PSEUDOGROUPS OF $C^1$ PIECEWISE PROJECTIVE HOMEOMORPHISMS

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The group $\text{PSL}_2\mathbb{R}$ acts transitively on the circle $S^1 = \mathbb{R} \cup \infty$, by linear fractional transformations. A homeomorphism $g: U \to V$ between open subsets of $\mathbb{R}$ is called $C^1$, piecewise projective if $g$ is $C^1$, and if there is some locally finite subset $S$ of $U$ such that, on each component of $U - S$, $g$ agrees with some element of $\text{PSL}_2\mathbb{R}$. Let $\Gamma_\mathbb{R}$ be the pseudogroup of such homeomorphisms. We show that the Haefliger classifying space $B\Gamma_\mathbb{R}$ is simply connected, and that there is a homology isomorphism $i: \#\text{PSL}_2\mathbb{R} \to B\Gamma_\mathbb{R}$. ($\text{PSL}_2\mathbb{R}$ is the universal cover of $\text{PSL}_2\mathbb{R}$, considered as a discrete group.) As a consequence, the classifying space of the discrete group of compactly supported, $C^1$ piecewise projective homeomorphisms of $\mathbb{R}$ is a “homology loop space” of $B\text{PSL}_2\mathbb{R}$.

1.1. Introduction. More generally, let $F \subset \mathbb{R}$ be a subfield of $\mathbb{R}$. $\text{PSL}_2F$ acts on the circle $\mathbb{R} \cup \infty$. The orbit of $1 \in F$ is $F \cup \infty$.

1.2. Definition. $\Gamma_F$ is the pseudogroup of $C^1$ homeomorphisms $g: U \to V$ between open subsets of $\mathbb{R}$, so that there is some locally finite subset $S$ of $U \cap (F \cup \infty)$ such that, on each connected component of $U - S$, $g$ agrees with some element of $\text{PSL}_2F$.

The set of restrictions of elements of $\text{PSL}_2F$ to open subsets of $\mathbb{R}$ forms a subpseudogroup of $\Gamma_F$ whose classifying space, the total space of the circle bundle over $B\text{PSL}_2F$, is homotopy equivalent to $B\text{PSL}_2F$, where $\text{PSL}_2F$ is defined as the pullback

$\text{PSL}_2F \to \text{PSL}_2\mathbb{R}
\downarrow \quad \downarrow
\text{PSL}_2F \to \text{PSL}_2\mathbb{R}$

Therefore, there is an inclusion map $i: B\text{PSL}_2F \to B\Gamma_F$.

1.3. Theorem. $\pi_1B\Gamma_F = 0$, and $i$ is a homology equivalence.

1.4. Definition. The group of compactly supported $\Gamma_F$ homeomorphisms, denoted $K_F$, is the group of elements of $\Gamma_F$ which are compactly supported homeomorphisms of the line $\mathbb{R}$. 
Following Segal's proof [Se2] of an extension of Mather's theorem [Ma] we find:

1.5. Proposition. There is a homology equivalence $BK_F \to \Omega B\Gamma_F$.

The proof of 1.5 involves the construction of a homology fibration [McS] $BK_F \to M \to B\Gamma_F$ where $M$ is contractible. Pulling this fibration back over $B\text{PSL}_2 F$ by the inclusion $i$ of 1.3 we obtain:

1.6. Corollary. There is a homology fibration $BK_F \to E \to B\text{PSL}_2 F$ where $E$ is acyclic, and the fundamental group of $B\text{PSL}_2 F$ acts trivially on the homology of the fiber.

1.7. Organization. In §2 Theorem 1.3 is proved, as an application of Corollary 1.10 of [G2]. In §3, 1.5 is proved, using a generalization of Segal's proof [Se2] of a generalization of Mather's theorem [Ma]. The generalization is outlined in §4.

2. Proof of 1.3. One may think of $\Gamma_F$ as constructed from the action of $\text{PSL}_2 F$ on $S^1$ by adding $C^1$ singularities at isolated points of $F$. As a consequence, 1.10 of [G2] says that $B\Gamma_F$ is weakly homotopy equivalent to the direct limit of the diagram

\[
\begin{array}{c}
BA \xrightarrow{j} BG^P \xrightarrow{l} BA \\
\downarrow \quad \quad \downarrow r \\
B\text{PSL}_2 F \\
\end{array}
\]

where $A$ is the discrete group of germs of projective maps fixing 0, and $G^P$ is the discrete group of germs of $\Gamma_F$ maps fixing 0. The map $j$ is inclusion, and $l$ and $r$ arise from the fact that an element of $G^P$, restricted to the left or right side of 0, can be identified with an element of $A$. Theorem 1.3 will follow from an analysis of diagram (2.1).

