MATRIX RINGS OVER \(*\)-REGULAR RINGS AND PSEUDO-RANK FUNCTIONS

Pere Ara
MATRIX RINGS OVER *-REGULAR RINGS
AND PSEUDO-RANK FUNCTIONS

PERE ARA

In this paper we obtain a characterization of those *-regular rings
whose matrix rings are *-regular satisfying LP ≤ RP. This result allows
us to obtain a structure theorem for the *-regular self-injective rings of
type I which satisfy LP ≤ RP matricially.

Also, we are concerned with pseudo-rank functions and their corre-
sponding metric completions. We show, amongst other things, that the
LP ≤ RP axiom extends from a unit-regular *-regular ring to its comple-
tion with respect to a pseudo-rank function. Finally, we show that the
property LP ≤ RP holds for some large classes of *-regular self-injective
rings of type II.

All rings in this paper are associative with 1.

Let $R$ be a ring with an involution *. Recall that * is said to be
\textit{n-positive definite} if $\sum_{i=1}^{n}x_i x_i^* = 0$ implies $x_1 = \cdots = x_n = 0$. The in-
volution * is said to be \textit{proper} if it is 1-positive definite; and if * is
\textit{n-definite positive} for all $n$, then we say that * is \textit{positive definite}.

Recall than an element $e \in R$ is said to be a \textit{projection} if $e^2 = e^* = e$ and
$R$ is called a \textit{Rickart *-ring} if for every $x \in R$ there exists a
projection $e$ in $R$ generating the right annihilator of $x$, that is $\iota(x) = eR$.
Because of the involution, we have $\ell(x) = Rf$ for some projection $f$.
Notice that $\iota(x) \cap x^* R = 0$, hence the involution * is proper and $R$ is
nonsingular. The above projections $e, f$ depend on $x$ only, $1 - e \ (1 - f)$
is called the right (left) projection of $x$ and, as usual, we shall write
$1 - e = \text{RP}(x), \ 1 - f = \text{LP}(x)$.

If $R$ is a *-ring, we denote by $P(R)$ the set of projections of $R$
partially ordered by $e \leq f$ iff $ef = e$. Thus, if $e \leq f$ we have $eR \subseteq fR$ and
$Re \subseteq Rf$. Recall [2, pg. 14] that if $R$ is Rickart, then $P(R)$ is a lattice.

Two idempotents $e, f$ of a ring $R$ are said to be \textit{equivalent}, $e \sim f$, if
there exist $x \in eRf, \ y \in fRe$ such that $xy = e, \ yx = f$. If $e, f$ are
projections in a ring with involution and we can choose $y = x^*$ then $e, f$
are said to be \textit{*-equivalent}, $e \overset{*}{\sim} f$. A ring is \textit{directly finite} if $e \sim 1$ implies
$e = 1$. A ring with involution is said to be \textit{finite} if $e \overset{*}{\sim} 1$ implies $e = 1$. 

209
A ring \( R \) is regular if for every \( a \in R \) there exists an element \( b \in R \) such that \( a = aba \). If \( R \), in addition, possesses a proper involution, then \( R \) is called a \(*\)-regular ring. By a theorem of von Neumann [14, Exercise 5, pg. 38] a regular ring with involution is \(*\)-regular iff it is a Rickart \(*\)-ring and in fact, if \( R \) is \(*\)-regular, then \( xR = LP(x)R \) and \( Rx = R(RP(x)) \) for every \( x \in R \).

If \( R \) is a \(*\)-regular ring and \( r \in R \) with \( e = RP(r) \), \( f = LP(r) \), then it is well-known [13] that \( e \sim f \), in fact there exists a unique \( s \in eRf \) (the relative inverse of \( r \)) such that \( sr = e \) and \( rs = f \).

1. The property \( LP \sim RP \) for \(*\)-regular rings. We say that a Rickart \(*\)-ring \( R \) satisfies the property \( LP \sim RP \) if \( LP(x) \sim RP(x) \) for every \( x \) in \( R \). Also, we say that \( R \) has partial comparability (PC) if for every \( e \), \( f \in P(R) \) such that \( eRf \neq 0 \) there exist nonzero subprojections \( e' \leq e \) and \( f' \leq f \) such that \( e' \sim f' \). Clearly, in any Rickart \(*\)-ring, we have \( LP \sim RP \Rightarrow (PC) \).

Lemma 1.1. For a \(*\)-regular ring \( R \), the following conditions are equivalent:

(a) \( R \) satisfies \( LP \sim RP \).
(b) Any two equivalent projections are \(*\)-equivalent.
(c) If \( xx^* \in eRe \) with \( e \in P(R) \), then there exists \( z \in eRe \) such that \( xx^* = zz^* \).

Proof. (a) \( \Leftrightarrow \) (b). Since \( LP(x) \sim RP(x) \) for every \( x \in R \).

(a) \( \Rightarrow \) (c). See [16, Theorem 1].

(c) \( \Rightarrow \) (a). First we show that \( R \) is directly finite. If \( xy = 1 \), then we can assume that \( yx = e \in P(R) \) and \( y \in eR \), \( x \in Re \). We have \( yy^* \in eRe \), so there exists \( z \in eRe \) such that \( yy^* = zz^* \). Now, we have \( 1 = xyy^*x^* = xzz^*x^* \). By [1, Theorem 3.1, (ii)], \( R \) is finite so \( z^*x^*xz = 1 \). This implies \( e = 1 \). Now, by [16, Theorem 1], the result follows.

Let \( R \) be a \(*\)-ring. We say that \( R \) is a Baer \(*\)-ring if for every subset \( S \subseteq R \) there exists a projection \( e \) in \( R \) such that \( e(S) = eR \) (and so \( e(S) = Pf \) for some projection \( f \) in \( R \)). Obviously, a Baer \(*\)-ring is Rickart and the partially ordered set \( P(R) \) is in fact a complete lattice.

An element \( w \in R \) is said to be a partial isometry if \( ww^*w = w \). In this case \( ww^* = e \) and \( w^*w = f \) are projections with \( wR = eR \) and \( w^*R = fR \). An element \( u \) is called unitary if \( uu^* = u^*u = 1 \).

It follows easily from Lemma 1.1 that the elements of a \(*\)-regular ring with \( LP \sim RP \) have weak polar decomposition, that is, if \( x \in R \) then...
Let $R$ be a Baer *-ring. We say that the *-equivalence is additive in $R$ if for any families $(e_i)_{i \in I}, (f_i)_{i \in I}$ of orthogonal projections of $R$ such that $e_i \sim f_i$, for all $i \in I$, we have $\bigvee_{i \in I} e_i \sim \bigvee_{i \in I} f_i$ (where $\bigvee$ denotes supremum). The partial isometries are addable in $R$ if for any family $(w_i)_{i \in I}$ of partial isometries such that $(w_i w_i^*)_{i \in I}$ and $(w_i^* w_i)_{i \in I}$ are families of orthogonal projections, there exists a partial isometry $w$ in $R$ such that $ww^* = \bigvee_{i \in I} (w_i w_i^*)$ and $w^* w = \bigvee_{i \in I} (w_i^* w_i)$.

**Lemma 1.2.** (i) If $R$ is a self-injective *-regular ring, then the partial isometries are addable in $R$.

(ii) If $R$ is a Baer *-regular ring, then the *-equivalence is additive in $R$.

**Proof.** (i) Set $e_i = w_i w_i^*$, $f_i = w_i^* w_i$, with $(e_i)_{i \in I}$ and $(f_i)_{i \in I}$ families of orthogonal projections. Consider the $R$-homomorphism $\varphi: \bigoplus_{i \in I} f_i R \to \bigoplus_{i \in I} e_i R$ for which $\varphi(f_i) = w_i$, all $i \in I$. Since $R$ is self-injective, $\varphi$ is given by left multiplication by some element, say $x$. Set $e = \bigvee_{i \in I} e_i$ and $f = \bigvee_{i \in I} f_i$. If $w = \text{exf}$ then it is easily seen that $e_i w = w f_i = w_i$ and $w^* w = \bigvee_{i \in I} (w_i w_i^*)$.

(ii) Since any Baer *-regular ring $R$ is complete, it follows from [13, Thm. 3, p. 535] that $R$ is a continuous ring. By [5, Thm. 13.17] $R = R_1 \times R_2$, where $R_1$ is self-injective and $R_2$ is an abelian continuous ring. Since a central idempotent of a Rickart *-ring is a projection, we have that $R_1$ and $R_2$ are *-regular. Moreover two *-equivalent projections in $R_2$ are equal so the *-equivalence is obviously additive in $R_2$. Since $R_1$ is self-injective and *-regular the partial isometries are addable in $R_1$. In particular the *-equivalence is additive in $R_1$. Therefore the *-equivalence is additive in $R$. $\square$

For a ring $R$, we denote by $Q_1(R), Q_1(R))$ the maximal ring of right (left) quotients of $R$. Recall that if $R$ is right nonsingular then $Q_1(R)$ is a regular right self-injective ring.

**Lemma 1.3.** Let $R$ be a nonsingular *-ring. Then, the involution $*$ extends to $Q_1(R)$ if and if $Q_1(R) = Q_1(R)$. In case $*$ extends to $Q_1(R)$, this extension is unique and if $*$ is $n$-positive definite on $R$, then the extended involution is also $n$-positive definite.
**Proof.** The proof is contained in [17, Thm. 3.2], except the \( n \)-positive definite part.

It is well-known that if \( x_1, \ldots, x_m \) are nonzero elements in \( Q_r(R) \), then there exist \( 1 \leq k \leq m \) and \( r \in R \) such that \( x_ir \in R \) for \( i = 1, \ldots, m \) and \( x_kr \neq 0 \). Assume that \( * \) is \( n \)-positive definite on \( R \) and let \( x_1, \ldots, x_m \) be nonzero elements in \( Q = Q_r(R) = Q_l(R) \), with \( m \leq n \). If \( k \) and \( r \) are as above, then we have \( (x_1r)^*+(x_1r) + \cdots + (x_mr)^*(x_mr) \neq 0 \), and so \( r^*(x_1^*x_1 + \cdots + x_m^*x_m)r \neq 0 \) (we also denote by \( * \) the extended involution). Hence \( * \) is \( n \)-positive definite on \( Q \).

**Remarks.** (1) In particular, if \( R \) is a nonsingular \(*\)-ring with proper involution and \( Q = Q_r(R) = Q_l(R) \), then \( Q \) is a self-injective \(*\)-regular ring.

(2) Recall that for a nonsingular ring \( R \) the condition \( Q_r(R) = Q_l(R) \) is equivalent to the Utumi’s conditions:

(a) For every right ideal \( I, \ell(I) = 0 \) implies \( I \subseteq eR \).

(b) For every left ideal \( I, \imath(I) = 0 \) implies \( I \subseteq eR \).

Obviously, (a) \( \Leftrightarrow \) (b) in any \(*\)-ring.

Let \( R \) be any \(*\)-ring. We say that \( R \) satisfies general comparability for \(*\)-equivalence (GC) if for every \( e, f \in P(R) \) there exists a central projection \( h \) in \( R \) such that \( he \leq hf \) and \( (1 - h)f \leq (1 - h)e \), cf. [2, p. 77].

**Theorem 1.4.** Let \( R \) be a \(*\)-regular ring such that \( Q = Q_r(R) = Q_l(R) \). Then \( R \) satisfies (PC) if and only if \( Q \) satisfies \( \text{LP} \sim \text{RP} \).

**Proof.** By Lemma 1.3, \( Q \) is a self-injective \(*\)-regular ring.

Assume that \( R \) satisfies (PC). Let \( e, f \) be two projections in \( Q \) such that \( eQf \neq 0 \). Since \( Q \) is regular, there exist nonzero subprojections \( e_1 \leq e \) and \( f_1 \leq f \) in \( Q \) such that \( e_1Q \equiv f_1Q \). Hence there exist \( x \in e_1Qf_1 \) and \( y \in f_1Qe_1 \) such that \( e_1 = xy \) and \( f_1 = yx \). Let \( I \) be a right ideal of \( R \) such that \( I \subseteq eR \) and \( yI \subseteq R \). We have \( yI = (ye_1)I = y(e_1I) \) and \( e_1I \leq e_1Q \). Choose a nonzero projection \( e_0 \) in \( R \) such that \( e_0 \in e_1I \). We note that \( ye_0 \neq 0 \), \( ye_0R \leq fQ \) and \( (ye_0)R \subseteq R \). Set \( f_0 = \text{LP}(ye_0) \), and note that \( f_0 \in P(R) \) and \( f_0 \leq f \). We observe that left multiplication by \( y \) induces an isomorphism from \( e_0R \) onto \( f_0R \) (since it is the restriction of an isomorphism from \( e_1Q \) onto \( f_1Q \)), and so \( e_0R \equiv f_0R \). Since \( R \) satisfies (PC), there exist nonzero projections \( e_0', f_0' \) in \( R \) such that \( e_0' \leq e_0 \leq e, f_0' \leq f_0 \leq f \) and \( e_0' \leq f_0' \). It follows that \( Q \) satisfies (PC). By Lemma 1.2 and [2, Prop. 4, p. 79], we have that \( Q \) satisfies (GC). Now it follows from [9, Prop. 3.2] that \( Q \) satisfies \( \text{LP} \sim \text{RP} \).
Conversely, assume that \( Q \) satisfies \( \text{LP} \sim \text{RP} \). Let \( e, f \) be projections in \( R \) such that \( eRF \neq 0 \). Then there exist nonzero projections \( e_0, f_0 \) in \( R \) such that \( e_0 \leq e, f_0 \leq f \) and \( e_0 \sim f_0 \). Thus, \( e_0 \sim f_0 \) in \( Q \), and so there exists \( x \) in \( Q \) such that \( xx^* = e_0, x^*x = f_0 \). Let \( I \) be a right ideal in \( R \) such that \( I \leq e \) and \( x^*I \leq R \). Choose a nonzero projection \( e' \) in \( R \) such that \( e' \in e_0R \cap I \) and note that \( f' = (x^*e')(e'x) \) is a projection in \( R \) such that \( e' \sim f' \). Inasmuch, \( e' \leq e_0 \leq e \) and \( f' \leq f_0 \leq f \). So, \( R \) satisfies (PC).

**Proposition 1.5.** Let \( R \) be a Rickart \(*\)-ring. Consider the following axioms for \( R \).

(a) \( R \) has \( \text{LP} \sim \text{RP} \).
(b) \( R \) has (PC).
(c) \( R \) satisfies general comparability for \(*\)-equivalence, (GC).
(d) The parallelogram law (P) \((e - e \wedge f \bar{\sim} e \vee f - f, \text{ for } e, f \in \text{P}(R))\).
(e) If \( e \sim f \), then there exists a unitary \( u \) in \( R \) such that \( f = ueu^* \).

