FIXED POINTS OF $S^1$-FIBRATIONS

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Let $M$ be a $S^1$-fibration over a space $B$ and $f: M \to M$ a map over $B$. We give some results when $f$ can be deformed over $B$ to a fixed point free map. When the fibration is principal then we compute $\tilde{H}^{n-1}(\text{Fix}(f), k)$ where $n = \dim M$ and we find $g$ homotopic to $f$ over $B$ which minimize the fixed points.

Introduction. In [1] or [2], A. Dold defined a fixed point index for fibre-preserving maps, i.e. for every map $f: U \subset E \to E$ which commutes with the projection $p: E \to B$ he defines an index $I(f)$ s.t. $I(f) \neq 0$ implies that every map $g$ homotopic to $f$ through a fibre-preserving homotopy has at least one fixed point. (We call fibre-preserving homotopy a homotopy over $B$). From [1] one can see that this index is not easy to compute even in the case where the fibration is $S^1 \times S^1 \xrightarrow{p} S^1$,

$p$ is the projection in the first coordinate and $f(x, y) = (x, xy)$. The purpose of this paper is to study the fixed point of a fibre-preserving map $f: M \to M$ where $M$ is a $S^1$-fibration over a space $B$ and $M, B$ are compact manifolds without boundary.

The paper is divided in 3 parts: In Part I we give a criterion, in terms of the fundamental group, for $f$ to be deformed over $B$ to a fixed point free map. This is Proposition 1.3. Some corollaries of this result are given. In Part II we look at orientable $S^1$-fibrations. We give a lower bound for the number of Nielsen classes over $B$ of $f$ as well as the topological dimension of this class. This is Theorem 2.5.

In Part III we state the question of realizing a homotopy class over $B$ by a map $f$ s.t. $\text{Fix}(f) = \{\text{set of fixed points of } f\}$ is minimal in the sense we will describe. We will answer this question in the case where the fibration and the total space are orientable. This is Theorem 3.7.

I would like to thank, Professors A. Dold, D. Sullivan, Drs. C. Biasi and O. Manzoli Neto for the help I had in writing this paper.

Part I. Detecting fixed points. Let

$$
S^1 \to M \\
\downarrow \\
B
$$
be a $S^1$-fibration over $B$ where $M, B$ are compact manifolds without boundary and $f: M \to M$ be a map over $B$ i.e. $p \circ f = p$. For the study of the category of spaces over $B$ see [1] and [2]. By a deformation over $B$ we mean a fibre-preserving homotopy. Let $M \times_B M$ be the fibre square which we denote by $S(M)$ and $\Delta$ the diagonal in $S(M)$. Now we will recall Proposition 2.4 of [3].

**Proposition 1.1.** The map $f$ can be deformed over $B$ to a fixed point free map if and only if there is a map $h: M \to S(M) - \Delta$ which makes the diagram below commutative up to homotopy.

$$
\begin{array}{ccc}
S(M) - \Delta & \xrightarrow{i} & S(M) \\
\downarrow & & \downarrow \\
M & \xrightarrow{(1,f)} & S(M)
\end{array}
$$

In [3], they consider fibrations $F \to M \to B$ where $F$ is a manifold of dimension greater than or equal to three. Under this hypothesis they show that the homotopy fibre of the inclusion $i: M \times M - \Delta \to M \times M$ is at least 1-connected. Therefore by general obstruction theory we can always lift the map $(1, f)$ over the 2-skeleton of $M$ and the obstructions to lift over the higher dimensional skeletons are cohomology classes. On the other hand if $F$ is the circle $S^1$ or a 2-dimensional surface, different from $S^2$ or $RP^2$, the obstructions to lift $(1, f)$ are no longer cohomology classes. In these cases the problem of lifting $(1, f)$ can be treated in terms of $\Pi_1$. The case where $F$ is a 2-dimensional surface is much more complicated than the case where $F = S^1$. We return to the case $F = S^1$.

Let $x_0 \in M$ be a base point of $M$ and let us assume that $f(x_0) \neq x_0$. Denote $(x_0, f(x_0))$ the base point of $S(M) - \Delta$ and $S(M)$.