Let $F^+$ be the positive, nonzero squares of $F$, considered as a group under multiplication. It is well known that $A$ is a subgroup of the one-dimensional affine group of $F$, an extension $F \to A \xrightarrow{d} F^+$ where $F^+$ acts on $F$ by multiplication. Since $d: A \to F^+$ is the derivative map, $G^P$ is the pullback

\[
\begin{array}{c}
G^P \xrightarrow{l} A \\
r \downarrow \quad \quad \downarrow d \\
A \xrightarrow{d} F^+ \\
\end{array}
\]
and therefore $G^p$ is an extension $F^2 \to G^p \to F^+$, with $F^+$ acting on $F^2$ by multiplication: $f(a, b) = (fa, fb)$.

Let $R$ be the pushout of

$$
\begin{array}{ccc}
BG^p & \to & BA \\
\downarrow r & & \downarrow \\
BA & \to & BF^+
\end{array}
$$

2.2. **Lemma.** The inclusion $j: BA \to BG^p$ induces a homology equivalence $BA \to R$.

Assuming 2.2 for now, we prove 1.3. By 2.2 and 2.1 it is clear that $B\text{PSL}_2 F \to B\Gamma_F$ is a homology equivalence. It remains to show that $\pi_1 B\Gamma_F = 0$.

We first compute $\pi_1 R$. By Van Kampen's theorem, $\pi_1 R = A \times_{G^p} A$. Elements in either $A$ factor with derivative 1 are equal to 1 in $\pi_1 R$. On the other hand, $\pi_1 R \to F^+$. It follows that $\pi_1 R$ is isomorphic to $F^+$.

Now by (2.1), $\pi_1 B\Gamma_F \simeq \text{PSL}_2 F \times_A F^+$, which is isomorphic to $\text{PSL}_2 F$ modulo the normal subgroup $N(F)$ generated by the subgroup $F$ of $\text{PSL}_2 F$. We now show that $N(F)$ is all of $\text{PSL}_2 F$.

Consider $\text{PSL}_2 F$ acting on $S^1 = \mathbb{R} / \mathbb{Z}$, and $\text{PSL}_2 F$ as acting on $\mathbb{R}$, so that $A$ is the subgroup of $\text{PSL}_2 F$ fixing each integer. Since [La] $\text{PSL}_2 F$ is simple, to show that $N(F) = \text{PSL}_2 F$, it suffices to prove that $N(F)$ contains the translation $t: x \mapsto x + 1$.

In fact, $N(\mathbb{Z})$ contains $t$. For $\text{PSL}_2 F$ contains $\text{PSL}_2 \mathbb{Z}$ as a subgroup, which contains $t$. Further, $\text{PSL}_2 \mathbb{Z}$ is generated by $a$, $b$ with $a^2 = b^3$, and $\mathbb{Z}$ is generated by $a^{-1} b$. Now $a(a^{-1} b) a^{-1} = ba^{-1}$, and $(ba^{-1})(a^{-1} b) = b$, so $N(\mathbb{Z}) \supset \text{PSL}_2 \mathbb{Z}$, and contains $t$.

**Proof of Lemma 2.2.** In fact, we show that the derivative maps $A \to F^+$, $G^p \to F^+$ induce isomorphisms on homology (and, therefore, because $\pi_1 R = F^+$, that

$$
\begin{array}{ccc}
BG^p & \to & BA \\
\downarrow r & & \downarrow \\
BA & \to & BF^+
\end{array}
$$

is both a pullback and a pushout). Considering the Serre spectral sequences of the extensions $F \to A \to F^+$ and $F^2 \to G^p \to F^+$, it suffices to prove that the groups $H_p(F^+; H_q F^2)$, $H_p(F^+; H_q F)$ are null for $q > 0$. The proof is essentially that of the “center kills” lemma [Sa].
The element \(4 \in F^+\) acts on \(H_q F\) and \(H_q F^2\) by multiplication by \(4^q\). Let this isomorphism (\(H_q F\) and \(H_q F^2\) are divisible and torsion free) be denoted \(e_q\). Then \(e_q - 1\) is also an isomorphism of \(H_q F\) and \(H_q F^2\), namely multiplication by \(4^q - 1\). Both \(e_q\) and \(e_q - 1\) induce the identity maps of \(H_p(F^+; H_q F), H_p(F^+; H_q F)\). Thus the latter groups must be zero.