If \( R \) is a unit-regular \(*\)-regular ring, then (a) \( \Leftrightarrow \) (d) \( \Leftrightarrow \) (e) and (c) \( \Rightarrow \) (a) \( \Rightarrow \) (b). If \( R \) is a Baer \(*\)-regular ring, then all these conditions are equivalent.

**Proof.** Assume that \( R \) is a unit-regular \(*\)-regular ring.

(a) \( \Rightarrow \) (d). Since \( R \) is regular we have \( e - e \wedge f \bar{\sim} e \vee f - f \) for all projections \( e, f \) in \( R \) [13, Lemma 1]. The result is immediate.

(d) \( \Rightarrow \) (a). This is a standard argument, cf. [10, Proof of Corollary 1.1, (g)].

(a) \( \Leftrightarrow \) (e). This is routine.

(c) \( \Rightarrow \) (a). For this, note that we can adapt the proof of [9, Prop. 3.2].

(a) \( \Rightarrow \) (b). Obvious.

If \( R \) is a Baer \(*\)-regular ring, then \( R \) is unit-regular. By Lemma 1.2 and [2, Prop. 4, p. 79], (b) \( \Rightarrow \) (c). This completes the proof.

If \( R \) is \(*\)-regular and \( I \) is a two-sided ideal of \( R \), then it is well-known that \( I \) is a \(*\)-ideal and the factor ring \( R/I \) is also \(*\)-regular with the natural involution. It is easy to see that if the involution on \( R \) is \( n \)-positive definite, then that on \( R/I \) is also \( n \)-positive definite.

**Lemma 1.6.** Let \( R \) be a \(*\)-regular ring and let \( I \) be a two-sided ideal of \( R \). Every projection in \( R/I \) has the form \( \bar{e} \), where \( e \in \text{P}(R) \). If \( \nu \) is any partial isometry in \( R/I \) and \( e, f \in \text{P}(R) \) are such that \( \bar{e} = \nu \nu^* \) and \( \bar{f} = \nu^* \nu \),
then there exists a partial isometry \( w \) in \( R \) such that \( \overline{w} = v \), \( \overline{ww^*} = e_1 \leq e \) and \( \overline{w^*w} = f_1 \leq f \). In particular, there exist orthogonal decompositions \( e = e_1 + e_2 \), \( f = f_1 + f_2 \) with \( e_1 \sim f_1 \) and \( e_2, f_2 \in I \).

Proof. Set \( \overline{R} = R/I \). From \( \overline{\text{LP}(x)R} = \overline{xR} = \text{LP}(\overline{x})R \) we deduce that \( \text{LP}(\overline{x}) = \overline{\text{LP}(x)} \) and similarly \( \overline{\text{RP}(x)} = \text{RP}(\overline{x}) \). So, any projection in \( R/I \) has the form \( \overline{e} \), where \( e \in \text{P}(R) \). If \( v \) is a partial isometry in \( \overline{R} \) and \( e, f \in \text{P}(R) \) are such that \( \overline{e} = vv^* \), \( \overline{f} = v^*v \) then we observe that we can choose \( w' \in eRf \) such that \( \overline{w'} = v \). We have

\[
(1) \quad w'w^* = e + y \quad \text{with} \quad y \in I.
\]

Put \( h = \text{LP}(y) \), and note that \( h \leq e \). By multiplying the relation (1) on right and left by \( e - h \), we obtain

\[
(2) \quad (e - h)w'w^*(e - h) = e - h.
\]

Set \( w = (e - h)w' \). Since \( h \in I \), we have \( \overline{w} = v \). Also, by (2), we have \( \overline{ww^*} = e - h \leq e \). Putting \( e_1 = e - h \), \( f_1 = w^*w = w'* (e - h)w' \), we have \( e_1 \leq e \), \( f_1 \leq f \) and \( e_1 \sim f_1 \). Moreover, \( \overline{e_1} = \overline{e} \) and \( \overline{f_1} = \overline{f} \) and so, if we put \( e_2 = h = e - e_1 \), \( f_2 = f - f_1 \), then we have \( e_2, f_2 \in I \). \( \square \)

It is obvious from the relations \( \text{LP}(\overline{x}) = \overline{\text{LP}(x)} \) and \( \text{RP}(\overline{x}) = \overline{\text{RP}(x)} \) that if \( R \) satisfies \( \text{LP} \prec \text{RP} \), then \( \overline{R} = R/I \) also satisfies \( \overline{\text{LP}} \prec \overline{\text{RP}} \). However, it is not true that property \( \text{(PC)} \) is preserved in factor rings, as the following example shows.

**Example 1.7.** There exists a *-regular ring \( R \) such that

(a) \( R \) is \( S_0 \)-continuous and \( S_0 \)-injective (see [5] for definitions) and \( Q_i(R) = Q_i(R) \).

(b) \( R \) has \( \text{(PC)} \) but \( R \) does not have \( \overline{\text{LP}} \prec \overline{\text{RP}} \).

(c) There exists a maximal two-sided ideal \( M \) such that the factor ring \( R/M \) does not satisfy \( \text{(PC)} \).

Proof. Let \( X \) be any uncountable infinite set. For \( i \in X \), set \( R_i = M_2(\mathbb{R}) \). Consider \( R = \{ x \in \prod_{i \in X} R_i | x_i \in M_2(\mathbb{Q}) \text{ for all but countably many } i \in X \} \). Obviously, \( R \) is a *-regular ring.

(a) If \( (e_n)_{n \in \mathbb{N}} \) is any sequence of projections of \( R \), then clearly \( \bigvee_{n \in \mathbb{N}} e_n \) exists in \( \prod_{i \in X} R_i \) and \( \bigvee_{n \in \mathbb{N}} e_n \in R \). So, since \( \prod_{i \in X} R_i \) is continuous, \( R \) is \( S_0 \)-continuous. Since \( R \cong M_2(S) \), where \( S = \{ x \in \prod_{i \in X} K_i | K_i = \mathbb{R} \text{ for all } i \in X, \text{ and } x_i \in \mathbb{Q} \text{ for all but countably many } i \in X \} \), it follows from [5, Corollary 14.13] that \( R \) is \( S_0 \)-injective. Clearly, \( Q_i(R) = Q_i(R) = \prod_{i \in X} R_i \).
(b) If $eRf \neq 0$, with $e, f \in \mathcal{P}(R)$, then there exist nonzero subprojections $e_1 \leq e, f_1 \leq f$ such that $e_1 \sim f_1$. There exist some $i \in X$ such that $e_{1i}$ is nonzero, and we observe that $e_{1i} \sim f_{1i}$ in $M_2(R)$. Define nonzero projections $e_2, f_2$ in $R$ by $e_{2j} = f_{2j} = 0$ if $j \in X$ and $j \neq i$; $e_{2i} = e_{1i}, f_{2i} = f_{1i}$. Clearly, $e_2 \leq e_1, f_2 \leq f_1$ and $e_2 \sim f_2$.

To show that $R$ does not satisfy $LP \preceq RP$, note first that the projections $(\frac{1}{2} \ 0 \ \frac{1}{2})$ and $(0 \ 0)$ are equivalent but not *-equivalent in $M_2(Q)$. Set $p_i = (\frac{1}{2} \ 0 \ \frac{1}{2})$ for all $i \in X$; $q_i = (0 \ 0)$ for all $i \in X$, and put $p = (p_i)_{i \in X}, q = (q_i)_{i \in X}$. Then, $p$ and $q$ are equivalent but not *-equivalent projections in $R$.

(c) Let $J = \{x \in R | x_i = 0$ for all but countable many $i \in X\}$. Clearly, $J$ is a proper two-sided ideal of $R$. Let $M$ be a maximal two-sided ideal of $R$ such that $J$ is contained in $M$. It follows from [5, Thm. 14.33] that $R/M$ is a simple self-injective *-regular ring. So, by Theorem 1.4, $R/M$ has $LP \preceq RP$ if and only if it has (PC). Consider the projections $p, q$ constructed in (b). We note that neither $p$ nor $q$ belong to $M$. We have $p \sim q$ in $R$ and so $\bar{p} \sim \bar{q}$ in $\bar{R} = R/M$. If $\bar{R}$ satisfies (PC), then $\bar{p} \sim \bar{q}$, and by applying Lemma 1.6, we see that there exist orthogonal decompositions $p = p' + p''$, $q = q' + q''$ with $p' \sim q'$ and $p'', q'' \in M$. Since all $p_i, q_i$ have rank one, we deduce that each $p_i$ is either 0 or $p_i$. It follows that $p', q' \in J$ and so $p, q \in M$. This is a contradiction. So, $R/M$ does not satisfy (PC).

\[\text{PROPOSITION 1.8. Let } R \text{ be a *-regular ring such that the intersection of the maximal two-sided ideals of } R \text{ is zero. If } R/M \text{ satisfies } (PC) \text{ for all maximal two-sided ideals } M \text{ of } R, \text{ then } R \text{ satisfies } (PC).\]

\[\text{Proof. It suffices to see that given two nonzero equivalent projections } e, f \text{ in } R, \text{ there exist nonzero subprojections } e_1 \leq e, f_1 \leq f \text{ such that } e_1 \sim f_1. \text{ Let } M \text{ be a maximal two-sided ideal of } R \text{ such that } e, f \notin M. \text{ Then, } e \text{ and } f \text{ are nonzero projections in } \bar{R} = R/M. \text{ By hypothesis, } \bar{R} \text{ satisfies } (PC) \text{ so there exist nonzero subprojections } \bar{e}' \leq \bar{e}, \bar{f}' \leq \bar{f} \text{ such that } \bar{e}' \sim \bar{f}' \text{ in } \bar{R}. \text{ Set } e'' = L\mathcal{P}(ee'), f'' = L\mathcal{P}(ff') \text{ and observe that } e'' = \bar{e}', f'' = \bar{f}', e'' \leq e, f'' \leq f. \text{ Thus, there exist orthogonal decompositions } e'' = e_1 + e_2, f'' = f_1 + f_2 \text{ with } e_1 \sim f_1 \text{ and } e_2, f_2 \in M. \text{ Clearly, } e_1 \text{ and } f_1 \text{ are nonzero *-equivalent projections and } e_1 \leq e, f_1 \leq f.\]

Proposition 1.8 and Example 1.7 suggest that maybe any *-regular ring such that the intersection of the maximal two-sided ideals is zero and the simple homomorphic images satisfy $LP \preceq RP$ has $LP \preceq RP$. However, this is not true and we offer a counterexample in §3.
Now, we examine property $\text{LP} \sim \text{RP}$ in matrix rings. Recall that if $R$ is a *-regular ring with $n$-positive definite involution, then the ring $M_n(R)$ of $n \times n$ matrices over $R$ is also *-regular with involution $A^* = (a_{ji}^*)$, where $A = (a_{ij})$ (the *-transpose involution). We shall assume in the rest of this section that $M_n(R)$ is endowed with this involution.

**Lemma 1.9.** Let $R$ be a *-regular ring with 2-positive definite involution. Set $S = M_2(R)$. If $E$ is a projection in $S$, then there exists an orthogonal decomposition $E = E_1 + E_2$, where $E_1 = \left( \begin{smallmatrix} 0 & 0 \\ 0 & q \end{smallmatrix} \right)$, with $p, q \in P(R)$ and $E_2 = \left( \begin{smallmatrix} a_1 & a_2 \\ a_2^* & a_3 \end{smallmatrix} \right)$, with $a_1R = a_2R$ and $a_2^*R = a_3R$.

**Proof.** Set $E = \left( \begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix} \right)$. We have

\begin{align*}
(1) & \quad a^2 + bb^* = a, \\
(2) & \quad c^2 + b^*b = c, \\
(3) & \quad ab + bc = b,
\end{align*}

and $a = a^*, c = c^*$.

Set $e = \text{LP}(a) = \text{RP}(a)$; $f = \text{LP}(c) = \text{RP}(c)$; $g = \text{LP}(b)$; $h = \text{LP}(b^*)$. From (1) and (2) we have $bb^* = a(1 - a)$ and $b^*b = c(1 - c)$ and so, $g \leq e$, $h \leq f$.

We claim that $ag = ga$. Set $d = bb^*$, and note that $ad = da$. We have $g = \text{LP}(d) = \text{RP}(d)$, and so $gad = da = ad$. Right multiplying this relation by $d$, the relative inverse of $d$, we obtain $gag = ag$. Analogously, $ga = gag$, and we conclude that $ag = ga$.

Similarly, we can show $hc = ch$.

Now, we have

\begin{align*}
(4) & \quad (e - g)a = a(e - g) = ((e - g)a)^*, \\
(5) & \quad (e - g)a^2(e - g) = (e - g)a(e - g).
\end{align*}

It follows that $(e - g)a$ is a projection. Note that $(e - g)aR = (e - g)eR = (e - g)R$. Hence,

\begin{equation}
(6) \quad e - g = (e - g)a
\end{equation}

and, similarly

\begin{equation}
(7) \quad f - h = (f - h)c.
\end{equation}

It follows from (1)–(7) that we have an orthogonal decomposition

\[
\begin{pmatrix}
    a & b \\
    b^* & c
\end{pmatrix}
= 
\begin{pmatrix}
    e - g & 0 \\
    0 & f - h
\end{pmatrix}
+
\begin{pmatrix}
    ga & b \\
    b^* & hc
\end{pmatrix}.
\]
Now, \((ga)R = geR = gR = bR\) and \((hc)R = hfR = hR = b^*R\). Putting
\[
E_1 = \begin{pmatrix} e - g & 0 \\ 0 & f - h \end{pmatrix}, \quad E_2 = \begin{pmatrix} ga & b \\ b^* & hc \end{pmatrix}
\]
we have the desired projections.

We note that the decomposition given in Lemma 1.9 is unique. Set
\(S = M_2(R)\). We say that a projection \(E\) of \(S\) is of type A if \(E = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\) with \(p, q \in P(R)\). We say that \(E\) is of type B if \(E = \begin{pmatrix} a^1 & a_2 \\ a_3 & a^3 \end{pmatrix}\) with \(a_1R = a_2R, a_3^*R = a_3R\). By Lemma 1.9, every projection of \(S\) is, in a unique way, an orthogonal sum of a projection of type A and a projection of type B.

We now construct some projections of type B. If \(e \in P(R)\) and \(w_1, w_2 \in R\), we say that \((w_1, w_2)\) is an isometric pair for \(e\) if \(w_1R = w_1^*R = w_2R = eR\) and \(w_1w_1^* + w_2w_2^* = e\). It is routine to verify that if \((w_1, w_2)\) is an isometric pair for \(e\), then
\[
E = \begin{pmatrix} w_1^*w_1 & w_1^*w_2 \\ w_2^*w_1 & w_2^*w_2 \end{pmatrix}
\]
is a projection of \(S\) of type B which is \(*\)-equivalent to \((e \ 0)\) (implemented by \((w_1 \ w_2)\)).

