees via $S$ each point of $N \cap S$. For convenience of notation, assume that $N \subseteq M \subseteq P$.

By a simple geometric argument, we may choose a point $b \in B \cap N$ such that $R(\ p', b)$ meets $N \cap L$ at some point $w$. Since $B \cap N \subseteq B \cap M \subseteq L_1$, $w \notin B$, so $(b, w)$ meets $\text{bdry}B$ at a point $c$. We have $c \in [b, w] \subseteq N$ and $c \in \text{bdry}B \subseteq S$, so $c \in N \cap S$. Therefore, by our choice of $p'$, $[p', c] \subseteq S$. Hence $b \in [p', c] \subseteq S$, impossible since $b \in B \subseteq P \sim S$. We have a contradiction, our supposition is false, and $(\text{conv} A_z) \cap P \subseteq \text{cl} L_2$. Thus Lemma 2 is proved.

**Lemma 3.** Let $S$ be a compact set in $\mathbb{R}^3$, and assume that $S$ is a finite union of polytopes. Let $P$ be a plane in $\mathbb{R}^3$, with $b$ a bounded component of $P \sim S$. For $z$ a point of local convexity of $\text{cl} B$, $z$ in edge $e \subseteq \text{rel} \text{bdry} \text{cl} B$, there exists a plane $H$ such that the following are true:

1. $H \cap P$ is a line containing $e$.
2. The two open halfspaces determined by $H$ can be denoted $H_1$ and $H_2$ in such a way that for $N$ any neighborhood of $z$ such that $(\text{cl} B) \cap N$ is convex, $B \cap N$ lies in $H_1$ while $A_z \subseteq \text{cl} H_2$. 


Proof. Notice that $S$ is locally starshaped at each of its points and that $\text{bdry} \, B$ is a closed polygonal curve in $P$. Let $J$ be a plane, $J \neq P$, such that $J$ contains edge $e$ of $\text{bdry} \, B$. If $N$ is any neighborhood of $z$ such that $(\text{cl} \, B) \cap N$ is convex, then $J$ supports $(\text{cl} \, B) \cap N$ at $e$, and $B \cap N$ necessarily lies in one of the open halfspaces $J_1$ determined by $J$. If $A_z \subseteq \text{cl} \, J_2$, then $J$ satisfies the lemma. Otherwise, $A_z \cap J_1 \neq \emptyset$.

For convenience of notation, let $P_1$ and $P_2$ denote distinct open halfspaces in $R^3$ determined by plane $P$, let $L = P \cap J$, and label the halfplanes in $P$ determined by $L$ so that $B \cap N \subseteq L_1 \equiv J_1 \cap P$. (See Figure 1.) Observe that $\text{conv} \, A_z$ is necessarily disjoint from one of $J_1 \cap P_1$ or $J_1 \cap P_2$, for otherwise $(\text{conv} \, A_z) \cap J_1 \cap P = (\text{conv} \, A_z) \cap L_1 \cap P \neq \emptyset$, contradicting Lemma 2. Thus we may assume that $(\text{conv} \, A_z) \cap J_1 \cap P_2 = \emptyset$, and since $(\text{conv} \, A_z) \cap L_1 = \emptyset$, $(\text{conv} \, A_z) \cap J_1 \subseteq P_1$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

Examine the points of $A_z \cap J_1 \subseteq P_1$. For $x \in A_z \cap J_1$, $x$ sees via $S$ a nondegenerate segment $s_z$ at $z$ contained in edge $e$, thus generating a planar set $T_x \equiv \text{conv}(s_x \cup \{x\})$. Since none of the $T_x$ sets lie in $P$, each determines with $\text{cl} \, L_1$ an angle of positive measure $m(x)$. Define $m \equiv \text{glb} \{m(x): x \in A_z \cap J_1\}$. Since $S$ is a finite union of polytopes, the $T_x$ sets lie in a finite union of polytopes, each meeting edge $e$ in a nondegenerate segment at $z$, each contained in $P_1 \cup L$. This forces $m$ to be positive. Using a standard argument, select sequence $\{x_t\}$ in $A_z \cap J_1$ so that $\{m(x_t)\}$ converges to $m$. Some subsequence of $\{x_t\}$ also converges, say to $x_0$. Moreover, the angle determined by $\text{conv}(e \cup \{x_0\})$ and $\text{cl} \, L_1$ has measure $m$, and $x_0 \in (\text{cl} \, A_z) \cap J_1 \subseteq P_1$. Let $H$ be the plane determined by $\text{conv}(e \cup \{x_0\})$. Of course $H \cap P = L$. Furthermore, for an
appropriate labeling of halfspaces determined by $H$, $L_1 \subseteq H_1$ so $B \cap N \subseteq H_1$.

