EIGENVALUE ESTIMATES WITH APPLICATIONS TO MINIMAL SURFACES

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We study eigenvalue estimates of branched Riemannian coverings of compact manifolds. We prove that if

\[ \varphi : M^n \to N^n \]

is a branched Riemannian covering, and \( \{ \mu_i \}_{i=0}^{\infty} \) and \( \{ \lambda_i \}_{i=0}^{\infty} \) are the eigenvalues of the Laplace-Beltrami operator on \( M \) and \( N \), respectively, then

\[ \sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}, \]

for all positive \( t \), where \( k \) is the number of sheets of the covering. As one application of this estimate we show that the index of a minimal oriented surface in \( \mathbb{R}^3 \) is bounded by a constant multiple of the total curvature. Another consequence of our estimate is that the index of a closed oriented minimal surface in a flat three-dimensional torus is bounded by a constant multiple of the degree of the Gauss map.

1. Introduction. Motivated by problems in the theory of minimal surfaces, we study the following question. Let

\[ \varphi : M^n \to N^n \]

be a branched Riemannian covering of compact manifolds, which has a singular set of codimension at least two. By this we mean that we endow \( M^n \) with the pullback metric

\[ \varphi^* (ds_N), \]

where \( \varphi^* \) is singular on a set of codimension at least two. We then want to estimate the eigenvalues of the Laplace-Beltrami operator on \( M \) in terms of the corresponding eigenvalues of \( N \). Note that we can speak of the eigenvalues of \( (M, \varphi^*(ds_N)) \) although the metric is possibly singular, since a singular set of codimension at least two will not affect the integrals of the variational characterization of the eigenvalues.

Our main theorem gives the estimate

\[ \sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}, \]

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for all \( t > 0 \), where \( k \) is the number of sheets of the covering \( \varphi \), and \( \{ \mu_i \} \) and \( \{ \lambda_i \} \) are the eigenvalues of the Laplace-Beltrami operator on \( M \) and \( N \), respectively. We then use this estimate to show that if \( M^2 \subseteq \mathbb{R}^3 \) is an oriented complete minimal surface of finite total curvature, then the index of \( M \) is bounded by a constant multiple of the total curvature. Here, the index of \( M \) is defined to be the limit of the indices of an increasing sequence of exhausting compact domains in \( M \). The index of a domain \( D \) is the number of negative eigenvalues of the eigenvalue problem

\[
(\Delta + |A|^2) \varphi + \lambda \varphi = 0 \quad \text{on } D, \quad \varphi|_{\partial D} = 0,
\]

where \( A \) is the second fundamental form of \( M \) as a submanifold of \( \mathbb{R}^3 \). Geometrically, the index of \( M \) can be described as the maximum dimension of a linear space of compactly supported deformations that decrease the area up to second order. Finally we also show that the index of a closed oriented minimal surface in a flat three-dimensional torus is bounded by a constant multiple of the degree of the Gauss map.

2. The eigenvalue estimate. Our main result is the following theorem.

**Theorem.** Let

\[
\varphi : M^n \to N^n
\]

be a \( k \)-sheeted branched Riemannian covering of compact manifolds, which has a singular set of codimension at least two. Let \( \{ \mu_i \}_{i=0}^{\infty} \) and \( \{ \lambda_i \}_{i=0}^{\infty} \) be the eigenvalues of the Laplace-Beltrami operator on \( M^n \) and \( N^n \), respectively. Then for all \( t > 0 \),

\[
\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}.
\]

**Remark.** Before proving the theorem we note that the main difficulty is that the fundamental comparison theorems of Cheng [1] do not carry through if the metric has singularities. We instead utilize the heat kernel on \( M \) and \( N \) to circumvent this difficulty.

**Proof.** We restrict \( \varphi \) of the theorem to \( \varphi_- : \)

\[
\varphi_- : M_- \to N_-,
\]

where

\[
M_- = M - E(\varepsilon),
\]
and $E(\epsilon)$ is an open set of volume less than $\epsilon$ with smooth boundary, containing the singular set. We then simply define $N_-$ to be the image under $\varphi$ restricted to $M_-$. 