**Proposition 1.10.** Let \(R\) be a \(*\)-regular ring with 2-positive definite involution such that \(S = M_2(R)\) satisfies \(LP \sim RP\). If \(E\) is a projection in \(S\), then there exists an orthogonal decomposition \(E = E_1 + E_2\), where \(E_1\) is a projection of type A and there exist a projection \(e\) in \(R\) and an isometric pair for \(e\), \((w_1, w_2)\), such that
\[
E_2 = \begin{pmatrix} w_1^*w_1 & w_1^*w_2 \\ w_2^*w_1 & w_2^*w_2 \end{pmatrix}.
\]

**Proof.** By Lemma 1.9, \(E = E_1 + E_2\), where \(E_1\) is type A and \(E_2\) is type B. Set \(E_2 = \begin{pmatrix} a^1 & a_2 \\ a_3 & a^3 \end{pmatrix}\), and put \(e = LP(a_1) = RP(a_1) = LP(a_2); f = LP(a_3) = RP(a_3) = LP(a_3^*)\). Set \(G = (e \ 0); G_1 = (e \ 0); G_2 = (f \ 0)\). It is not difficult to see that
\[
G \cdot S = G_1 \cdot S \oplus G_2 \cdot S = G_1 \cdot S \oplus E_2 \cdot S = G_2 \cdot S \oplus E_2 \cdot S.
\]

We conclude that \(G_1 \cdot S \equiv G_2 \cdot S \equiv E_2 \cdot S\). Since, by hypothesis, \(S\) satisfies \(LP \sim RP\), we have \(E_2 \sim G_1\). Let \(W\) be a partial isometry of \(S\) implementing this \(*\)-equivalence. It is easy to see that \(W\) has the form \((w_1 \ w_2)\) for \(w_1, w_2 \in R\). An easy computation shows that \((w_1, w_2)\) is an isometric pair for \(e\). \(\square\)
PROPOSITION 1.11. Let $R$ be a $^*$-regular ring with 2-positive definite involution and satisfying $LP \simeq RP$. Set $S = M_2(R)$. Then, $S$ satisfies $LP \simeq RP$ if and only if for every projection $(\frac{a}{b}, \frac{c}{d})$ of $S$ of type $B$ with $e = LP(a) = LP(b)$, we have $(\frac{a}{b}, \frac{c}{d}) \simeq (\frac{e}{0}, \frac{0}{0})$.

Proof. We first observe that every subprojection of a projection of type $B$ is itself of type $B$. This follows from Lemma 1.9 by observing that a projection of type $B$ cannot contain a nonzero projection of type $A$. For, if $(\frac{0}{q}) \leq (\frac{a}{b}, \frac{c}{d})$, where $(\frac{a}{b}, \frac{c}{d})$ is of type $B$, then $pa = p$, $pb = 0$, $qb^* = 0$, $qc = q$. But $aR = bR$ implies $\ell(a) = \ell(b)$, so $pa = 0 = p$, and similarly $qc = 0 = q$.

If $E = (\frac{e}{0}, \frac{0}{0})$, then we say $E$ is type $A_1$ and if $E = (\frac{0}{q}, \frac{0}{0})$, then we say that $E$ is type $A_2$. Note that every projection in $S$ is an orthogonal sum of projections of types $A_1$, $A_2$ and $B$. Also, note that any subprojection of a projection $E$ of type $A_1$, $A_2$ or $B$ is itself of the same type as $E$.

Suppose that $E$, $F$ are two equivalent projections in $S$. We will show that $E \sim F$ provided $S$ satisfies the stated condition. Let $E = E_1 + E_2 + E_3$ be the decomposition of $E$ into projections $E_1$, $E_2$ and $E_3$ of types $A_1$, $A_2$ and $B$ respectively. Since $E \sim F$, there exists an orthogonal decomposition $F = F_1 + F_2 + F_3$, with $E_1 \sim F_1$, $E_2 \sim F_2$ and $E_3 \sim F_3$. For $i = 1, 2, 3$, we have orthogonal decompositions $F_i = F_{i1} + F_{i2} + F_{i3}$ of $F_i$ into projections of types $A_1$, $A_2$ and $B$ respectively. Returning to $E$, we obtain $E_i = E_{i1} + E_{i2} + E_{i3}$ with $E_{ij} \sim F_{ij}$ for $i, j = 1, 2, 3$. So, we have decomposed $E$ and $F$ into nine orthogonal projections, each one of pure type. It follows that it suffices to consider the following cases:

(a) $E$ is type $A_1$ and $F$ is type $A_1$.

(b) $E$ is type $A_1$ and $F$ is type $A_2$.

(c) $E$ is type $A_1$ and $F$ is type $B$.

(d) $E$ is type $B$ and $F$ is type $B$.

Case (a). If $E = (\frac{e}{0}, \frac{0}{0})$, $F = (\frac{e'}{0}, \frac{0}{0})$ with $p$, $p' \in P(R)$, then it follows that $p \sim p'$ in $R$. Since $R$ satisfies $LP \simeq RP$, we have $p \sim p'$, and so $E \sim F$.

Case (b). Similar to case (a).

Case (c). By hypothesis, $F = (\frac{e}{b}, \frac{c}{d}) \simeq (\frac{e}{0}, \frac{0}{0})$, where $eR = aR = bR$. So, $(\frac{e}{0}, \frac{0}{0}) \sim E$. By case (a), $(\frac{e}{0}, \frac{0}{0}) \simeq E$, and so, $E \sim F$. 
Case (d). Each one of $E, F$ is $*$-equivalent, by hypothesis, to a projection of type $A_1$ and so, case (a) applies.

If $S$ satisfies $LP \precsim RP$, then it follows as in the proof of Proposition 1.10 that for a projection $E = (\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix})$ of $S$ of type $B$, with $e = LP(a)$, we have $E \precsim (\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix})$. □

Recall that a $*$-ring is said to be $*$-Pythagorean if for every $x, y$ in $R$ there exists $z \in R$ such that $xx^* + yy^* = zz^*$. Following [11], we say than an element $a$ in $R$ is a norm in $R$ if it has the form $a = xx^*$, with $x \in R$. Clearly, in a $*$-Pythagorean ring any sum of norms is a norm.

The following theorem is an extension of some results of Handelman, cf. [9, Theorem 4.5] and [11; Theorem 4.9, Corollary 4.10].

**Theorem 1.12.** Let $R$ be a $*$-regular ring with 2-positive definite involution and satisfying $LP \precsim RP$. Then, $M_2(R)$ satisfies $LP \precsim RP$ if and only if $R$ is $*$-Pythagorean. In this case, $*$ is positive definite and $M_n(R)$ satisfies $LP \precsim RP$ for all $n \geq 1$.

**Proof.** The “only if” part follows from [16, Lemma 1].

Assume now that $R$ is $*$-Pythagorean. By Proposition 1.11, it suffices to see that for any projection $E = (\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix})$ in $M_2(R)$ with $aR = bR$, $b^*R = cR$, $e = LP(a)$, we have $E \precsim (\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix})$. We have $a = a^2 + bb^* = aa^* + bb^*$, so there exists $w$ in $R$ such that $a = ww^*$. Since $R$ has $LP \precsim RP$, we see from Lemma 1.1 that we can choose $w \in eRe$. Let $\bar{w}$ be the relative inverse of $w$ and note that

\[(1) \quad w\bar{w} = \bar{w}w = e.\]

Consider the relation

\[(2) \quad ww^*ww^* + bb^* = ww^*.\]

By multiplying the relation (2) on the left by $\bar{w}$ and on the right by $\bar{w}^* = \bar{w}^*$ and using (1), we get

\[(3) \quad w^*w + \bar{w}bb^*\bar{w}^* = e.\]

Hence,

\[\left(\begin{array}{cc} w^* & \bar{w}b \\ 0 & 0 \end{array}\right)\left(\begin{array}{cc} w & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} e & 0 \\ 0 & 0 \end{array}\right)\]

and so $(w^* \bar{w}b)$ is a partial isometry. It follows that

\[F = \left(\begin{array}{cc} w & 0 \\ b^*\bar{w} & 0 \end{array}\right)\left(\begin{array}{cc} w^* & \bar{w}b \\ 0 & 0 \end{array}\right)\]
is a projection in $S$ and we compute that
\[ F = \begin{pmatrix} a & b \\ b^* & b^*w^*wb \end{pmatrix}. \]

Note that $b^*w^*wbR = b^*R = cR$, so $F$ is of type B. To see that $E = F$, we observe that for any projection $(a_1^*, a_2^*)$ of type B, $a_3$ is uniquely determined by $a_1$ and $a_2$. For, note that $a_2 = a_1a_2 + a_2a_3$. Let $\bar{a}_2$ be the relative inverse of $a_2$. Multiplying the above relation on the left by $\bar{a}_2$, and observing that $f = \bar{a}_2a_2 = RP(a_2) = LP(a_2^*) = LP(a_3)$, we get $f = \bar{a}_2a_1a_2 + a_3$, so $a_3 = \bar{a}_2(1 - a_1)a_2$.

Clearly, if $R$ is *-Pythagorean, then * is positive definite. By applying [16, Theorem 3], we see that $M_{2^*}(R)$ is *-Pythagorean for all $n \geq 0$, and so, $M_{2^*}(R)$ satisfies LP $\prec$ RP for all $n \geq 0$. Since any ring $M_m(R)$ is a corner in some ring $M_{2^*}(R)$, it follows that $M_m(R)$ satisfies LP $\prec$ RP for all $m \geq 1$.

Let $R$ be a *-ring such that $M_n(R)$ is Rickart for all $n \geq 1$. We say that $R$ satisfies LP $\prec$ RP matricially if $M_n(R)$ satisfy LP $\prec$ RP for all $n \geq 1$.

**Corollary 1.13.** Let $R$ be a *-regular ring with 2-positive definite involution. Then, $R$ is a *-regular ring satisfying LP $\prec$ RP matricially if and only if $R$ satisfies the following condition

If $aa^* + bb^* \in eRe$, where $a, b \in R$, $e \in P(R)$, then there exists $z \in eRe$ such that $aa^* + bb^* = zz^*$.

If $R$ is a self-injective *-regular ring, we see from Propositions 1.5 and 1.8 that $R$ satisfies LP $\prec$ RP if and only if all simple homomorphic images of $R$ satisfy LP $\prec$ RP. Now we obtain a characterization of the self-injective *-regular rings of type I which satisfy LP $\prec$ RP matricially. The background of the structure theory for regular, right self-injective rings can be found in [5, Chapter 10].

**Corollary 1.14.** Let $R$ be a *-regular self-injective ring of type I. Then, $M_m(R)$ is a *-regular self-injective ring of type I satisfying LP $\prec$ RP, for all $m \geq 1$, if and only if $R$ is *-isomorphic to a direct product $\prod_{n=1}^\infty M_n(A_n)$, where each $A_n$ is an abelian self-injective *-regular ring and all its simple homomorphic images are *-Pythagorean division rings with positive definite involution.
Proof. If \( R = \prod_{n=1}^{\infty} M_n(A_n) \), where each \( A_n \) is an abelian self-injective *-regular ring with all division ring images *-Pythagorean and with positive definite involution, we see from 1.5, 1.8 and 1.12 that \( R \) satisfies \( LP \sim RP \) matricially. Also, it is well-known that \( M_m(R) \) is a regular self-injective ring of type I, for all \( m \geq 1 \).

For the converse, note that by [5, Thm. 10.24] there exist regular, self-injective rings \( R_1, R_2, \ldots \) such that \( R = \prod_{n=1}^{\infty} R_n \) and each \( R_n \) is of type \( I_n \). It follows that there exist orthogonal central projections \( e_1, e_2, \ldots \) in \( R \) with \( \sqrt{}_n e_n = 1 \), and orthogonal projections \( f_{i1}, f_{i2}, \ldots, f_{ii} \) for \( i = 1, 2, \ldots \) such that \( f_{i1} \sim f_{i2} \sim \cdots \sim f_{ii} \) and \( e_i = f_{i1} + f_{i2} + \cdots + f_{ii} \) for \( i = 1, 2, \ldots \). Since \( R \) satisfies \( LP \sim RP \), also \( e_i R \) satisfies \( LP \sim RP \) and so \( f_{i1} \sim f_{i2} \sim \cdots \sim f_{ii} \). Set \( A_n = f_{n1} R f_{n1} \), and observe that \( e_n R \cong M_n(A_n) \).

We deduce that \( R = \prod_{n=1}^{\infty} M_n(A_n) \) and \( A_n \) are abelian self-injective *-regular rings with positive definite involution and satisfying \( LP \sim RP \) matricially. Since all simple homomorphic images of an abelian regular ring are division rings, the result follows. \( \square \)

2. Pseudo-rank functions on *-regular rings. In this section, we study property \( LP \sim RP \) for completions of *-regular rings with respect to pseudo-rank functions. In particular, we show that if \( R \) is a *-regular unit-regular ring satisfying \( LP \sim RP \) and \( N \) is a pseudo-rank function on \( R \), then its \( N \)-completion also satisfies \( LP \sim RP \). In [3], Burke showed this holds for an irreducible *-regular rank ring with order \( k \), with \( k \geq 4 \), in which comparability holds, which turns out to be a very special case of the result here. Our result follows from Theorem 2.8, which is also used in §3.

A pseudo-rank function on a regular ring \( R \) is a map \( N: R \to [0, 1] \) such that

(a) \( N(1) = 1 \)
(b) \( N(xy) \leq N(x) \) and \( N(xy) \leq N(y) \)
(c) \( N(e + f) = N(e) + N(f) \) for all orthogonal idempotents \( e, f \in R \).

A rank function on \( R \) is a pseudo-rank function with the additional property

(d) \( N(x) = 0 \) implies \( x = 0 \).

If \( N \) is a pseudo-rank function on \( R \), then the rule \( \delta(x, y) = N(x - y) \) defines a pseudo-metric on \( R \). Clearly, \( \delta \) is a metric iff \( N \) is a rank function. The Hausdorff completion of \( R \) with respect to \( \delta, \overline{R}, \) is showed [5, Chapter 19] to be a right and left self-injective regular ring which is complete with respect to the \( \overline{N} \)-metric, where \( \overline{N} \) is the unique extension of \( N \) to \( \overline{R} \).
If $R$ is $*$-regular, it follows as in [8, Prop. 1] that we can extend $*$ in a natural way to the $N$-completion of $R$, $\overline{R}$, so that $\overline{R}$ becomes a $*$-regular ring.