3. Proof of 1.5. In §4 we outline a proof of the following fact:

4.8. Proposition. Let \(\Gamma\) be a pseudogroup of orientation preserving homeomorphisms of \(\mathbb{R}\). Let \(K\) be the discrete group of elements of \(\Gamma\) which are compactly supported homeomorphisms of \(\mathbb{R}\). Assume that the orbit of any element of \(\mathbb{R}\) under \(\Gamma\) is dense in \(\mathbb{R}\). Further, assume:

\(3.1\) Suppose \(g\) is the germ of an element of \(\Gamma\) with domain \(x \in \mathbb{R}\), and let \(t \in \mathbb{R}\) such that \(t > x\), \(gx\) (or \(t < x, gx\)). Then there is an element \(\bar{g} \in \Gamma\) whose domain is connected and includes \(t\) and \(x\), and such that \(\bar{g} \equiv \text{id} near t, \ \bar{g} \equiv g near x\).

Then there is a homology equivalence \(BK \to \Omega B\Gamma\).

To prove 1.5, therefore, we must verify condition 3.1 for the pseudogroups \(\Gamma_F\). We rephrase 3.1 as the following lemma, using the fact that \(F\) is dense in \(\mathbb{R}\).

3.2. Lemma. Let \(g \in \text{PSL}_2 F, x \in F\), and assume that \(g(x) \neq \infty\).

\(a\) Let \(z = \max(x, gx)\). Let \(\epsilon > 0\). Then there is some \(\epsilon'\), \(0 < \epsilon' < \epsilon\), \(\delta > 0\), and \(s \in \Gamma_F\) with domain \((x - 2\epsilon', \infty)\) such that \(s(t) = gt, t \leq x + \delta\), and \(s(t) = t, t \geq z + \epsilon'\).

\(b\) Let \(z = \min(x, gx)\). Let \(\epsilon > 0\). Then there is some \(\epsilon'\), \(0 < \epsilon' < \epsilon\), \(\delta > 0\), and an \(s \in \Gamma_F\) with domain \((-\infty, x + 2\epsilon')\) such that \(s(t) = gt, t \geq x - \delta, s(t) = t, t \leq z - \epsilon'\).

For the proof we first recall some facts about \(\text{PSL}_2 F\). A circle in the upper half plane which is tangent to the \(x\)-axis is called a horocycle. The action of \(\text{PSL}_2 F\) on \(\mathbb{R} \cup \infty\) extends to an action on the upper half plane which takes horocycles to horocycles. Let \(f \in F\). The subgroup \(T_f \subset \text{PSL}_2 F\) of elements which fix \(f\) and have unit derivative at \(f\) takes each horocycle at \(f\) to itself. \(T_f\) is isomorphic to the translation group \(F\) and acts transitively on \((F \cup \infty)/f\).

We prove 3.2(a); the proof of 3.2(b) follows in parallel.
Assume that \( x \geq gx \) so that \( z = x \). If this is not true, simply follow the proof for the germ of \( g^{-1} \) at \( gx \). Pick \( \epsilon' \in F, 0 < \epsilon' < \epsilon \), so that \( g \) is noninfinite on the interval \((x - 2\epsilon', x + 2\epsilon')\). Let \( y = x + \epsilon' \). There are three cases.

(i) \( y = gy \). In this case pick \( \epsilon' \) slightly smaller so as to drop to case (ii) or (iii).

(ii) \( y > gy \) (Fig. 3.3). Let \( H \) be a horocycle tangent to \( y \), and let \( gH \) be its image, tangent to \( gy \). Pick \( a_1 \in F, gx < a_1 < gy \), close enough to \( gy \), and pick \( h \in T_{a_1} \) so that \( hgy \) is large enough, so that the base \( a_2(a_1, h) \) of the horocycle \( C \) tangent to \( hgy \), \( H \) and \( R \) (and to the left of \( H \)) is between \( gy \) and \( y \). Pick \( h' \) belonging to the subgroup of \( \text{PSL}_2F \) fixing the horocycles based at \( a_2 \), and so that \( h'y = hgy \). Then \( h'H = hgy \), so that \( h'^{-1}h \in T_y \). Consequently, \( a_2 \in F \) and \( h' \in \text{PSL}_2F \).

Now define

\[
s(t) = \begin{cases} 
  g(t), & t \leq g^{-1}a_1, \\
  hgy(t), & g^{-1}a_1 \leq t \leq (hg)^{-1}a_2, \\
  h'^{-1}hg(t), & (hg)^{-1}a_2 \leq t \leq y, \\
  t, & t \geq y.
\end{cases}
\]

By construction, \( s \in \Gamma_F \).