Now fixing $x \in M_-$, we consider 

$$H : y \mapsto H_{N_-}(\varphi(x), \varphi(y), t), \quad y \in M_-, \ t > 0,$$

where $H_{N_-}$ is the heat kernel on $N_-$, with Dirichlet boundary conditions. Since $\varphi_-$ is the local isometry, the function $H$ solves the heat equation on $M_-$. As $t$ tends to zero we obtain 

$$H_{N_-}(\varphi(x), \varphi(y), t) \to \sum_{\varphi(x_i) = \varphi(x)} \delta_{x_i}.$$ 

On the other hand, for the heat kernel $H_{M_-}$ on $M_-$ with Dirichlet boundary conditions, we have as $t$ tends to zero 

$$H_{M_-}(x, y, t) \to \delta_x.$$ 

Hence, at $t = 0$ we have in the sense of distributions 

$$H_{M_-}(x, y, 0) \leq H_{N_-}(\varphi(x), \varphi(y), 0).$$ 

By the maximum principle for the heat equation, we then have 

$$(1) \quad H_{M_-}(x, y, t) \leq H_{N_-}(\varphi(x), \varphi(y), t),$$ 

for all $t > 0$. Inequality (1) holds for all $x$ and $y$ in $M_-$ so we can let $x = y$ and integrate over $M_-$: 

$$\int_{M_-} H_{M_-}(x, x, t) \, dV(x) \leq \int_{M_-} H_{N_-}(\varphi(x), \varphi(x), t) \, dV(x).$$ 

Since $\varphi_-$ is a $k$-sheeted covering, we have 

$$\int_{M_-} H_{N_-}(\varphi(x), \varphi(x), t) \, dV(x) = k \int_{N_-} H_{N_-}(z, z, t) \, dV(z).$$ 

Again using the maximum principle for the heat equation, we obtain 

$$\int_{N_-} H_{N_-}(z, z, t) \, dV(z) \leq \int_{N} H_{N}(z, z, t) \, dV(z),$$ 

where $H_{N}$ denotes the heat kernel of $N$. We have therefore shown that 

$$\int_{M_-} H_{M_-}(x, x, t) \, dV(x) \leq k \int_{N} H_{N}(z, z, t) \, dV(z).$$ 

Finally, letting the volume $\epsilon$ of $E(\epsilon)$ tend to zero, we obtain 

$$(2) \quad \int_{M} H_{M}(x, x, t) \, dV(x) \leq k \int_{N} H_{N}(z, z, t) \, dV(z),$$
where $H_M$ is the heat kernel of $M$. Using separation of variables, one shows that the heat kernels $H_M$ and $H_N$ have the representations

$$H_M(x, y, t) = \sum_{i=0}^{\infty} e^{-\mu_i t} \psi_i(x) \psi_i(y)$$

$$H_N(x, y, t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y),$$

where

$$\Delta \psi_i + \mu_i \psi_i = 0, \quad i = 0, 1, 2, \ldots,$$

and

$$\Delta \varphi_i + \lambda_i \varphi_i = 0, \quad i = 0, 1, 2, \ldots,$$

are the eigenvalues and eigenfunctions of $M$ and $N$, respectively, normalized so that $\{\psi_i\}_{i=0}^{\infty}$ and $\{\varphi_i\}_{i=0}^{\infty}$ form orthonormal systems. Using these representations in inequality (2), we obtain

$$\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t},$$

finishing the proof of the theorem.

### 3. Applications to minimal surfaces.

In [2], D. Fisher-Colbrie shows that a complete minimal oriented surface $M$ in $\mathbb{R}^3$ has finite index if and only if it has finite total curvature (see the introduction for the definition of index). A natural question to ask then is how the index varies with the total curvature. Using our eigenvalue estimate, we can show that the index is bounded by a constant multiple of the total curvature.