We now show the analogue of [5, Lemma 19.5] for projections in $*$-regular rings.

**Lemma 2.1.** Let $R$ be a $*$-regular ring with pseudo-rank function $N$, let $\overline{R}$ be its $N$-completion and let $\varphi: R \to \overline{R}$ be the natural map. If $p, q \in P(\overline{R})$ are orthogonal, then there exists a sequence $\{ (p_n, q_n) \} \subseteq R \times R$ such that

(a) $\varphi(p_n) \to p$, $\varphi(q_n) \to q$.

(b) For all $n$, $p_n$ and $q_n$ are orthogonal projections.

**Proof.** By [5, Lemma 19.5], there exists a sequence $\{ (e_n, f_n) \} \subseteq R \times R$ such that $\varphi(e_n) \to p$, $\varphi(f_n) \to q$ and for all $n$, $e_n$ and $f_n$ are orthogonal idempotents. Set $p_n = LP(e_n)$, $q_n = RP(f_n)$, and note that $p_n e_n = e_n$, $e_n p_n = p_n$, $q_n f_n = q_n$, $f_n q_n = f_n$. We have $q_n p_n = q_n f_n e_n p_n = 0$, so, for all $n$, $p_n$ and $q_n$ are orthogonal projections in $R$.

Given $\varepsilon > 0$, we can choose $M$ such that $\overline{N}(p - \varphi(e_n)) < \varepsilon/2$ and $\overline{N}(p - \varphi(f_n)) < \varepsilon/2$ for $n > M$. Now, we have

\[
N(p_n - e_n) = N(p_n e_n^* - p_n e_n) \leq N(e_n^* - e_n)
\]

\[
\leq \overline{N}(\varphi(e_n^*) - p) + \overline{N}(p - \varphi(e_n)) < \varepsilon \quad \text{if } n > M.
\]

It follows that $\varphi(p_n) \to p$, and similarly $\varphi(q_n) \to q$. $\square$

**Proposition 2.2.** (a) Let $R$ be a regular ring and let $N$ be a pseudo-rank function on $R$. Let $\varphi: R \to \overline{R}$ be the natural map from $R$ to its $N$-completion, $\overline{R}$. If $e, f$ are equivalent idempotents in $\overline{R}$, then there exist sequences $\{ e_n \}, \{ f_n \}$ such that, for all $n$, $e_n$ and $f_n$ are equivalent idempotents in $R$ and $\varphi(e_n) \to e$, $\varphi(f_n) \to f$.

(b) In (a), if $e$ and $f$ are orthogonal, then we can choose $\{ e_n \}, \{ f_n \}$ such that $e_n$ and $f_n$ are equivalent orthogonal idempotents for all $n$.

(c) If $R$ is $*$-regular and $p, q$ are (orthogonal) equivalent projections in $\overline{R}$, then there exist $\{ p_n \}, \{ q_n \}$ such that, for all $n$, $p_n$ and $q_n$ are (orthogonal) equivalent projections in $R$ and $\varphi(p_n) \to p$, $\varphi(q_n) \to q$.

**Proof.** (a) It suffices to see that given $\varepsilon > 0$, there exist equivalent idempotents $h, g$ in $R$ such that $\overline{N}(e - \varphi(h)) < \varepsilon$ and $\overline{N}(f - \varphi(g)) < \varepsilon$. We observe that we can get idempotents $e', f'$ in $R$, and elements
$x \in e'Rf'$ and $y \in f'Re'$ such that $\overline{N}(e - \varphi(e')) < \varepsilon/2$, $\overline{N}(f - \varphi(f')) < \varepsilon/2$ while $N(e' - xy) < \varepsilon/6$ and $N(f' - yx) < \varepsilon/6$. Note that $xy \in e'Re'$. Clearly, $xyR + (e' - xy)R = e'R$ and so there exists an idempotent $h$ in $R$ such that $e'h = he' = h$, $hR = xyR$ and $(e' - h)R \leq (e' - xy)R$. Thus, we have $N(e' - h) < \varepsilon/6$.

Let $\lambda \in Rh$ with $xy\lambda = h$. We have

$$N(e'\lambda - e') \leq N(e'\lambda - h) + N(h - e') = N((e' - xy)\lambda) + N(h - e') < \varepsilon/6 + \varepsilon/6 = \varepsilon/3.$$ 

Set $g = y\lambda x$. Clearly, $g$ is idempotent, $g$ is equivalent to $h$ and $g \leq f'$. We have

$$N(f' - g) = N(f' - y\lambda x) \leq N(f' - yx) + N(yx - y\lambda x) < \varepsilon/6 + N(y(e' - e'\lambda)x) < \varepsilon/6 + \varepsilon/3 = \varepsilon/2.$$ 

So, $g$ and $h$ are equivalent idempotents and

$$\overline{N}(e - \varphi(h)) \leq \overline{N}(e - \varphi(e')) + N(e' - h) < \varepsilon/2 + \varepsilon/6 < \varepsilon,$$

$$\overline{N}(f - \varphi(g)) \leq \overline{N}(f - \varphi(f')) + N(f' - g) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

(b) We note that, by [5, Lemma 19.5] we can choose the idempotents $e', f'$ in the proof of (a) to be orthogonal. Since $h \in e'Re'$, $g \in f'Rf'$, $h$ and $g$ are orthogonal and so the result follows.

(c) If $p, q$ are (orthogonal) equivalent projections in $\overline{R}$, then by ((b))

(a) there exist $\{e_n\}, \{f_n\}$ with $\varphi(e_n) \to p$, $\varphi(f_n) \to q$, and for all $n$, $e_n$ and $f_n$ (orthogonal) equivalent idempotents in $R$. Set $p_n = LP(e_n)$, $q_n = RP(f_n)$. As in the proof of Lemma 2.1, we obtain $\varphi(p_n) \to p$ and $\varphi(q_n) \to q$. Also, it is easily shown that, for all $n$, $p_n$ and $q_n$ are (orthogonal) equivalent projections in $R$.

Let $R$ be any *-ring. We say that $R$ satisfies the *-cancellation law for projections (briefly, $R$ has *-cancellation) if whenever $e \sim f$ with $e, f \in P(R)$, we have $1 - e \sim 1 - f$. This is equivalent to saying that two *-equivalent projections in $R$ are unitarily equivalent. Also, it is easy to see that if $R$ has *-cancellation and $e, f, g, h \in P(R)$ are such that $e$ and $f$ are orthogonal, $g$ and $h$ are orthogonal, $e + f \sim g + h$ and $f \sim h$, then $e \sim g$.

Examples of *-regular rings with *-cancellation are the *-regular rings with general comparability for *-equivalence. Also, the *-regular rings with primitive factors artinian and the *-regular self-injective rings of type I satisfy the *-cancellation law. The key to prove this is the following lemma.
**Lemma 2.3.** Let $R$ be any simple artinian ring with proper involution $\ast$. Then, $R$ satisfies the $\ast$-cancellation law.

**Proof.** We note that $R$ is $\ast$-regular. Since $R$ is simple artinian, there exist orthogonal equivalent idempotents $e_1, e_2, \ldots, e_n$ such that $e_1 + \cdots + e_n = 1$ and each $e_iR$ is a simple $R$-module. Since $R$ is $\ast$-regular, we can assume that $e_1, e_2, \ldots, e_n$ are projections, so that $e_1Re_1 = D$ is a division ring with involution. Choose $x_i \in e_iRe_i$, $y_i \in e_iRe_1$, $i = 1, \ldots, n$, such that $x_iy_i = e_1$, $y_ix_i = e_i$ for $i = 1, \ldots, n$. Endow $M_n(D)$ with an involution $\#$ given by $(a_{ij})^\# = (b_{ij})$, where $b_{ij} = (x_ix_i^*)(y_jy_j^*)$, $i, j = 1, \ldots, n$. The map $R \to M_n(D)$ given by $a \mapsto (x_ia y_j)$ is a $\ast$-isomorphism from $R$ onto $M_n(D)$ with inverse map $(a_{ij}) \mapsto \Sigma_i^n j=1 y_i a_{ij} x_j$. Note that $x_i x_i^*$, $y_j y_j^* \in e_1 Re_1 = D$ are such that $(x_i x_i^*)(y_j y_j^*) = (y_j y_j^*)(x_i x_i^*) = e_1 = 1_D$. So, $x_i x_i^* = (y_j y_j^*)^{-1}$ in $D$. Thus, if we put $t_i = y_j y_j^*$ for $i = i, \ldots, n$ we have $t_i = t_i^\ast$ and $b_{ij} = t_i^{-1} a_{ij} t_j^\ast$, where $(a_{ij})^\# = (b_{ij})$.

If $x_1, \ldots, x_n$ are in $D$, and some $x_i$ is nonzero, then, since $\#$ is a proper involution on $M_n(D)$, we have $x_1^t t_1 x_1 + \cdots + x_n^t t_n x_n \neq 0$. Define $\langle , \rangle : D^n \times D^n \to D$ by

$$\langle a, b \rangle = \langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1^t t_1 b_1 + \cdots + a_n^t t_n b_n.$$

$\langle , \rangle$ has the following properties:

1. $\langle a, b + c \rangle = \langle a, b \rangle + \langle a, c \rangle$,
2. $\langle a, b \rangle = \langle b, a \rangle^\ast$,
3. $\langle a, b \lambda \rangle = \langle a, b \rangle \lambda$,
4. $\langle a, a \rangle = 0$ iff $a = 0$

for $a, b, c \in D^n$, $\lambda \in D$.

So, $\langle , \rangle$ is a nonsingular hermitian form over $D^n$. It is easy to verify that $\langle Tx, y \rangle = \langle x, T^\ast y \rangle$ for $T \in M_n(D)$, $x, y \in D^n$, and so isometric spaces in $D^n$ correspond to $\ast$-equivalent projections in $M_n(D)$. So, the result follows from Witt’s theorem for division rings with involution [12, pg. 162].

**Proposition 2.4.** Let $R$ be a $\ast$-regular ring and assume that either $R$ has all primitive factor rings artinian or $R$ is self-injective of type I. Then, $R$ satisfies the $\ast$-cancellation law.

**Proof.** Let $R$ be a $\ast$-regular ring with all primitive factor rings artinian. By [5, Corollary 6.7], all indecomposable factor rings of $R$ are simple artinian. Thus, by Lemma 2.3, they satisfy the $\ast$-cancellation law. Also, note that we can write the $\ast$-cancellation law in equational terms. So, we can proceed as in [5, Thm. 6.10].
If $R$ is a $\ast$-regular, self-injective ring of type I, then $R \cong \prod_{n=1}^{\infty} R_n$, where each $R_n$ is of type $I_n$ and so, $R_n$ has all primitive factor rings artinian. Thus, each $R_n$ satisfies the $\ast$-cancellation law and so, also $R$ satisfies the $\ast$-cancellation law.

We note that the $\ast$-cancellation law is preserved in direct products and direct limits of $\ast$-rings. If $R$ is $\ast$-regular and $R$ satisfies the $\ast$-cancellation law, then, by Lemma 1.6, $R/I$ has $\ast$-cancellation and unitaries in $R/I$ lift to unitaries in $R$, for every two-sided ideal $I$ of $R$.

**Lemma 2.5** ([3, Lemma 6.5]). Let $R$ be a $\ast$-regular ring with $\ast$-cancellation and let $N$ be a pseudo-rank function on $R$. Let $e_1, e_2, f_1, f_2 \in P(R)$ such that $e_1 \ast f_1, e_2 \ast f_2$ and let $u_1$ be a unitary such that $f_1 = u_1 e_1 u_1^\ast$. Then, there exists a unitary $u_2$ such that $u_2 e_2 u_2^\ast = f_2$ and $N(u_2 - u_1) \leq 2(N(e_2 - e_1) + N(f_2 - f_1))$.

**Proof.** We first observe that if $e, f \in P(R)$ are such that $eR \cap fR = 0$, then $eR \leq (e - f)R, fR \leq (e - f)R$ and so $N(e) + N(f) \leq 2N(e - f)$. Set $f_3 = u_1 e_2 u_1^\ast$, and note that $f_3 \ast f_2$ and

$$N(f_3 - f_2) = N(u_1(e_2 - e_1)u_1^\ast) = N(e_2 - e_1).$$

So,

$$(1) \quad N(f_3 - f_2) \leq N(f_3 - f_1) + N(f_2 - f_1) = N(e_2 - e_1) + N(f_2 - f_1).$$

We have orthogonal decompositions $f_2 = f_2 \wedge f_3 + f_2', f_3 = f_2 \wedge f_3 + f_3'$, where $f_2', f_3' \in P(R)$. Note that $f_2'R \cap f_3'R = 0$.

Since $R$ has $\ast$-cancellation, $f_2' \ast f_3'$. Set $g = f_2' \vee f_3'$. Then, there exists $u_3' \in gRg$ such that $u_3' u_3'^\ast = u_3'^\ast u_3' = g$ and $u_3' f_2' u_3'^\ast = f_3'$. Set $u_3 = u_3' + 1 - g$ and note that $u_3 f_2 u_3'^\ast = f_3$ and $1 - u_3 = (1 - u_3)g = g(1 - u_3)$.

Finally, define $u_2 = u_3^\ast u_1$. We have $u_2 e_2 u_2^\ast = u_3^\ast u_1 e_2 u_1^\ast u_3 = u_3^\ast f_3 u_3 = f_2$, and

$$N(u_2 - u_1) = N(u_3^\ast u_1 - u_1) = N(1 - u_3) = N((1 - u_3)g) \leq N(g) = N(f_2') + N(f_3') \leq 2N(f_2' - f_3') = 2N(f_2 - f_3) \leq 2(N(e_2 - e_1) + N(f_2 - f_1)).$$

So, the result follows. □

**Lemma 2.6.** Let $R$ be a $\ast$-regular ring with pseudo-rank function $N$. Let $\overline{R}$ be the $N$-completion of $R$ and let $\varphi: R \to \overline{R}$ denote the natural map. If $w$ is a partial isometry in $\overline{R}$, then there exists a sequence $\{w_n\} \subseteq R$ such that
φ(wn) → w and, for all n, wn is a partial isometry in R. If, in addition, R satisfies the *-cancellation law, then the group of unitaries of R is dense in that of \( \overline{R} \). (These groups are endowed with the relative pseudo-rank-metric topology and they are topological groups.)