(iii) \( gy > y \) (Fig. 3.4). Let \( a_0 = g(x + \delta), \delta = (y - gx)/10 \), and let \( k \in T_{a_0} \) so that \( kgy < y \). Let \( H \) be a horocycle tangent to \( y \), and let \( kgH \) be its image at \( kgy \). Pick \( a_1 \in F, a_1 < kgy \) close enough to \( kgy \), and pick
h ∈ T_a, so that hkgy is large enough, so that the base a_2(a_1, h) < y of the horocycle C tangent to H, hkgH and R (and left of H) is between kg and y. Let h' ∈ T_a so that h' = hkgy. Note then that h'H = hkgH, so that h'^{-1}hkg ∈ T_y. One can show that a_2 ∈ F, h' ∈ PSL_2F. Then define

\[
s(t) = \begin{cases} 
  g(t), & t \leq x + \delta, \\
  kg(t), & x + \delta \leq t \leq (kg)^{-1}a_1, \\
  hkg(t), & (kg)^{-1}a_1 \leq t \leq (hkg)^{-1}a_2, \\
  h'^{-1}hkg(t), & (hkg)^{-1}a_2 \leq t \leq y, \\
  t, & t \geq y.
\end{cases}
\]

By construction, s ∈ Γ_F.

4. Groups of compactly supported homeomorphisms. In this section we specify a condition on a pseudogroup which allows one to mimic Segal's proof [Se2] of a generalization of Mather's theorem [Ma]. We work in the context of groupoids of homeomorphisms. References for topological categories are [Se1], [Se3].

4.1. Definition. A groupoid Γ etale over R is a topological groupoid Γ whose space of objects is R, in which the domain and range maps D, R: Γ → R are locally homeomorphisms (abusing notation, we let Γ denote the space of morphisms of the topological groupoid Γ).

Given a pseudogroup Γ on R, one can construct an associated groupoid Γ etale over R, whose space of morphisms is the sheaf of germs of elements of the pseudogroup. Taking the geometric realization (in the "thick" sense of [Se1], App.) of the nerve of the groupoid, we obtain a classifying space BΓ, which is weakly homotopy equivalent to the classifying space of the pseudogroup.

We make the following assumption throughout §4 of the paper. Let Γ be a groupoid of homeomorphisms of R.

4.2. Assumption. (a) For any x ∈ R the orbit of x under Γ is dense in R.

(b) If g ∈ Γ, and t < Dg, Rg (or t > Dg, Rg) then there is a section s: U → Γ of the domain map, over an open interval U containing Dg and t, such that s(Dg) = g, and s(t) = id_Γ.

The following proposition gives what is needed to mimic Segal's proofs.
4.3. Proposition. (a) Let \( a < b < c < d \), so that \( a \) and \( b \), and likewise \( c \) and \( d \), are in the same \( \Gamma \)-orbit. Then there is a section \( s : [a, d] \to \Gamma \) of \( D \) so that \( Rs(a) = b \), \( Rs(d) = c \).

(b) If \( a < b < c < d \), \( \varepsilon > 0 \) there is a section \( s : [a, d] \to \Gamma \) of \( D \) so that \( s(a) = \text{id}_a \), \( s(d) = \text{id}_d \), and \( |Rs(b) - a| < \varepsilon \), \( |Rs(c) - d| < \varepsilon \).

Proof. (a) Let \( s_1 \in \Gamma \), with \( Ds_1 = a \) and \( Rs_1 = b \), and \( s_2 \in \Gamma \) so that \( Ds_2 = d \), \( Rs_2 = c \). Then 4.2 guarantees a section \( s \) of \( D \), over some interval containing \([a, d]\), so that \( s(a) = s_1 \), \( s(d) = s_2 \), and \( s|_{(b+\varepsilon,c-\varepsilon)} = \text{id} \).

(b) Let \( s_1 \in \Gamma \) so that \( Ds_1 = b \), \( Rs_1 \in (a, a + \varepsilon) \), and \( Rs_1 < b \), and let \( s_2 \in \Gamma \) with \( Ds_2 = c \), \( Rs_2 \in (d - \varepsilon, d) \) and \( Rs_2 > c \). Then 4.2 guarantees a section \( s \) of \( D \), over some interval containing \([a, d]\), so that \( s(a) = \text{id}_a \), \( s(d) = \text{id}_d \), \( s(b) = s_1 \), \( s(c) = s_2 \), and \( s|_{(b+\varepsilon,c-\varepsilon)} = \text{id} \).