**Theorem.** Let $M^2$ be a complete oriented minimal surface in $\mathbb{R}^3$. Set

$$k = \frac{1}{4\pi} \int_M (-K) \, dV,$$

where $K$ is the Gaussian curvature of $M$. Then

$$\text{index of } M \leq (7.68183) \cdot k.$$

**Proof.** Without loss of generality, we can assume that $k$ is finite. By Osserman’s classical theorem, we then know that $M$ is conformally a compact Riemann surface $\tilde{M}$, punctured at a finite set of points. Also, the Gauss map extends to a conformal map

$$G : \tilde{M} \to S^2.$$
For a minimal surface in $\mathbb{R}^3$, $|A|^2 = -2K$. Now, the number of negative eigenvalues for
\[
\Delta + |A|^2 = \Delta - 2K,
\]
on any domain $D$ in $M$, is the same as the number of negative eigenvalues of the corresponding domain in $\overline{M}$ for the operator
\[
\Delta_M + 2,
\]
where we use the pullback metric from $S^2$ on $\overline{M}$. This follows from the fact that
\[
G^* (ds^2_{S^2}) = (-K) \cdot ds^2_M,
\]
and $\Delta_M = (-K)\Delta_{\overline{M}}$. Since the index of $M$ is the limit of the indices of an exhausting sequence of domains $D$ in $M$, we can conclude, by the domain monotonicity of eigenvalues, that the index of $M$ is bounded by the number of negative eigenvalues of $\Delta_{\overline{M}} + 2$ on $\overline{M}$, or equivalently, by the number of eigenvalues of $\Delta_{\overline{M}}$ that are strictly less than two.

Now, $G$ is a holomorphic mapping so it establishes $\overline{M}$ as a $k$-sheeted branched cover of $S^2$. The singular set of this covering is the set of isolated points where $K = 0$. We can therefore apply our eigenvalue estimate and conclude that
\[
\sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}, \quad \text{all } t > 0,
\]
where $\{\mu_i\}_{i=0}^{\infty}$ and $\{\lambda_i\}_{i=0}^{\infty}$ are the eigenvalues of $\overline{M}$ and $S^2$, respectively. Since the index of $M$ is bounded by the number of $\mu_i$'s that are strictly less than two, we conclude that
\[
(\text{index of } M) \cdot e^{-2t} \leq \sum_{\mu_i < 2} e^{-\mu_i t} \leq \sum_{i=0}^{\infty} e^{-\mu_i t} \leq k \sum_{i=0}^{\infty} e^{-\lambda_i t}.
\]
Hence
\[
\text{index of } M \leq \left( e^{2t} \sum_{i=0}^{\infty} e^{-\lambda_i t} \right) \cdot k.
\]
The $i$th distinct eigenvalue of $S^2$ is known to be $i(i + 1)$, with multiplicity $2i + 1$. Using this, we find that $t = 0.4506 \ldots$ gives the smallest possible value of $7.68182 \ldots$ for the coefficient of $k$, proving the theorem.

As another application of our eigenvalue estimate, we consider the case of minimal surfaces in a flat three-dimensional torus. Let $N$ be such a torus, which we know we can write isometrically as
\[
N = \mathbb{R}^3 / \Lambda,
\]
where $\Lambda$ is a cocompact lattice, and let $M$ be a closed minimal oriented surface immersed in $N$. We can define the Gauss map

$$G : M \to S^2$$

by viewing $M$ as a minimal surface in $\mathbb{R}^3$, periodic with respect to the lattice $\Lambda$.

The index of $M$ is the number of negative eigenvalues of

$$\Delta + |A|^2 = \Delta - 2K$$

on $M$, where $A$ denotes the second fundamental form of $M$ in $N$, and $K$ denotes the Gaussian curvature of $M$. We endow $M$ with the pullback metric from $S^2$ via $G$ and conclude, using the same argument as in the preceding example, that

$$\text{index of } M \leq (7.68183) \cdot k,$$

where $k$ is the degree of the Gauss map.

References


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