Proof. Set \( e = ww^* \in P(\overline{R}) \). Choose sequences \( \{e_n\}, \{\alpha_n\} \) such that \( e_n \in P(R) \), \( \alpha_n \in R \), for all \( n \) and \( \varphi(e_n) \to e \), \( \varphi(\alpha_n) \to w \). Note that we can assume that \( \alpha_n \in e_nR \) for all \( n \). Set \( \gamma_n = e_n - \alpha_n \alpha_n^* \). Then, \( \varphi(\gamma_n) \to e - ww^* = 0 \). Put \( e'_n = RP(\gamma_n) = LP(\gamma_n) \), all \( n \). Clearly, \( \varphi(e'_n) \to 0 \). Consequently, \( e''_n = e_n - e'_n \) are projections in \( R \) and \( \varphi(e''_n) \to e \). Now, we note that \( 0 = e''_n \gamma_n e''_n = e''_n - e''_n \alpha_n \alpha_n^* e''_n \). So, \( e''_n = (e''_n \alpha_n)(e''_n \alpha_n)^* \). We deduce that \( w_n = e''_n \alpha_n \) are partial isometries such that \( \varphi(w_n) \to ew = w \).

Clearly, the group of unitaries of \( R \) and that of \( \overline{R} \) are topological groups (see [8, Prop. 8]). If \( u \) is a unitary in \( \overline{R} \), then there exists a sequence \( \{w_n\} \) such that each \( w_n \) is a partial isometry and \( \varphi(w_n) \to u \). If \( R \) has *-cancellation, then there exist unitaries \( u_n \) such that \( w_n w_n^* u_n = w_n \) for all \( n \). Since \( \varphi(w_n w_n^*) \to 1 \), we obtain \( \varphi(u_n) \to u \). □

In the next theorem, we show that the *-cancellation law extends from \( R \) to \( \overline{R} \). This is not new in case \( \overline{R} \) is type I, by Proposition 2.4.

**Theorem 2.7.** Let \( R \) be a *-regular ring with pseudo-rank function \( N \). Let \( \overline{R} \) be the \( N \)-completion of \( R \). If \( R \) satisfies the *-cancellation law, then so does \( \overline{R} \).

Proof. Let \( \varphi: R \to \overline{R} \) denote the natural map.

Let \( e, f \) be two *-equivalent projections in \( \overline{R} \), and let \( w \) be a partial isometry in \( \overline{R} \) such that \( ww^* = e \) and \( w^* w = f \). By Lemma 2.6, there exists a sequence \( \{w_n\} \) of partial isometries in \( R \) such that \( \varphi(w_n) \to w \). Set \( e_n = w_n w_n^* \) and \( f_n = w_n^* w_n \) and note that \( e_n, f_n \in P(R) \) and \( \varphi(e_n) \to e \), \( \varphi(f_n) \to f \). By passing to subsequences of \( \{e_n\} \) and \( \{f_n\} \), we can assume that \( N(e_{n+1} - e_n) < 2^{-n} \) and \( N(f_{n+1} - f_n) < 2^{-n} \). Let \( u_1 \) be a unitary in \( R \) with \( u_1 e_1 u_1^* = f_1 \). We construct, by using Lemma 2.5, a sequence of unitaries \( \{u_n\} \) in \( R \) such that \( u_n e_n u_n^* = f_n \) and

\[
N(u_{n+1} - u_n) \leq 2(N(e_{n+1} - e_n) + N(f_{n+1} - f_n)) < 2(2^{-n} + 2^{-n}) = 2^{-n+2}.
\]

It follows that \( \{u_n\} \) is a Cauchy sequence. Let \( u = \lim_{n \to \infty} \varphi(u_n) \in \overline{R} \). Clearly, \( u e u^* = f \) and so, \( e \) and \( f \) are unitarily equivalent in \( \overline{R} \). □
Next, we show the following technical, but useful, result.

**Theorem 2.8.** Let $R$ be a *-regular ring with *-cancellation and let $N$ be a pseudo-rank function on $R$. Let $\overline{R}$ be its $N$-completion. Then, $\overline{R}$ satisfies $LP \preceq RP$ if and only if given $\varepsilon > 0$ and equivalent projections $e, f$ in $R$, there exist subprojections $e' \leq e, f' \leq f$ such that $e' \sim f'$ and $N(e - e') < \varepsilon, N(f - f') < \varepsilon$.

**Proof.** Let $\varphi: R \to \overline{R}$ denote the natural map.

Assume that $\overline{R}$ satisfies $LP \preceq RP$. If $e, f$ are equivalent projections in $R$, then $\varphi(e) \sim \varphi(f)$ and, since $\overline{R}$ satisfies $LP \preceq RP$, we have $\varphi(e) \sim \varphi(f)$. Let $w$ be a partial isometry in $\overline{R}$ such that $ww^* = \varphi(e)$ and $w^*w = \varphi(f)$. We observe that, in this situation, we can choose the partial isometries $\{w_n\}$ constructed in the proof of Lemma 2.6 in such a way that $w_n \in eRf$. Set $e_n = w_nw_n^*, f_n = w_n^*w_n$. Clearly, $\varphi(e_n) \to \varphi(e)$ and $\varphi(f_n) \to \varphi(f)$, and $e_n \sim f_n$ for all $n$. It follows that $N(e - e_n) \to 0$ and $N(f - f_n) \to 0$. So, given $\varepsilon > 0$, there exist $e', f'$ such that $e' \leq e, f' \leq f, e' \sim f'$ and $N(e - e') < \varepsilon, N(f - f') < \varepsilon$.

Conversely, assume that $e$ and $f$ are equivalent projections in $\overline{R}$. By Proposition 2.2, (c), there exist sequences $\{e_n\}, \{f_n\}$, with $e_n, f_n \in P(R)$, $\varphi(e_n) \to e, \varphi(f_n) \to f$, and $e_n \sim f_n$ for all $n$. Thus, by application of our hypothesis with $\varepsilon_n = 2^{-n}$, we have that there exist, for each $n$, subprojections $e'_n \leq e_n, f'_n \leq f_n$ such that $e'_n \sim f'_n$, $N(e_n - e'_n) < 2^{-n}$ and $N(f_n - f'_n) < 2^{-n}$. It follows that $\varphi(e'_n) \to e$ and $\varphi(f'_n) \to f$. Now, as in the proof of Theorem 2.7, we get a unitary $u$ in $\overline{R}$ such that $ueu^* = f$. In particular, we obtain that $e \sim f$. □

So, if $R$ has *-cancellation, then $\overline{R}$ satisfies $LP \preceq RP$ iff any two equivalent projections $e, f$ in $R$ can be "well approximated" with respect to $N$ by *-equivalent subprojections in $R$. Since any *-regular unit-regular ring with $LP \preceq RP$ obviously satisfies the *-cancellation law, we have

**Theorem 2.9.** Let $R$ be a *-regular unit-regular ring with pseudo-rank function $N$, and let $\overline{R}$ be its $N$-completion. If $R$ satisfies $LP \preceq RP$, then so does $\overline{R}$. □

**Remark.** Let $R$ be any regular ring. Denote by $P(R)$ the set of pseudo-rank functions of $R$. Define ([6]), if $P(R) \neq \emptyset$, $N^*(x) = \sup\{ P(x) | P \in P(R) \}$ and $N^*(x) = 0$ if $P(R) = \emptyset$. Then, $N^*$ induces a
pseudo-metric $\delta(x, y) = N^*(x - y)$ on $R$ and the completion of $R$ with respect to $\delta$, $S$, is a regular ring, called the $N^*$-completion of $R$. If $R$ is *-regular, then $S$ is also *-regular in a natural way. It can be seen that the results of this section also hold for the $N^*$-completion of a *-regular ring. In particular, the *-cancellation law and, if $R$ is unit-regular, the LP $\cong$ RP axiom, extends from $R$ to $S$.

3. Applications to the study of property LP $\cong$ RP for certain *-regular self-injective rings. Let $R$ be a *-regular ring with positive definite involution. We assume throughout in this section that $M_n(R)$ is endowed with the *-transpose involution (see §1). We proceed to construct a Grothendieck group for $R$ which is attached to the *-equivalence of projections in the rings $M_n(R)$. We shall call this group $K^*_0(R)$. For to construct it, we follow the construction in [7] for $C^*$-algebras. Set $P_\infty(R) = \bigcup_{n=1}^\infty P(M_n(R))$. For $e, f \in P_\infty(R)$, set $e \oplus f = (e_{ij} f_{ij}) \in P_\infty(R)$. If $e, f \in P_\infty(R)$, then we say that $e$ and $f$ are *-equivalent, $e \approx f$, if $(e_{ij} f_{ij})$ is in some ring $M_m(R)$, for some suitably-sized zero matrices. Also, define $e, f \in P_\infty(R)$ to be stably *-equivalent, written $e \approx f$, provided $e \oplus g \approx f \oplus g$ for some $g \in P_\infty(R)$. Let $P_\infty(R)/\approx$ denote the family of all the equivalence classes defined by $\approx$ (which is clearly an equivalence relation). For $e \in P_\infty(R)$, we use $[e]_\approx$ to denote the equivalence class of $e$ with respect to $\approx$. It follows easily that $P_\infty(R)/\approx$, with the operation $[e]_\approx + [f]_\approx = [e \oplus f]_\approx$, is an abelian semigroup with cancellation. So, we may formally adjoin inverses to $P_\infty(R)/\approx$, obtaining an abelian group, denoted by $K^*_0(R)$.

Recall that, if we use in the above construction equivalence instead of *-equivalence, we obtain the group $K_0(R)$, which can also be defined by using finitely generated projective modules over $R$ (see [5, Chapter 15]).

We have a map $\Phi: K^*_0(R) \to K_0(R)$ given by $\Phi([e]_\approx) = [e]$ where $[e]$ denotes the corresponding equivalence class of $e$ in $K_0(R)$. This map is clearly a group homomorphism from $K^*_0(R)$ onto $K_0(R)$.

Define a cone $C$ in $K^*_0(R)$ by $C = K^*_0(R) + = \{[e]_\approx | e \in P_\infty(R)\}$. It follows from [1, Thm. 3.1, (b)] that $(K^*_0(R), [1]_\approx)$ is a partially ordered group with order unit ([5, pg. 203]) for any *-regular ring $R$ with positive definite involution. Also, we may view $\Phi: (K^*_0(R), [1]_\approx) \to (K_0(R), [1])$ as a morphism in the category $\mathcal{P}$ defined in [5, pg. 203].

Now, we study $K^*_0(F)$, where $F$ is any *-field with positive definite involution. In this case, $K^*_0(F)$ and $K_0(F)$ admit in a natural way a structure of ring, where the product is induced by the tensor product. Recall that $M_n(F) \otimes M_m(F) \cong M_{nm}(F)$ and the usual isomorphism is in
fact a \(*\)-isomorphism of \(*\)-algebras, if we define \((x \otimes y)^* = x^* \otimes y^*\) for \(x \in M_n(F)\) and \(y \in M_m(F)\). Also, note that \(K_0(F) \cong \mathbb{Z}\), and so \(\Phi\): \(K_0^*(F) \rightarrow K_0(F)\) induces a ring map \(r\): \(K_0^*(F) \rightarrow \mathbb{Z}\) given by \(r([e]_* - [f]_*) = \text{rank}(e) - \text{rank}(f)\). If we set \(K = \text{Ker}(r)\), we have an exact sequence of groups

\[0 \rightarrow K \rightarrow K_0^*(F) \rightarrow \mathbb{Z} \rightarrow 0\]

Hence, \(K_0^*(F) \cong \mathbb{Z} \oplus K\) as abelian groups. In fact, \(K_0^*(F)\) is the ring generated by \([1]_*\) and \(K\). Since \(K\) is an ideal of \(K_0^*(F)\), this is the unitification of the (non unital) ring \(K\).

We now relate \(K_0^*(F)\) with the Witt ring of \(F\), \(W(F)\). The construction of \(W(F)\) can be found in [15]. There are no extra difficulties in constructing \(W(F)\) using hermitian forms instead of symmetric bilinear forms. We now fix some notation.

For any \(*\)-field \(F\), an hermitian form over \(F\) is a map \(\Phi: V \times V \rightarrow F\), where \(V\) is a finite-dimensional vector space over \(F\), such that

1. \(\Phi(e_1 + e_2, v) = \Phi(e_1, v) + \Phi(e_2, v)\),
2. \(\Phi(\lambda e, v) = \lambda \Phi(e, v)\) for \(\lambda \in F\),
3. \(\Phi(e, v) = \Phi(v, e)^*\).

Let \(F_s\) denote the fixed field of \(F\), that is \(F_s = \{x \in F | x = x^*\}\). For \(a \in V\), we note that \(\Phi(a, a) \in F_s\). We define \(D_F(\Phi) = \{\lambda \in \bar{F} | \lambda = \Phi(a, a)\text{ for some } a \in V\} \subseteq \bar{F}_s\).

Each hermitian form \(\Phi\) is isometric to a form \(\langle a_1, \ldots, a_n\rangle\), with \(a_1, \ldots, a_n \in D_F(\Phi)\), where \(\langle a_1, \ldots, a_n\rangle\) denotes the hermitian form \(\psi: F^n \times F^n \rightarrow F\) defined by \(\psi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = a_1x_1y_1^* + \cdots + a_nx_ny_n^*\).

If \(\text{ch}(F) \neq 2\), then we construct \(W(F)\) as in [15, Chapter 2] using hermitian forms instead of symmetric bilinear forms. Recall [15, Prop. II.1.4] that

1. The elements of \(W(F)\) are in one-one correspondence with the isometry classes of all anisotropic hermitian forms.
2. Two nonsingular hermitian forms \(\Phi, \Phi'\) represent the same element in \(W(F)\) iff the anisotropic part of \(\Phi, \Phi_a\), is isometric to the anisotropic part of \(\Phi', \Phi'_a\); in symbols, \(\Phi_a \simeq \Phi'_a\).
3. If \(\dim \Phi = \dim \Phi'\) (where \(\Phi, \Phi'\) are nonsingular) then \(\Phi\) and \(\Phi'\) represent the same element in \(W(F)\) iff \(\Phi \simeq \Phi'\).

We now return to the case where \(*\) is positive definite. For \(e \in P(M_n(F))\), we have an hermitian form associated \(H(e) = (e(F^n), h_e)\), where \(h_e\) is the restriction to \(e(F^n)\) of the hermitian form \(\langle x, y \rangle = x_1y_1^* + \cdots + x_ny_n^*\) over \(F^n\). Set \(-H(e) = (e(F^n), -h_e)\); and note that \(-H(e) = -\{H(e)\}\), where \(\{\Phi\}\) denotes the class of \(\Phi\) in \(W(F)\).
PROPOSITION 3.1. (a) There exists an injective ring map \( \varphi: K^*_0(F) \rightarrow W(F) \) such that \( \varphi([e]_* - [f]_*) = \{ H(e) \oplus (-H(f)) \} \), for \( e, f \in P_\infty(F) \).