**Proposition 1.2.** The map $h$ exists if and only if

$$(1, f)_\#(\Pi_1(M, x_0)) = i_\#(\Pi_1(S(M) - \Delta; (x_0, f(x_0))))$$

**Proof.** Let $\overline{M}$ be the covering space of $S(M)$ which corresponds to the subgroup $(1, f)_\#(\Pi_1(M, x_0))$. So we have the commutative diagram:
where $\tilde{f}$ is a lifting of $(1, f)$ which exists by elementary properties of covering spaces.

Now let us assume that

$$(1, f)_#(\Pi_1(M, x_0)) = i_#(\Pi_1(S(M) - \Delta; (x_0, f(x_0)))).$$ 

Then there is a map $j: S(M) - \Delta \to \tilde{M}$ which is a lifting of $i$. By Proposition 2.1. of [3] we have

$$\Pi_i(S(M), S(M) - \Delta) \approx \Pi_i(S^1, S^1 - y_0) = 0, \quad i > 1.$$ 

So $j$ induces isomorphisms in all homotopy groups. Since $S(M) - \Delta$ and $\tilde{M}$ are CW-complexes, there exists $l: \tilde{M} \to S(M) - \Delta$ which is a homotopy inverse of $j$. Take $h = l \circ \tilde{f}$.

Now suppose that $\Lambda$ exists. Since $i \circ h$ is homotopic to $(1, \tilde{f})$, it follows that

$$(1, f)_#(\Pi_1(M, x_0)) \subset i_#(\Pi_1(S(M) - \Delta, (x_0, f(x_0)))).$$ 

Let $p_1: S(M) \to M$ be the projection on the first coordinate. We have the fibration

$$S^1 - \{x_0\} \to S(M) \quad \downarrow \quad M$$ 

Therefore $p_1: \Pi_1(S(M)) \to \Pi_1(M)$ is an isomorphism. Since $p_1 \circ h \approx \text{id}$ we have that $h_#$ is an isomorphism and the equality $(1, f)_#(\Pi_1(M, x_0)) = i_#(\Pi_1(S(M) - \Delta, (x_0, f(x_0))))$ follows.

**Proposition 1.3.** A map $f$ can be deformed over $B$ to a fixed point free map if and only if

$$(1, f)_#(\Pi_1(M, x_0)) = i_#(\Pi_1(S(M) - \Delta, (x_0, f(x_0)))).$$ 

**Proof.** This follows directly from Proposition 1.1. and 1.2.

Let us consider a fibre preserving map $A: M \to M$ whose restriction to each fibre is the antipodal map. Such a map exists because $M \to B$ is a locally trivial $S^1$-fibration. Without loss of generality let us assume that $f(x_0) = A(x_0)$.

**Proposition 1.4.** If $\text{im}(i_#) = \text{im}(1, f)_#$ then $A_# = f_#$. Conversely if $\Pi_1(S^1) \to \Pi_1(M)$ is injective or surjective then $A_# = f_#$ implies $\text{im}(i_#) = \text{im}(1, f)_#$. 


Proof. The map $(1, A) : M \to S(M)$ is a left inverse of $p_1$. Since
$\Pi_1(S(M) - \Delta) \to \Pi_1(M)$ is an isomorphism, (see the proof of Proposition 2.2) it follows that
$$(1, A)_# : \Pi_1(M) \to \Pi_1(S(M) - \Delta)$$
is an isomorphism. Therefore $\text{im}(i_#) = \text{im}((1 \circ (1, A))_#$. But
$$\text{im}(i \circ (1, A))_# = \text{im}(1, f)_#$$is equivalent to $(i \circ (1, A))_#(\alpha) = (1, f)_#(\alpha)$ for every $\alpha \in \Pi_1(M)$. This implies $A_#(\alpha) = f_#(\alpha)$.

For the second part let us assume first that
$$\Pi_1(S^1) \to \Pi_1(M)$$
is injective. From the diagram below

$$\begin{array}{c}
\Pi_1(S^1) \downarrow j_2 \\
\Pi_1(S(M)) \quad p_2 \\
\Pi_1(M) \downarrow p_1
\end{array}$$

we have
$$p_1 \circ (1, A)_#(\alpha) = p_1(1, f)_#(\alpha) \Rightarrow [(1, A)_#(\alpha) - (1, f)_#(\alpha)] = j_2(\beta)$$for some $\beta \in \Pi_1(S^1)$. So
$$p_2[[(1, A)_#(\alpha) - (1, f)_#(\alpha)] = A_#(\alpha) - f_#(\alpha) = j_#(\beta) = 0.$$Since $j_#$ is injective we have $\beta = 0$. Therefore $(1, A)_#(\alpha) = (1, f)_#(\alpha)$ and the result follows. Finally let us assume that $j_# : \Pi_1(S^1) \to \Pi_1(M)$ is surjective. We have the diagram:

$$\begin{array}{c}
S^1 \xrightarrow{(1, A)_{|S^1}} S^1 \times S^1 \\
\downarrow j \\
M \xrightarrow{(1, f)} S(M)
\end{array}$$