It remains to show that $A_z \subseteq \text{cl} H_2$. Suppose on the contrary that $y \in A_z \cap H_1$. If $y \in P_1$, then the angle $m$ chosen above would not be minimal. If $y \in P$, then $y \in A_z \cap P \cap L_1$, contradicting Lemma 2. If $y \in P_2$, then since $y \in P_2 \cap H_1$ and $x_0 \in P_1 \cap H$, $[y, x_0]$ would meet $P \cap H_1 = L_1$. Moreover, since $x_0 \in \text{cl} A_z$, there would be a point $x'_0 \in A_z$ sufficiently near $x_0$ that $[y, x'_0]$ would meet $P \cap H_1 = L_1$ also, say at point $w$. Then $w \in (\text{conv} A_z) \cap P \cap L_1$, again contradicting Lemma 2. We conclude that $A_z \cap H_1 = \emptyset$, and $A_z \subseteq \text{cl} H_2$, finishing the proof of Lemma 3.

The final lemma follows immediately from [4, Theorem 1].

**Lemma 4 (Lawrence, Hare, Kenelly Lemma).** Let $S$ be a closed set in $R^d$. Assume that every finite set $F$ in $\text{bdry} S$ may be partitioned into two sets $F_1$ and $F_2$ such that each point of $F_i$ is clearly visible from a common point of $S$. Then $\text{bdry} S$ may be partitioned into two sets $S_1$ and $S_2$ such that for every finite set $F$ in $\text{bdry} S$, each point of $F \cap S_i$ is clearly visible from a common point of $S$, $i = 1, 2$.

We are ready to prove the following theorem.

**Theorem 1.** Let $S \subseteq R^3$. If $S$ satisfies property $P_2$, then $S$ is a union of two starshaped sets.

**Proof.** Using Lemma 4, select a partition $S_1$, $S_2$ for $\text{bdry} S$ such that for every finite set $F$ in $\text{bdry} S$, each point of $F \cap S_i$ is clearly visible via $S$ from a common point. For $i = 1, 2$, define $\mathcal{T}_i = \{\text{cl} A_z : z \in S_i\}$. Then each $\mathcal{T}_i$ is a collection of compact subsets of $S$. Moreover, by our choice of $S_1$ and $S_2$, each $\mathcal{T}_i$ has the finite intersection property. Hence $\bigcap\{T : T \text{ in } \mathcal{T}_1\} \neq \emptyset$, and we may select points $c$ and $d$ with $c \in \bigcap\{T : T \text{ in } \mathcal{T}_1\}$ and $d \in \bigcap\{T : T \text{ in } \mathcal{T}_2\}. \text{ Observe that for } z \in \text{bdry} S = S_1 \cup S_2, \text{ one of } c \text{ or } d, \text{ say } c, \text{ belongs to } \text{cl} A_z. \text{ Then } [c, z] \subseteq S. \text{ We conclude that each boundary point of } S \text{ sees via } S \text{ either } c \text{ or } d.$

We will show that each point of $S$ sees via $S$ either $c$ or $d$. Portions of the argument will resemble the proof of [1, Theorem 1]. Let $x \in S$ and suppose on the contrary that neither $c$ nor $d$ sees $x$, to reach a contradiction. Certainly $x \not\in \{c, d\}$, and by a previous observation. $x \in \text{int} S$. As in [1, Theorem 1], choose the segment at $x$ in $S \cap L(c, x)$ having maximal length, and let $p$ and $q$ denote its endpoints, with the order of
the points $c < p < x < q$. Then $p, q \in \text{bdry} S$, neither is seen by $c$, so $d$ sees via $S$ both $p$ and $q$. Notice that $d \notin L(c, x)$ since $d$ cannot see $x$. Similarly, choose a segment at $x$ in $S \cap L(d, x)$ having maximal length, and let $r$ and $s$ denote its endpoints, $d < r < x < s$. Then point $c$ sees via $S$ both $r$ and $s$. (See Figure 2.)

Since points $c, d, x$ are not collinear, they determine a plane $P$ in $R^3$. In the next part of our proof, we restrict our attention to $P$. Since $[d, x] \not\subseteq S$, there is a segment in $(d, r) \sim S$, and this segment lies in a bounded component $K$ of $P \sim S$, $K \subseteq \text{rel int} \text{conv} \{d, p, q\}$. Likewise, there is a segment in $(c, p) \sim S$ belonging to a bounded component $J$ of $P \sim S$, $J \subseteq \text{rel int} \text{conv} \{c, s, r\}$. Letting $L(c, r) \cap L(d, p) = \{v\}$, it is not hard to show that $J$ and $K$ lie in opposite open halfplanes of $P$ determined by $L(v, x)$.