(b) The hermitian form \( H(e) \oplus (-H(f)) \) is isotropic if and only if there exist nonzero subprojections \( e' \leq e, f' \leq f \) such that \( e' \sim f' \) in \( P_\infty(F) \).

Proof. Define \( \varphi': K^*_0(F)^+ \rightarrow W(F) \) by \( \varphi'( [e]_* ) = \{ H(e) \} \). We show that \( \varphi' \) is well-defined, \( \varphi'([e]_* + [f]_*) = \varphi'([e]_*) + \varphi'([f]_*) \) and \( \varphi'([e]_* \cdot [f]_*) = \varphi'([e]_*) \cdot \varphi'([f]_*) \), for \( e, f \in P_\infty(F) \). For, assume that \( [e]_* = [f]_* \), with \( e \in M_n(F), f \in M_m(F) \). There exist \( g \in P_\infty(F) \) and suitably-sized zero matrices such that

\[
\begin{pmatrix}
  e & 0 & 0 \\
  0 & g & 0 \\
  0 & 0 & 0 \\
\end{pmatrix}
\sim
\begin{pmatrix}
  f & 0 & 0 \\
  0 & g & 0 \\
  0 & 0 & 0 \\
\end{pmatrix}
\]

in some ring \( M_k(F) \). By Lemma 2.3, \( M_k(F) \) has \( * \)-cancellation, so \( (0 \ 0 \ 0) \sim (0 \ 0 \ 0) \) in \( M_k(F) \). It follows easily that \( (e(F^n), h_e) \) is isometric to \( (f(F^m), h_f) \). So, \( \{ H(e) \} = \{ H(f) \} \) and \( \varphi' \) is well-defined. If \( e, f \in P_\infty(F) \), then

\[
\varphi'([e]_* + [f]_*) = \varphi'([e \oplus f]_*) = \{ H(e \oplus f) \}
= \{ (e \oplus f)(F^{n+m}), h_{e \oplus f} \} = \{ (e(F^n), h_e) \} + \{ (f(F^m), h_f) \}
= \{ H(e) \} + \{ H(f) \} = \varphi'([e]_*) + \varphi'([f]_*)
\]

Since the products in \( K_0(F) \) and in \( W(F) \) are both induced by the tensor product, we obtain similarly \( \varphi'([e]_* \cdot [f]_*) = \varphi'([e]_*) \cdot \varphi'([f]_*) \).

From this, we deduce that we can define \( \varphi: K^*_0(F) \rightarrow W(F) \) such that \( \varphi([e]_* - [f]_*) = \varphi([e]_*) - \varphi([f]_*) \). So,

\[
\varphi([e]_* - [f]_*) = \{ H(e) \} - \{ H(f) \} = \{ H(e) \} + \{-H(f) \}
= \{ H(e) \oplus (-H(f)) \}
\]

We note that, since the involution on \( F \) is positive definite, \( H(e) \) is anisotropic for every \( e \in P_\infty(F) \).

Suppose that \( \varphi([e]_* - [f]_*) = 0 \). Then, \( \{ H(e) \} = \{ H(f) \} \) and so, \( H(e) = H(e)_a = H(f)_a = H(f) \). It follows that \( e \sim f \) in \( P_\infty(F) \) and so, \( [e]_* = [f]_* \).

(b) Assume that \( H(e) \oplus (-H(f)) \) is isotropic. Then, there exist nonzero vectors \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_m) \) such that \( u \in e(F^n), v \in f(F^m) \) and \( u_1u_1^* + \cdots + u_nu_n^* = v_1v_1^* + \cdots + v_mv_m^* \). We infer that there exist (nonzero) subprojections \( e' \leq e \) and \( f' \leq f \) with \( e'(F^n) = uF \) and \( f'(F^m) = vF \). It follows that \( e' \sim f' \).
Conversely, assume that $e' \leq e$, $f' \leq f$ are nonzero *-equivalent projections. Then, $H(e')$ and $H(f')$ are nonzero isometric subspaces of $H(e)$ and $H(f)$ respectively. So, $H(e) \oplus (-H(f))$ is isotropic. □

We define $D_F(m) = D(m\langle 1 \rangle)$ and $D_F(\infty) = \cup_{m=1}^\infty D_F(m)$. Let $W_i(F)$ denote the subgroup of additive torsion of $W(F)$. Clearly, $W_i(F)$ is an ideal and by [15, Corollary XI.3.2], $W_i(F)$ is a 2-primary group. If $w \in D_F(\infty)$, let $2^n$ be the smallest power of 2 for which $w \in D_F(2^n)$. Then, by [15, Prop. XI.1.3], the additive order of the form $\langle 1, -w \rangle$ is precisely $2^n$. So, $\langle 1, -w \rangle \in W_i(F)$ if $w \in D_F(\infty)$ and, by [15, Prop. XI.3.3 and supplement], $W_i(F)$ coincides with the ideal generated by these elements.

**Proposition 3.2.** Let $K$ be the kernel of the map $r: K_0^*(F) \rightarrow \mathbb{Z}$ given by $r([e]_* - [f]_*) = \text{rank}(e) - \text{rank}(f)$ and let $\varphi: K_0^*(F) \rightarrow W(F)$ be the map defined in Proposition 3.1. Then, $\varphi(K) \subseteq W_i(F)$ and so, $K$ is a 2-primary group. Moreover, $\varphi(K) = \tilde{W}_i(F)$, where $\tilde{W}_i(F)$ is the (non unital) subring of $W(F)$ generated by $\{\langle 1, -w \rangle | w \in D_F(\infty)\}$ and $K_0^*(F)$ is ring isomorphic, via $\varphi$, to the unitification of $\tilde{W}_i(F)$.

**Proof.** We first observe that $K$ is generated by the elements $[1]_* - [e]_*$, where $e \in P_\infty(F)$ is of rank 1. If $e \in M_n(F)$, then we deduce that $\varphi([1]_* - [e]_*) = \{\langle 1, -w \rangle\}$, where $w \in D_F(n)$. Thus, clearly $\varphi(K) = \tilde{W}_i(F)$. We have a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & K_0^*(F) & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W_i(F) & \rightarrow & W(F) & \rightarrow & W(F)/W_i(F) & \rightarrow & 0
\end{array}
\]

So, $K_0^*(F) = \mathbb{Z} \oplus K \cong \mathbb{Z} \oplus \tilde{W}_i(F) \subseteq W(F)$ and clearly $K_0^*(F)$ is ring isomorphic to the unitification of $\tilde{W}_i(F)$. □

If $D_F(\infty)$ induces a total ordering on $F$, that is, if $F = D_F(\infty) \cup \{0\} \cup (-D_F(\infty))$, then $K_0^*(F) \cong W(F)$. On the other hand, if $F$ is *-Pythagorean, then $W_i(F) = \tilde{W}_i(F) = 0$ and $K_0^*(F) \cong \mathbb{Z}$.

**Definitions.** Let $(F, *)$ be a field with positive definite involution. A *-algebra $A$ over $F$ is said to be matricial if $A$ is isomorphic as *-algebra to $M_{n(1)}(F) \times \cdots \times M_{n(r)}(F)$ for some positive integers $n(1), \ldots, n(r)$. The *-algebra is ultramatricial if $A$ contains a sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of matricial *-algebras such that $\bigcup_{n=1}^\infty A_n = A$. 

Conversely, assume that $ef < e, f' < f$ are nonzero *-equivalent projections. Then, $H(e')$ and $H(f')$ are nonzero isometric subspaces of $H(e)$ and $H(f)$ respectively. So, $H(e) \oplus (-H(f))$ is isotropic. □

We define $D_F(m) = D(m\langle 1 \rangle)$ and $D_F(\infty) = \cup_{m=1}^\infty D_F(m)$. Let $W_i(F)$ denote the subgroup of additive torsion of $W(F)$. Clearly, $W_i(F)$ is an ideal and by [15, Corollary XI.3.2], $W_i(F)$ is a 2-primary group. If $w \in D_F(\infty)$, let $2^n$ be the smallest power of 2 for which $w \in D_F(2^n)$. Then, by [15, Prop. XI.1.3], the additive order of the form $\langle 1, -w \rangle$ is precisely $2^n$. So, $\langle 1, -w \rangle \in W_i(F)$ if $w \in D_F(\infty)$ and, by [15, Prop. XI.3.3 and supplement], $W_i(F)$ coincides with the ideal generated by these elements.

**Proposition 3.2.** Let $K$ be the kernel of the map $r: K_0^*(F) \rightarrow \mathbb{Z}$ given by $r([e]_* - [f]_*) = \text{rank}(e) - \text{rank}(f)$ and let $\varphi: K_0^*(F) \rightarrow W(F)$ be the map defined in Proposition 3.1. Then, $\varphi(K) \subseteq W_i(F)$ and so, $K$ is a 2-primary group. Moreover, $\varphi(K) = \tilde{W}_i(F)$, where $\tilde{W}_i(F)$ is the (non unital) subring of $W(F)$ generated by $\{\langle 1, -w \rangle | w \in D_F(\infty)\}$ and $K_0^*(F)$ is ring isomorphic, via $\varphi$, to the unitification of $\tilde{W}_i(F)$.

**Proof.** We first observe that $K$ is generated by the elements $[1]_* - [e]_*$, where $e \in P_\infty(F)$ is of rank 1. If $e \in M_n(F)$, then we deduce that $\varphi([1]_* - [e]_*) = \{\langle 1, -w \rangle\}$, where $w \in D_F(n)$. Thus, clearly $\varphi(K) = \tilde{W}_i(F)$. We have a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & K_0^*(F) & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & W_i(F) & \rightarrow & W(F) & \rightarrow & W(F)/W_i(F) & \rightarrow & 0
\end{array}
\]

So, $K_0^*(F) = \mathbb{Z} \oplus K \cong \mathbb{Z} \oplus \tilde{W}_i(F) \subseteq W(F)$ and clearly $K_0^*(F)$ is ring isomorphic to the unitification of $\tilde{W}_i(F)$. □

If $D_F(\infty)$ induces a total ordering on $F$, that is, if $F = D_F(\infty) \cup \{0\} \cup (-D_F(\infty))$, then $K_0^*(F) \cong W(F)$. On the other hand, if $F$ is *-Pythagorean, then $W_i(F) = \tilde{W}_i(F) = 0$ and $K_0^*(F) \cong \mathbb{Z}$.

**Definitions.** Let $(F, *)$ be a field with positive definite involution. A *-algebra $A$ over $F$ is said to be matricial if $A$ is isomorphic as *-algebra to $M_{n(1)}(F) \times \cdots \times M_{n(r)}(F)$ for some positive integers $n(1), \ldots, n(r)$. The *-algebra is ultramatricial if $A$ contains a sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ of matricial *-algebras such that $\bigcup_{n=1}^\infty A_n = A$. 

Conversely, assume that $e' \leq e$, $f' \leq f$ are nonzero *-equivalent projections. Then, $H(e')$ and $H(f')$ are nonzero isometric subspaces of $H(e)$ and $H(f)$ respectively. So, $H(e) \oplus (-H(f))$ is isotropic. □
In [7, Prop. 16.1], it is shown that a \(*\)-algebra \(A\) is ultramatricial iff \(A\) is isomorphic as \(*\)-algebra to a direct limit (in the category of \(*\)-algebras) of a sequence of matricial \(*\)-algebras and \(*\)-algebra maps.

The \(*\)-algebra \(A\) is standard matricial if \(A = M_{n(1)}(F) \times \cdots \times M_{n(r)}(F)\) for some positive integers \(n(1),\ldots,n(r)\); (see [7, Chapter 17]).

If \(A = M_{n(1)}(F) \times \cdots \times M_{n(k)}(F)\) and \(B = M_{m(1)}(F) \times \cdots \times M_{m(i)}(F)\) are standard matricial \(*\)-algebras, then a standard map from \(A\) to \(B\) is any map which sends the element \((a_1,\ldots,a_k)\) of \(A\) to

\[
\begin{bmatrix}
  a_1 & & \\
  & \ddots & \\
  & & a_k
\end{bmatrix}
\quad \text{to} \quad
\begin{bmatrix}
  a_1 & & \\
  & \ddots & \\
  & & a_k
\end{bmatrix}
\]

where \(s_{ij}\) are nonnegative integers such that \(s_{i1}n(1) + \cdots + s_{ik}n(k) = m(i)\) for all \(i\). Clearly any standard map is a \(*\)-algebra map. We observe that the maps we obtain by iterated composition of standard ones are precisely the “block diagonal” maps.

A standard ultramatricial \(*\)-algebra is a direct limit of a sequence \(A_1 \to A_2 \to A_3 \to \cdots\) of standard matricial \(*\)-algebras \(A_n\) and standard maps \(\Phi_n: A_n \to A_{n+1}\).

**Proposition 3.3.** If \(F\) is \(*\)-Pythagorean then every ultramatricial \(*\)-algebra over \(F\) is isomorphic as \(*\)-algebra to a standard ultramatricial \(*\)-algebra. Moreover, if \(A\) and \(B\) are ultramatricial \(*\)-algebras over \(F\), then \(A\) and \(B\) are isomorphic as rings if and only if they are isomorphic as \(*\)-algebras.

**Proof.** We know that property LP \(\leq\) RP holds in \(M_n(F)\) for all \(n\). So we can adapt the proofs of [7, Prop. 17.2] and [7, Thm. 20.6]. \(\square\)

We do not know if Proposition 3.3 remains true for arbitrary fields with positive definite involution. By using [5, Thm. 15.26] one can show that any ultramatricial algebra over a field \(F\) is isomorphic as \(F\)-algebra to a standard ultramatricial algebra.

Now we proceed to study completions of direct limits of direct systems of standard matricial \(*\)-algebras and standard maps with respect to a pseudo-rank function. We need a lemma which gives a characteriza-
tion of those pseudo-rank functions $N$ on a regular ring $R$ such that the $N$-completion of $R$ is type II.

**Lemma 3.4.** Let $R$ be a regular ring with pseudo-rank function $N$ and let $\overline{R}$ be its $N$-completion. Then, $\overline{R}$ is type II if and only if for each idempotent $e$ in $R$, for each $\epsilon > 0$, and for each $m \geq 1$ there exist equivalent orthogonal idempotents $e_1, e_2, \ldots, e_m \in R$ such that $e_i e = e_i = e_i$ for all $i$, and $N(e - (e_1 + \cdots + e_m)) < \epsilon$.