Let \( X \subset Y \) be open intervals such that \( \partial X \cap \partial Y = \emptyset \), and such that \( \partial X \cup \partial Y \) is contained in a single \( \Gamma \)-orbit.

4.4. Definition.

\[
M(Y) = \{ m : Y \to \Gamma : m \text{ continuous}, \ Dm = \text{id}, \ RmY \subseteq Y \} \\
M(Y, X) = \{ m \in M(Y) : RmX \subseteq X \} \\
M(\bar{Y}) = \{ m : \bar{Y} \to \Gamma : \ Dm = \text{id}, \ Rm\bar{Y} \subseteq \bar{Y}, \ m \text{ continuous} \} \\
M(\bar{Y}, X) = \{ m \in M(\bar{Y}) : RmX \subseteq X \}
\]

These four sets are monoids of embeddings of \( Y \); give them the discrete topology. Notice that \( M(\bar{Y}) \) is the monoid of embeddings of \( \bar{Y} \), with a germ of an extension to a neighborhood of \( \bar{Y} \). As a consequence of 4.3(a) and [G1], 2.8 there is a weak homotopy equivalence \( BM(Y) \to B\Gamma \).

There are extension and restriction homomorphisms

\[
M(Y) \xleftarrow{i} M(Y, X) \xrightarrow{r} M(X) \\
M(\bar{Y}) \xleftarrow{i} M(\bar{Y}, X) \xrightarrow{r} M(\bar{X})
\]

4.5. Proposition. The homomorphisms \( i, \tilde{i}, r, \tilde{r} \) induce homotopy equivalences of classifying spaces.

Proof. Follow [Se2], 2.7.

4.6. Proposition. The restrictions \( M(\bar{Y}, X) \to M(Y, X) \) and \( M(\bar{X}) \to M(X) \) induce homotopy equivalences of classifying spaces.
Proof. Following Segal, consider the sequence of homomorphisms $M(\overline{Y}, X) \to M(Y, X) \to M(\overline{X}) \to M(X)$. Note that the composition of any two arrows induces a homotopy equivalence of classifying spaces, by 4.5. The result follows.

4.7. Definition. $K(X) = \{ g \in M(\overline{X}) : Rg\overline{X} = \overline{X}, \text{ and } g|_{\partial\overline{X}} = \text{id} \}$. $K(X)$ is the group of $\Gamma$-homeomorphisms with compact support in $X$.

4.8. Proposition. There is a homology equivalence $BK(X) \to \Omega BT$.

Proof. Follow 2.11 in [Se2], where, in fact, a homology fibration $K(X) \to M \to B\Gamma$ is constructed, with $M$ contractible.

4.9. Corollary. There is a homology equivalence $BK(R) \to \Omega B\Gamma$.

Proof. We construct a continuous section of the domain map $s : R \to \Gamma$ so that $Rs$ is a $\Gamma$-homeomorphism from $R$ onto $X$, conjugating $K(R)$ to $K(X)$. Let $x_n, y_n, n \in \mathbb{Z}$, be members of a single $\Gamma$-orbit such that (i) $x_n < x_{n+1}, y_n < y_{n+1}, n \in \mathbb{Z}$, and (ii) $\bigcup_n (x_{-n}, x_n) = X, \bigcup_n (x_{-n}, y_n) = R$. Further, we assume that $x_0 = y_0$, that $x_n > y_n$ for $n > 0$, and that $x_n < y_n$ for $n < 0$.

Because the $x_n$ and $y_n$ belong to a single orbit, there are $s_n \in \Gamma$ with $Ds_n = x_n, Rs_n = y_n$; we take $s_0 = \text{id}$. Define $s$ so that $s(x_n) = s_n$, as follows. Suppose $n \geq 0$. By 4.2 there is a continuous section $f : [x_n, x_{n+1}] \to \Gamma$ of the domain map such that $f(x_n) = s_n, f(x_{n+1}) = \text{id}$. Also, there is a continuous section of the domain map $g : [y_n, x_{n+1}] \to \Gamma$ such that $g(y_n) = \text{id}, g(x_{n+1}) = s_{n+1}$. Define $s$ to be $g \circ f$ on $[x_n, x_{n+1}]$; note that $s(x_n) = s_n$ and $s(x_{n+1}) = s_{n+1}$. Similarly, define $s$ on the intervals $[x_n, x_{n+1}]$ for $n < 0$.

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**CENTRO DE INVESTIGACION Y ESTUDIOS**
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**MEXICO 14 DF, CP-07000**
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