For future reference, observe that for any line $U$ from $c$ meeting $K$, $d \notin U$, $d$ cannot see via $S$ all points of $\text{bdry} K$ on the opposite side of $U$ from $d$, so $c$ sees via $S$ some of these points. Thus if line $U'$ from $c$ supports $\text{conv} K$, by a convergence argument, $c$ sees via $S$ some point of $U' \cap (\text{bdry} K)$. We will use this observation in the next part of the proof.
Define line $L'$ and associated point $t$ as follows: Clearly $L(c,v) \cap J = \emptyset$. In case $L(c,v) \cap K \neq \emptyset$, let $L_1$ denote the open halfplane of $P$ determined by $L(c,v)$ and containing $J$. Let $L'$ be the line from $c$ supporting $\text{conv}K$ at a point of $L_1$. Using our previous observation, $L' \cap (\text{bdry conv}K)$ contains some point $t$ of $\text{bdry}K$ such that $[c,t] \subseteq S$. In case $L(c,v) \cap K = \emptyset$, rotate $L(c,v)$ about $c$ toward $d$ until $\text{bdry}K$ is met. Let $L'$ be the corresponding rotated line. Again using our observation, there is some $t \in L' \cap (\text{bdry conv}K) \cap (\text{bdry}K)$ with $[c,t] \subseteq S$. Of course, in each case $t$ may be chosen to be the furthest point from $c$ having the required property. Moreover, $[c,t] \cap J = \emptyset$, and we may label the open halfplanes of $P$ determined by $L'$ so that $J \subseteq L'$. Then $K \cup \{d\}$ lies in the opposite halfplane $L'_2$.

Since $S$ is a finite union of polytopes, $\text{bdry}K$ is necessarily a simple closed polygonal curve in plane $P$. By our choice of $t$, clearly $t$ is a point of local convexity of $\text{cl}K$. Also, $t$ must be a vertex of $\text{bdry}K$, so $\text{bdry}K$ contains two edges $e_1$ and $e_2$ at $t$. Moreover, for an appropriate labeling of these edges, $e_1 \subseteq \text{cl} L'_2$, $e_2 \subseteq L'_2 \cup \{t\}$, and for any neighborhood $N$ of $t$ with $(\text{cl}K) \cap N$ convex, $K \cap N$ and $c$ lie in the same open halfplane of $P$ determined by $L(e_2)$.

Using Lemma 3, select a plane $H$ such that $H \cap P$ is a line containing $e_2$, $K \cap N \subseteq H_1$, and $A_t \subseteq \text{cl} H_2$. Similarly, select plane $M$ for $e_1$ so that $K \cap N \subseteq M_1$ and $A_t \subseteq \text{cl} M_2$. Recall that by our choice of $c$ and $d$, at least one of these points lies in $\text{cl} A_t \subseteq \text{cl} H_2 \cap \text{cl} M_2$. Since $c$ and $K \cap N$ are in the same open halfplane of $P$ determined by $L(e_2)$, $c \in H_1$. This forces $d$ to belong to $\text{cl} H_2 \cap \text{cl} M_2 \cap P$. However, clearly $\text{cl} H_2 \cap \text{cl} M_2 \cap P \subseteq \text{cl} L'_1$, while $d \in L'_2$. We have a contradiction, our supposition is false, and every point of $S$ must see via $S$ either $c$ or $d$. Hence $S$ is a union of two starshaped sets, and Theorem 1 is established.

**Theorem 2.** For $k \geq 1$ and $d \geq 1$, let $\mathcal{F}(k,d)$ denote the family of all compact unions of $k$ (or fewer) starshaped sets in $\mathbb{R}^d$, $\mathcal{C}(k,d)$ the subfamily of $\mathcal{F}(k,d)$ whose members are finite unions of $d$-polytopes. Then $\mathcal{C}(k,d)$ is dense in $\mathcal{F}(k,d)$, relative to the Hausdorff metric. Moreover, $\mathcal{F}(k,d)$ is closed, relative to the Hausdorff metric.

**Proof.** In the proof, $h$ will denote the Hausdorff metric on compact subsets of $\mathbb{R}^d$. That is, if $(A)_\delta = \{x: \text{dist}(x,A) < \delta\}$, then for $A$ and $B$ compact in $\mathbb{R}^d$, $h(A,B) = \inf\{\delta: A \subseteq (B)_\delta \text{ and } B \subseteq (A)_\delta, \delta > 0\}$.