**Proof.** Let $\varphi : R \to \overline{R}$ denote the natural map.

Assume that for each idempotent $e \in R$, $\epsilon > 0$, and $m \geq 1$, there exist equivalent orthogonal idempotents $e_1, \ldots, e_m$ such that $e e_i = e_i = e_i$ for all $i$, and $N(e - (e_1 + \cdots + e_m)) < \epsilon$. If $\overline{R}$ is not type II then there exists a central idempotent $h \in \overline{R}$ such that $h \neq 0$ and $hR$ is type $I_n$ for some $n \geq 1$. Set $\epsilon = N(h)$, where $N$ denotes the natural extension of $N$ to $\overline{R}$. There exist equivalent orthogonal idempotents $e_1, e_2, \ldots, e_{n+1} \in R$ such that $N(1 - (e_1 + \cdots + e_{n+1})) < \epsilon$. We observe that $h\varphi(e_1), \ldots, h\varphi(e_{n+1})$ are equivalent orthogonal idempotents of $\overline{R}$. We have

$$N(h(1 - (\varphi(e_1) + \cdots + \varphi(e_{n+1})))) \leq N(1 - (e_1 + \cdots + e_{n+1})) < \epsilon = N(h).$$

In particular $h(\varphi(e_1) + \cdots + \varphi(e_{n+1})) \neq 0$. So $h\varphi(e_1), \ldots, h\varphi(e_{n+1})$ are nonzero equivalent orthogonal idempotents in $hR$. This contradicts [5, Thm. 7.2] and consequently we deduce that $\overline{R}$ is type II.

Conversely, assume that $\overline{R}$ is type II. First we show that for each $e \in R$, for each $\epsilon > 0$, and for each $n \geq 1$, there exist $2^n$ equivalent orthogonal idempotents $e_1, e_2, \ldots, e_{2^n} \in R$ such that $e e_i = e_i = e_i$ for all $i$, and $N(e - (e_1 + \cdots + e_{2^n})) < \epsilon$. We proceed by induction on $n$. Set $n = 1$. If $N(e) = 0$ then the result is trivial. So assume that $N(e) \neq 0$ and consider the pseudo-rank function $N'$ on $eRe$ defined by $N'(z) = N(z)/N(e)$ for $z \in eRe$. Then the $N'$-completion of $eRe$ is precisely $\varphi(e)\overline{R}\varphi(e)$ which is also type II. So we can assume without loss of generality that $e = 1$. Since $\overline{R}$ is type II it follows from [5, Prop. 10.28] that there exist equivalent orthogonal idempotents $g_1, g_2 \in \overline{R}$ such that $1 = g_1 + g_2$. By Proposition 2.2, (b) we can choose sequences $\{g_{1r}\}, \{g_{2r}\}$ such that, for each $r$, $g_{1r}$ and $g_{2r}$ are equivalent orthogonal idempotents in $R$ and $\varphi(g_{1r}) \to g_1$, $\varphi(g_{2r}) \to g_2$. Consequently there exist equivalent orthogonal idempotents $e_1, e_2 \in R$ such that $N(g_1 - \varphi(e_1)) < \epsilon/2$ and $N(g_2 - \varphi(e_2)) < \epsilon/2$. Hence

$$N(1 - (e_1 + e_2)) \leq N(g_1 - \varphi(e_1)) + N(g_2 - \varphi(e_2)) < \epsilon.$$
Now assume that the result is true for $1 \leq k < n$ with $n \geq 2$. Taking $k = 1$ we see that there exist equivalent orthogonal idempotents $e'_1$, $e'_2 \in R$ such that $e'_1 + e'_2 \leq \varepsilon$ and $N(e - (e'_1 + e'_2)) < \varepsilon/3$. Taking now $k = n - 1$ we obtain $2^{n-1}$ equivalent orthogonal idempotents $e_1, \ldots, e_{2^{n-1}} \in R$ such that $e_1 + \cdots + e_{2^{n-1}} \leq e_1'$ and $N(e_1' - (e_1 + \cdots + e_{2^{n-1}})) < \varepsilon/3$. Since $e_1' \sim e_2'$ there exist equivalent orthogonal idempotents $e_{2^{n-1}+1}, \ldots, e_{2^n} \in R$ such that $e_{2^{n-1}+1} + \cdots + e_{2^n} \leq e_2'$ and $e_1 \sim e_{2^{n-1}+1} \sim \cdots \sim e_{2^n}$. We have

$$N(e_2' - (e_{2^{n-1}+1} + \cdots + e_{2^n})) = N(e_2') - N(e_{2^{n-1}+1}) - \cdots - N(e_{2^n}) = N(e_1') - N(e_1) - \cdots - N(e_{2^{n-1}}) < \varepsilon/3.$$ 

So, $e_1, \ldots, e_{2^n}$ are $2^n$ equivalent orthogonal idempotents such that $e_1 + \cdots + e_{2^n} \leq \varepsilon$ and

$$N(e - (e_1 + \cdots + e_{2^n})) \leq N(e - (e_1' + e_2')) + N(e_1' - (e_1 + \cdots + e_{2^{n-1}})) + N(e_2' - (e_{2^{n-1}+1} + \cdots + e_{2^n})) < \varepsilon.$$

Now let $e \in R$ be an idempotent and let $\varepsilon > 0$, $m \geq 1$. Choose $n \geq 1$ such that $m/2^n < \varepsilon/2$ and put $2^n = mr + k$ where $r \geq 0$ and $0 \leq k < m$. As we have seen there exist equivalent orthogonal idempotents $e_1', \ldots, e_{2^n}' \in R$ such that $e_i'e = ee_i' = e_i'$ for all $i$, and $N(e - (e_1' + \cdots + e_{2^n}')) < \varepsilon/2$. Observe that $N(e_i') \leq 2^{-n}$ for all $i$. Define $e_i = e_{(i-1)r+1} + \cdots + e_{ir}$ for $i = 1, \ldots, m$. Then $e_1, \ldots, e_m$ are equivalent orthogonal idempotents of $R$ such that $e_i'e = ee_i' = e_i$ all $i$. Moreover we have

$$N(e - (e_1 + \cdots + e_m)) = N(e - (e_1' + \cdots + e_{mr}')) \leq N(e - (e_1' + \cdots + e_{2^n}')) + N(e_{mr+1}' + \cdots + e_{2^n}') < \varepsilon/2 + kN(e_{2^n}') \leq \varepsilon/2 + m/2^n < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

Hence $N(e - (e_1 + \cdots + e_{2^n})) < \varepsilon$ as desired. 

**Theorem 3.5.** Let $F$ be a *-field with positive definite involution. Let \{ $R_i, \Phi_{ji}$ \}_{i, j} \in I \text{ be a direct system such that, for every } i \in I, R_i \text{ is a standard matricial *-algebra over } F \text{ and, if } i \leq j, \Phi_{ji} : R_i \to R_j \text{ is a composition of standard maps. Let } R \text{ be the direct limit of } \{ R_i, \Phi_{ji} \} \text{ and let } N \text{ be a pseudo-rank function on } R. \text{ Then the type II part of the } N\text{-completion of } R \text{ satisfies } LP \prec RP \text{ matricially.}
Proof. It suffices to see that the type II part of the $N$-completion of $R$ satisfies $LP \simeq RP$.

Let $\overline{R}$ be the $N$-completion of $R$ and let $\varphi: R \to \overline{R}$ denote the natural map. There exists a unique decomposition $\overline{R} = R_1 \times R_2$ where $R_1$ is type I and $R_2$ is type II. Let $\overline{N}$ be the natural extension of $N$ to $\overline{R}$, and note that $\overline{N}$ is a rank function on $\overline{R}$. If $R_1$ and $R_2$ are nonzero, then there exists a central projection $h \neq 0, 1$ such that $h\overline{R} = R_1$ and $(1 - h)\overline{R} = R_2$. By [5, Prop. 16.4] there exist unique rank functions $N'_1, N'_2$ on $R_1, R_2$ such that

$$\overline{N}(x) = \overline{N}(h)N'_1(hx) + \overline{N}(1 - h)N'_2((1 - h)x)$$

for all $x \in \overline{R}$. For $y \in R$, define $N_2(y) = N'_2((1 - h)\varphi(y))$. Then, it is easily seen that $N_2$ is a pseudo-rank function on $R$. Also, one can see that the map $\psi: R \to R_2$ defined by $\psi(y) = (1 - h)\varphi(y)$ is the natural map from $R$ to its $N_2$-completion, so that the completion of $(R, N_2)$ is precisely $(R_2, N'_2)$.

If $R_2 = 0$, there is nothing to prove. If $R_2 \neq 0$, then we see from the above discussion that $R_2$ is the completion of $R$ with respect to a certain pseudo-rank function on $R$. So, we can assume without loss of generality that $\overline{R}$ is of type II.

Since each $R_i$ has *-cancellation, so does $R$. Thus, by Theorem 2.8, it suffices to prove that given $\varepsilon > 0$ and equivalent projections $e, f$ in $R$, there exist subprojections $e' \preceq e, f' \preceq f$ such that $e' \sim f'$ and $N(e - e') < \varepsilon, N(f - f') < \varepsilon$. For $i \in I$, let $\theta_i: R_i \to R$ be the natural map from $R_i$ to the direct limit. There exist $i \in I$ and projections $g, h$ in $R_i$ such that $\theta_i(g) = e, \theta_i(h) = f$ and $g \sim h$ in $R_i$. Since $R_i$ is a standard matricial *-algebra, there exist some positive integers $c(1), \ldots, c(n)$ such that $R_i = \bigotimes_{\alpha=1}^n M_{c(\alpha)}(F)$. Clearly, we may assume without loss of generality that $g = (0, \ldots, 0, g', 0, \ldots, 0)$ and $h = (0, \ldots, 0, h', 0, \ldots, 0)$ where $g'$ and $h'$ are projections of rank one in some ring $M_{c(\alpha)}(F)$ for some $1 \leq \alpha \leq n$.

Let $k$ be the additive order of $[g']_* - [h']_*$ in $K_0^*(F)$. By Proposition 3.2, $k$ is a power of 2. Moreover, since $M_n(F)$ has *-cancellation for all $n$, we have

$$\begin{bmatrix}
g'
g' & k & \cdots & g'
g' & k & \cdots & g'
g' & k & \cdots & g'
\end{bmatrix} \sim \begin{bmatrix}
h'
h' & k & \cdots & h'
h' & k & \cdots & h'
h' & k & \cdots & h'
\end{bmatrix}.$$
Let \( l \) be a positive integer with \( 1/7 < \varepsilon/2 \), and set \( m = kl \). By Lemma 3.4 (and a standard argument) there exist \( m \) orthogonal equivalent projections \( e_1, \ldots, e_m \) in \( R \) such that \( e_1 + \cdots + e_m \leq e \) and \( N(e - (e_1 + \cdots + e_m)) < \varepsilon/2 \). Now, there exist \( j \in I \) such that \( j \geq i \) and \( m \) orthogonal equivalent projections \( g_1, \ldots, g_m \) in \( R_j \) such that \( g_p \leq \Phi_{ji}(g) \) and \( \theta_j(g_p) = e_p \) for \( p = 1, \ldots, m \). There exist positive integers \( d(1), \ldots, d(r) \) such that \( R_j = M_{d(1)}(F) \times \cdots \times M_{d(r)}(F) \). Set \( g_p = (g_{p1}, \ldots, g_{pr}) \) for \( p = 1, \ldots, m \), and note that, for each \( q = 1, \ldots, r \), \( g_{1q}, \ldots, g_{mq} \) are \( m \) orthogonal equivalent projections in \( M_{d(q)}(F) \). Without loss of generality, we can assume that \( g_{11}, \ldots, g_{1r} \neq 0 \) and \( g_{1r+1} = \cdots = g_{1r} = 0 \). Set \( \Phi_{ji}(g) = (e_1', \ldots, e_r') \). We note that
\[
N(\theta_j(0, \ldots, 0, e_{r+1}', \ldots, e_r')) < N(\theta_j(\Phi_{ji}(g) - (g_1 + \cdots + g_m)))
= N(e - (e_1 + \cdots + e_m)) < \varepsilon/2.
\]
Since \( \Phi_{ji} \) is a composition of standard maps, each \( e_q' \) has the form
\[
\begin{bmatrix}
0 \\
g' \\
0 \\
g' \\
\vdots
\end{bmatrix}
\]
for suitably-sized zero matrices.

Since \( g_{1q} + \cdots + g_{mq} \leq e_q' \) for \( q = 1, \ldots, r \), we have \( \text{rank}(e_q') \geq m \) for \( q = 1, \ldots, r' \). If we put \( \Phi_{ji}(h) = (f_1', \ldots, f_{r'}') \) we see that \( \text{rank}(f_q') \geq m \) for \( q = 1, \ldots, r' \).

For \( q = 1, \ldots, r' \), set \( t(q) = \text{rank}(e_q') \) and note that \( t(q) \) is precisely the number of copies of \( g' \) that appear in the expression of \( e_q' \). Put \( t(q) = s(q)k + t'(q) \) with \( 0 \leq t'(q) < k \). We observe that \( m \leq s(q)k \).

For each \( q = 1, \ldots, r' \), let \( e_q'' \) be the projection of \( M_{d(q)}(F) \) which has \( s(q)k \) \( g' \)'-blocks in the same places as the first \( s(q)k \) \( g' \)'-blocks of \( e_q' \) and zeroes elsewhere, that is
\[
e_q'' = \begin{bmatrix}
0 \\
g' \\
0 \\
\vdots
\end{bmatrix}.
\]
For \( q = 1, \ldots, r' \), let \( f''_q \) be the projection of \( M_{d(q)}(F) \) formed in the same way as \( e''_q \) but with \( h' \) instead of \( g' \).

Set \( e' = \theta_j((e''_1, \ldots, e''_r, 0, \ldots, 0)) \), \( f' = \theta_j((f''_1, \ldots, f''_r, 0, \ldots, 0)) \). Clearly, \( e' \leq e \) and \( f' \leq f \). Since \( e''_q \rightarrow f''_q \) for \( q = 1, \ldots, r' \), we have \( e' \rightarrow f' \).

Set \( N_j = N\theta_j \). Then, \( N_j \) is a pseudo-rank function on \( R_j \) and by [5, Corollary 16.6], we have that there exist nonnegative real numbers \( \alpha_1, \ldots, \alpha_r \) with \( \alpha_1 + \cdots + \alpha_r = 1 \) such that
\[
N_j((x_1, \ldots, x_r)) = \alpha_1 \text{rank}(x_1)/d(1) + \cdots + \alpha_r \text{rank}(x_r)/d(r)
\]
For \( q = 1, \ldots, r' \) we have
\[
\text{rank}(e'_q - e''_q)/d(q) = t'(q)/d(q)
\]
\[
\leq t'(q)/m < k/m = k/(kl) = 1/l < \epsilon/2.
\]
Finally,
\[
N(e - e') = N(\theta_j(e'_1 - e''_1, \ldots, e'_r - e''_r, e'_{r+1}, \ldots, e'_r))
\]
\[
\leq N_j((e'_1 - e''_1, \ldots, e'_r - e''_r, 0, \ldots, 0))
\]
\[
+ N_j((0, \ldots, 0, e'_{r+1}, \ldots, e'_r))
\]
\[
< N_j((e'_1 - e''_1, \ldots, e'_r - e''_r, 0, \ldots, 0)) + \epsilon/2 = \alpha_1 \text{rank}(e'_1 - e''_1)/d(1)
\]
\[
+ \cdots + \alpha_r \text{rank}(e'_r - e''_r)/d(r') + \epsilon/2 < (\alpha_1 + \cdots + \alpha_r)\epsilon/2 + \epsilon/2 \leq \epsilon.
\]

Similarly, \( N(f - f') < \epsilon \). So, the proof is complete. \( \square \)

As a consequence of Theorem 3.5, we see that if \( F \) is any \(*\)-field with positive definite involution, then there exists a simple, \(*\)-regular, self-injective ring of type II satisfying \( \text{LP} \rightarrow \text{RP} \) whose center is \( F \). For example, let \( n(1) < n(2) < \cdots \) be positive integers such that \( n(k)|n(k+1) \) for all \( k \), and set \( S = \lim M_{n(k)}(F) \) (with respect to the obvious standard maps). Let \( R \) be the completion of \( S \) with respect to the unique rank function on \( S \). Then, \( R \) is a simple, \(*\)-regular, self-injective ring of type II whose center is \( F \) ([4, Thm. 2.8]). By Theorem 3.5, \( R \) satisfies \( \text{LP} \rightarrow \text{RP} \) matrixially.

Next, we shall construct a simple, \(*\)-regular, self-injective ring of type II which does not satisfy \( \text{LP} \rightarrow \text{RP} \). In [9, pg. 31, Example 1] Handelman tries to offer an example of a simple, \(*\)-regular, type II self-injective ring \( R \) which does not satisfy \( \text{LP} \rightarrow \text{RP} \) and a Baer \(*\)-subring \( S \) of \( R \) which
contains all the partial isometries of $R$ and does not satisfy neither $LP \simeq RP$ nor the (EP)-axiom. The ring $R$ constructed by Handelman is the completion of $\lim M_{2^n}(\mathbb{Q}(x))$ with respect to its unique rank function. So, it follows from Theorem 3.5 that $R$ satisfies $LP \simeq RP$ and therefore, also the Baer *-subring $S$ has $LP \simeq RP$. It is true, however, that they do not satisfy the (SR)-axiom of \cite[pg. 66]{2}.

**Example 3.6.** There exists a simple, *-regular, self-injective ring of type II which does not satisfy $LP \simeq RP$.

**Proof.** Let $F$ be a formally real field such that $D_F(1) \subsetneq D_F(2) \subsetneq \cdots$ (for example we can take $F = \mathbb{R}(x_1, x_2, \ldots)$, \cite[Exercise 6, pg. 315]{15}). Set $S = \prod_{n=1}^{\infty} M_{2^n}(F)$. Let $M$ be a maximal two-sided ideal of $S$ which contains the direct sum $\bigoplus_{n=1}^{\infty} M_{2^n}(F)$. Set $R = S/M$. By \cite[Thm. 10.30]{5} $R$ is a simple, regular, right and left self-injective ring of type II. Clearly, both $R$ and $S$ are *-regular rings (here, the involution on $F$ is the identity). For $n \geq 1$, choose $w_n \in D_F(2^n) - D_F(2^{n-1})$. From Propositions 3.1 and 3.2, we see that there exist rank one projections $f_{n,i} \in M_{2^n}(F)$, $i = 1, \ldots, 2^n$ such that for each $n$, $f_{n,i}$ are $2^n$ orthogonal *-equivalent projections adding to the identity in $M_{2^n}(F)$, that is $f_{n,1} + \cdots + f_{n,2^n} = 1_{2^n}$, and $\varphi([f_{n,i}]) = \langle \langle w_n \rangle \rangle$ for $i = 1, \ldots, 2^n$. Set

$$g_{n,1} = f_{n,1} + \cdots + f_{n,2^n-1}$$
$$g_{n,2} = f_{n,2^{n-1}+1} + \cdots + f_{n,2^n};$$
$$h_{n,1} = \text{diag}\left(\frac{2^n-1}{1, \ldots, 1, 0, \ldots, 0}\right);$$
$$h_{n,2} = \text{diag}\left(0, \ldots, 0, 1, \ldots, 1\right).$$

From \cite[Corollary X.1.6]{15} and 3.1 (b) we deduce that for each $n$, $g_{n,1}$ and $h_{n,1}$ does not have nonzero *-equivalent subprojections. Set $g_1 = (g_{1,1}, g_{2,1}, \ldots); \ g_2 = (g_{1,2}, g_{2,2}, \ldots); \ h_1 = (h_{1,1}, h_{2,1}, \ldots); \ h_2 = (h_{1,2}, h_{2,2}, \ldots)$. We have $g_1 \simeq g_2$, $h_1 \simeq h_2$ and $g_1 + g_2 = h_1 + h_2 = 1$. Note that $g_1 \sim h_1$ and $g_2 \sim h_2$ in $S$. So, in $R$ we have $\bar{g}_1 \sim \bar{h}_1$ and $\bar{g}_2 \sim \bar{h}_2$. Clearly, $\bar{g}_1, \bar{h}_1 \neq 0$.

Suppose that $\bar{g}_1 \simeq \bar{h}_1$. By Lemma 1.6, there exist orthogonal decompositions $g_1 = g'_1 + g''_1$, $h_1 = h'_1 + h''_1$ such that $g'_1 \simeq h'_1$ and $g''_1, h''_1 \in M$. But $g_{n,1}$ does not have any nonzero subprojection *-equivalent to a subprojection of $h_{n,1}$. We conclude that $g'_1 = h'_1 = 0$, and so $g_1$, $h_1 \in M$ which is a contradiction. So, $\bar{g}_1$ and $\bar{h}_1$ are equivalent but not *-equivalent projections in $R$ and we conclude that $R$ does not have $LP \simeq RP$. \qed

We now consider the special case in which $F$ is chosen to be a formally real number field.
LEMMA 3.7. Let $F$ be a formally real number field and let $e, f$ be two projections in $M_n(F)$. Then, if $e \sim f$, there exist subprojections $e' \leq e$, $f' \leq f$ such that $e' \sim f'$ and $\text{rank}(e - e') < 4$, $\text{rank}(f - f') < 4$.

Proof. If $\text{rank}(e) < 4$, then the result is trivial. If $\text{rank}(e) \geq 4$, set $q = H(e)$. By [15, Thm. XI.1.4] we see that $q$ represents 1 (since $\text{dim } q \geq 4$) and so $q \cong \langle 1 \rangle \perp q'$. Thus, we conclude that we can get a quadratic form $r$ such that $\text{dim } r = 3$ and

$$q = \left( \begin{array}{c} 1, \ldots, 1 \\ \end{array} \right) \perp r.$$ 

This implies that there exists an orthogonal decomposition

$$e = e' + e'' \quad \text{with } e' \sim \text{diag} \left( \begin{array}{c} 1, \ldots, 1, 0, \ldots, 0 \end{array} \right).$$

Similarly,

$$f = f' + f'' \quad \text{with } f' \sim \text{diag} \left( \begin{array}{c} 1, \ldots, 1, 0, \ldots, 0 \end{array} \right).$$

So, $e' \sim f'$ and $\text{rank}(e - e') = \text{rank}(e'') = \text{rank}(f'') = \text{rank}(f - f') = 3$. \hfill \Box

PROPOSITION 3.8. Let $F$ be a formally real number field.

(a) Let $\{ R_i, \Phi_{ji} \}_{i,j \in I}$ be any direct system where each $R_i$ is a matricial $*$-algebra over $F$ (with the identity involution on $F$). Set $R = \lim\limits_{\longrightarrow} R_i$ and let $N$ be a pseudo-rank function on $R$. Then, the type II part of the $N$-completion of $R$ satisfies $LP \sim RP$ matricially.

(b) Set $S = \prod_{i=1}^{\infty} M_{n(i)}(F)$ with $n(1) < n(2) < \cdots$, and let $M$ be any maximal two-sided ideal of $S$ which contains $\bigoplus_{i=1}^{\infty} M_{n(i)}(F)$. Then, the factor ring $S/M$ is a simple, $*$-regular, self-injective ring of type II satisfying $LP \sim RP$ matricially.

Proof. (a) The proof is analogous to that of Theorem 3.5, using Lemma 3.7 adequately.

(b) Set $R = S/M$. By [5, Thm. 10.30], $R$ is a simple, regular, right and left self-injective ring of type II. Also, $R$ is $*$-regular with positive definite involution. It suffices to show that $R$ satisfies $LP \sim RP$.

Let $e, f$ be two nonzero equivalent projections in $R$. By Proposition 1.5, we only have to prove that there exist nonzero subprojections $e' \leq e$, $f' \leq f$ such that $e' \sim f'$. Let $n$ be any integer such that $n \geq 6$. By [5, 10.28] (and a standard argument), there exist $n$ orthogonal equivalent projections $e_1, \ldots, e_n$ in $R$ such that $e = e_1 + \cdots + e_n$. 


Choose equivalent projections \( p, q \in S \) such that \( \bar{p} = e \) and \( \bar{q} = f \). By applying [5, Prop. 2.18] we obtain orthogonal projections \( p_1', \ldots, p_n' \in S \) such that \( p_j' \preceq p \) and \( \bar{p}_j' = e_j \) for \( j = 1, \ldots, n \). By [5, Prop. 2.19] there exist projections \( p_j \preceq p \) such that \( p_j < p \) and \( 
abla p_j = e \) for \( j = 1, \ldots, n \). By \([5, \text{Prop. 2.19}]\) there exist projections \( p_j \preceq p_j \) such that \( p \sim p \) and \( p_j = p_j \) for \( j = 1, \ldots, n \). Since \( \bar{h} \preceq f \) and \( R \) is directly finite, we obtain \( \bar{h} = f \). Summarizing we have \( \bar{g} = e, \bar{h} = f, g \sim h \) and \( g = p_1 + \cdots + p_n \) where the \( p_i \) are equivalent orthogonal projections.

Set \( g = (g_1, g_2, \ldots), h = (h_1, h_2, \ldots) \) where \( g_i, h_i \in \mathcal{P}(M_{n(i)}(F)) \). Note that \( g_i \sim h_i \) in \( M_{n(i)}(F) \) and that each \( g_i \) (and so each \( h_i \)) is the sum of \( n \) equivalent orthogonal projections. By Lemma 3.7 we can choose subprojections \( g_i' \preceq g_i, h_i' \preceq h_i \), for \( i = 1, 2, \ldots \) such that \( g_i' \preceq h_i' \), \( \operatorname{rank}(g_i - g_i') \leq 4 \) and \( \operatorname{rank}(h_i - h_i') \leq 4 \). Set \( g_i'' = g_i - g_i', h_i'' = h_i - h_i' \). Since \( n \geq 6 \) we have \( g_i'' \preceq g_i' \) and \( h_i'' \preceq h_i' \) for \( i = 1, 2, \ldots \). Set \( g' = (g_i'), h' = (h_i'), g'' = (g_i''), h'' = (h_i''). \) We have \( g' \preceq h', g' + g'' = g, h' + h'' = h, g'' \preceq g' \) and \( h'' \preceq h' \). Hence \( \bar{g}' \preceq \bar{h}', \bar{g} \preceq \bar{g} = e \) and \( \bar{h}' \preceq \bar{h} = f \). It only remains to prove that \( g' \notin M \). If \( g' \in M \) then since \( g'' \preceq g' \) we have \( g'' \in M \) and so \( g \in M \) which is a contradiction. Therefore \( g' \neq 0 \) and this completes the proof.

**Example 3.9.** There exists a \(*\)-regular ring such that

(a) The intersection of the maximal two-sided ideals is zero.

(b) For every maximal two-sided ideal \( M \) of \( R \), \( R/M \) satisfies \( \text{LP} \preceq \text{RP} \) matricially, but \( R \) does not satisfy \( \text{LP} \preceq \text{RP} \).

**Proof.** Set \( R = \{ x \in \prod_{n=1}^{\infty} M_n(R) | x_n \in M_n(Q) \text{ for all but finitely many } n \} \). Clearly the intersection of the maximal two-sided ideals of \( R \) is zero. If \( M \) is a maximal two-sided ideal of \( R \) such that \( M \) does not contain the direct sum \( \bigoplus_{n=1}^{\infty} M_n(R) \), then \( R/M \preceq M_m(R) \) for some \( m \) and so \( R/M \) satisfies \( \text{LP} \preceq \text{RP} \) matricially. If \( M \) contains the direct sum \( \bigoplus_{n=1}^{\infty} M_n(R) \) then \( R/M \preceq \prod_{n=1}^{\infty} M_n(Q)/(M \cap \prod_{n=1}^{\infty} M_n(Q)) \) and so, by Proposition 3.8, (b), \( R/M \) satisfies \( \text{LP} \preceq \text{RP} \) matricially. On the other hand it is clear that \( R \) does not satisfy \( \text{LP} \preceq \text{RP} \). \( \Box \)

**References**


Received June 3, 1985. This work was partially supported by CAICYT grant 3556/83.

UNIVERSITAT AUTONOMA DE BARCELONA
BELLATERRA (BARCELONA)
SPAIN
Pere Ara, Matrix rings over \(*\)-regular rings and pseudo-rank functions ........ 209
Lindsay Nathan Childs, Representing classes in the Brauer group of quadratic number rings as smash products .................. 243
Dicesar Lass Fernandez, Vector-valued singular integral operators on $L^p$-spaces with mixed norms and applications .......... 257
Louis M. Friedler, Harold W. Martin and Scott Warner Williams, Paracompact $C$-scattered spaces .................................. 277
Daciberg Lima Gonçalves, Fixed points of $S^1$-fibrations .............. 297
Adolf J. Hildebrand, The divisor function at consecutive integers .......... 307
George Alan Jennings, Lines having contact four with a projective hypersurface .................................................. 321
Tze-Beng Ng, 4-fields on $(4k + 2)$-dimensional manifolds ............... 337
Mei-Chi Shaw, Eigenfunctions of the nonlinear equation $\Delta u + vf(x, u) = 0$ in $R^2$ ...................................................... 349
Roman Svirsky, Maximally resonant potentials subject to $p$-norm constraints .................................................. 357
Lowell G. Sweet and James A. MacDougall, Four-dimensional homogeneous algebras ........................................... 375
William Douglas Withers, Analysis of invariant measures in dynamical systems by Hausdorff measure ................................ 385