To see that $\mathcal{C}(k,d)$ is dense in $\mathcal{F}(k,d)$, let $S \in \mathcal{F}(k,d)$. For an arbitrary $\delta > 0$, we must find some $C$ in $\mathcal{C}(k,d)$ for which $h(S,C) < \delta$. Assume that each point of $S$ is visible via $S$ from one of $s_1, \ldots, s_k$. Form
an open cover for $S$, using interiors of $d$-simplices whose diameters are at most $\delta/2$. Using the compactness of $S$, reduce to a finite subcover, say $\{\text{int} P_j : 1 \leq j \leq m\}$, where $P_j$ is a $d$-simplex. For $1 \leq i \leq k$, define $C_i = \bigcup \{\text{conv}(s_i \cup P_j) : s_i \text{ sees via } S \text{ some point of } P_j, 1 \leq j \leq m\}$. Certainly set $C = C_1 \cup \cdots \cup C_k$ is a union of $k$ starshaped sets as well as a finite union of $d$-polytopes. Thus $C \in \mathcal{C}(k, d)$.

Clearly $S \subseteq C$, so $S \subseteq (C)_\delta$. To see that $C \subseteq (S)_\delta$, let $x \in C \sim S$. Then $x \in \text{conv}(s_i \cup P_j)$ for some $i$ and $j$. Moreover, for an appropriate $i$ and $j$, there is some $y' \in P_j \cap S$ with $[s_i, y'] \subseteq S$. If $x, s_i, y'$ are collinear, then since $x \not\in S$, $x$ must belong to $P_j$, and $\text{dist}(x, y') \leq \delta/2$. Thus $x \in (S)_\delta$. If $x, s_i, y$ are not collinear, assume $x \in [s_i, y]$ where $y \in P_j$, and let $x'$ be the point of $[s_i, y']$ such that $[x, x']$ and $[y, y']$ are parallel. Then $x' \in S$ and $\text{dist}(x, x') \leq \text{dist}(y, y') \leq \delta/2$. Again $x \in (S)_\delta$.

We conclude that $C \subseteq (S)_\delta$, $h(S, C) < \delta$, and $\mathcal{C}(k, d)$ is indeed dense in $\mathcal{F}(k, d)$.

Finally, to see that $\mathcal{F}(k, d)$ is closed, let $\{S_i\}$ be a sequence in $\mathcal{F}(k, d)$ converging to the compact set $S_0$. To show that $S_0 \in \mathcal{F}(k, d)$ also. For convenience of notation, for $i \geq 1$, let $S_i$ be a union of $k$ starshaped sets whose compact kernels are $A_{i1}, A_{i2}, \ldots, A_{ik}$, respectively. Then by standard results concerning the Hausdorff metric [6], $\{A_{i1} : i \geq 1\}$ has a subsequence $\{A'_{i1}\}$ converging to some compact convex set $A_1$. Pass to the associated subsequence $\{S'_i\}$ of $\{S_i\}$, and repeat the argument for corresponding kernels $\{A'_{i2}\}$. By an obvious induction, in $k$ steps we obtain subsequences $\{A'_{i1}^{(k)}\}, \{A'_{i2}^{(k)}\}, \ldots, \{A'_{ik}^{(k)}\}$ converging to compact convex sets $A_1, \ldots, A_k$, respectively. It is a routine matter to show that $S_0$ is a union of $k$ or fewer compact starshaped sets having kernels $A_1, \ldots, A_k$.

**Theorem 3.** Let $S$ be a compact union of $k$ starshaped sets in $R^d$, $k \geq 1$, $d \geq 3$. Then there is a sequence $\{S_j\}$ converging to $S$ (relative to the Hausdorff metric) such that each $S_j$ satisfies property $P_k$. That is, using the notation of Theorem 2, sets having property $P_k$ are dense in $\mathcal{F}(k, d)$.

**Proof.** As in the proof of Theorem 2, $h$ will denote the Hausdorff metric on compact subsets of $R^d$. For any $\delta > 0$, we must find some $C$ having property $P_k$ for which $h(S, C) < \delta$.

Assume that each point of $S$ is visible via $S$ from one of the distinct points $s_1, \ldots, s_k$. Form an open cover for $S$ using spheres of radius $\delta/4$, centered at points of $S$. Reduce to a finite subcover, and choose the center of each sphere. Say these centers are the points $t_1, \ldots, t_m$. Partition
\{t_1, \ldots, t_m\} \text{ into } k \text{ subsets } V_1, \ldots, V_k \text{ such that the following is true: If } t \in V_i \text{, then } s_i \text{ is a point of } \{s_1, \ldots, s_k\} \text{ closest to } t \text{ with } [s_j, t] \subseteq S. \text{ Define } T_i = \bigcup\{[s_j, t]: t \in V_i\}. \text{ Observe that } s_i \notin T_j \text{ for } i \neq j; \text{ Otherwise, } s_i \in (s_j, t] \text{ for some } t \in V_j, [s_i, t] \subseteq (s_j, t] \subseteq S, \text{ and } s_i \text{ would be closer to } t \text{ than } s_j \text{ is to } t, \text{ impossible by the definition of } V_j.

In case the sets } T_1, \ldots, T_k \text{ are pairwise disjoint, let } T_i' = T_i, 1 \leq i \leq k, \text{ and define } T \text{ to be their union. Otherwise, suppose } T_i \text{ meets } T_2 \cup \cdots \cup T_k. \text{ Then for some point in } V_1, \text{ call it } t_1 \text{ (for convenience of notation), } (s_1, t_1] \text{ meets } T_2 \cup \cdots \cup T_k. \text{ Using the facts that each } T_i \text{ set is a finite union of edges at } s_i, s_1 \notin T_2 \cup \cdots \cup T_k, \text{ and } d \geq 3, \text{ it is not hard to show that there exists an edge } [s_1, t_1'] \text{ not collinear with } [s_1, t_1] \text{ such that } [s_1, t_1'] \text{ is disjoint from } T_2 \cup \cdots \cup T_k \text{ and } \text{dist}(t_1, t_1') < \delta/4. \text{ Thus } h([s_1, t_1], [s_1, t_1']) < \delta/4, \text{ also. Repeating the procedure for each edge of } T_1, \text{ in finitely many steps we obtain a new set } T_i' \text{ starshaped at } s_1 \text{ such that } T_i' \text{ is disjoint from } T_2 \cup \cdots \cup T_k \text{ and } h(T_1, T_1') < \delta/4.

Continuing the process for } T_2, \ldots, T_k, \text{ by an obvious induction we obtain pairwise disjoint starshaped sets } T_1', T_2', \ldots, T_k' \text{ with } h(T_i, T_i') < \delta/4, 1 \leq i \leq k. \text{ Define } T = T_1' \cup \cdots \cup T_k'. \text{ Standard arguments reveal that }

\begin{align*}
h(S, T_1 \cup \cdots \cup T_k) &< \frac{\delta}{4}, \\
h(T_1 \cup \cdots \cup T_k, T) &< \frac{\delta}{4},
\end{align*}

and hence } h(S, T) < \delta/2.

Finally, we extend the sets } T_1', \ldots, T_k' \text{ to finite unions of } d\text{-polytopes, define } m = \min\{h(T_i', T_j'): i \neq j\}. \text{ Using techniques from Theorem 2, select set } C \equiv C_1 \cup \cdots \cup C_k \text{ in } \mathcal{B}(k, d) \text{ with } h(T_i, C_i) < \min\{\delta/2, m/2\} \text{ and with } s_i \in \ker C_i, 1 \leq i \leq k. \text{ Since } h(T_i, C_i) < m/2, \text{ certainly the } C_i \text{ sets must be pairwise disjoint. Therefore, each boundary point of } C \text{ is clearly visible from some } s_i, 1 \leq i \leq k, \text{ and } C \text{ has property } P_k. \text{ Moreover, }

\begin{align*}
h(S, C) &\leq h(S, T) + h(T, C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\end{align*}

Theorem 3 is established.

It is interesting to observe that while Theorem 3 holds when } d \geq 3, \text{ it fails in the plane, as the following easy example reveals.

EXAMPLE 1. Let } S \text{ be the set in Figure 3. Then } S \text{ is a union of two starshaped sets with kernels } \{c\}, \{d\}, \text{ respectively. However, sets sufficiently close to } S \text{ fail to satisfy the clear visibility condition required for property } P_2.
Finally, the characterization theorem for unions of two starshaped sets in $R^3$ is an easy consequence of our previous results.

**Corollary 1.** Let $S \subseteq R^3$. Then $S$ is a compact union of two starshaped sets if and only if there is a sequence $\{S_j\}$ converging to $S$ (relative to the Hausdorff metric) such that each set $S_j$ satisfies property $P_2$.

*Proof.* The necessity follows immediately from Theorem 3. For the sufficiency, Theorem 1 implies that each set $S_j$ is a compact union of two starshaped sets in $R^3$. By Theorem 2, their limit $S$ is a compact union of two starshaped sets as well.

**References**


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**University of Oklahoma**

**Norman, OK 73019**
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