From the fact that $f_# = A_#$ and using the long exact sequence in homotopy of the fibration $S^1 \to M \to B$ we have that $(A_{|S^1})_# = (f_{|S^1})_#$. Given $\alpha \in \Pi_1(M)$, there exists $\beta \in \Pi_1(S^1)$ s.t. $j_#(\beta) = \alpha$. So we have
$$(1, A)_#(\alpha) + (1, f)_#(\alpha) = (1, A)_#j_#(\alpha) - (1, f)_#j_#(\alpha)$$
$$= (j_1 \times j_2)_#(1, A_{|S^1})_#(\beta) - (1, f_{|S^1})(\alpha) = 0$$and the result follows.
COROLLARY 1.5. Let $S^1 \to K \to S^1$ be the $S^1$-fibration where $K$ is the Klein bottle. Then the $1_K: K \to K$ $1_K$ = identity map cannot be deformed over $B$ to a fixed point free map.

COROLLARY 1.6. Let

$$f: B \times S^1 \to B \times S^1$$

be a fibre-preserving map. Then $f$ can be deformed over $B$ to a fixed point free map if and only if $f = (1, g)$ where $g: B \times S^1 \to S^1$ is homotopic to $p_2: B \times S^1 \to S^1$ defined by $p_2(x, y) = y$.

Proof. The “if” part is clear. So let us assume that $f$ can be deformed over $B$ to a fixed point free map. By Proposition 1.4. we have $f_\# = A_\#$ and therefore $p_2 \# f_\# = p_2 \# A_\#$ or $(p_2 \circ f)_\# = (p_2 \circ A)_\#$. But this means that

$$(p_2 \circ f)^* = (p_2 \circ A)^*: H^1(S^1) \to H^1(B \times S^1)$$

and consequently $p_2 \circ f$ is homotopic to $p_2 \circ A$ which is homotopic to $p_2$. So the result follows.

REMARK. In general $f_\# = A_\#$ does not imply $\text{im}(1, f)_\# = \text{im}(i)_\#$. We can construct a counter-example with $B = S^1 \times S^1 \times S^2$ and the fibration is the induced fibration from the universal $S^1$-fibration by the map $g: S^1 \times S^1 \times S^2 \to K(Z, 2)$ which is represented by $\alpha_2 \otimes 1 + 2 \otimes \beta_2 \in H^2(S^1 \times S^1 \times S^2)$, $\alpha_2$, $\beta_2$ being generators of $H^2(S^1 \times S^1)$, $H^2(S^2)$ respectively.

Part II. The homology of the fixed points set. We will start by recalling some results of [4].

Let $x, y \in \text{Fix}(f)$, where $f$ is a fibre-preserving map and $S^1 \to M \to B$ is a $S^1$-fibration. We say that $x$ is equivalent to $y$ over $B$ if there is a path $\lambda: [0, 1] \to M$ s.t. $\lambda(0) = x$, $\lambda(1) = y$ and $\lambda$ is homotopic to $f(\lambda)$ rel$\{x, y\}$ over $B$.

DEFINITION 2.1. The equivalence classes are called the Nielsen classes of $f$ over $B$.

PROPOSITION 2.2. If $M$ is compact then the number of Nielsen classes over $B$ is finite.
Proof. This is Lemma 2.1 of [4].

In [4] he also defines essential Nielsen classes and the Nielsen number of $f$. Now we will define a lower bound for the number of non-empty Nielsen classes of $f$ over $B$ for the case of a principal $S^1$-fibration. I believe it would be interesting to compare this number with the Nielsen number as defined in [4].

Let $S^1 \rightarrow M \rightarrow B$ be an orientable $S^1$-fibration and $\Theta: S^1 \times M \rightarrow M$ the $S^1$-action. Given $f: M \rightarrow M$ a fibre-preserving map, there is a map $\theta_f: M \rightarrow S^1$, which satisfies the equation $f(x) = \theta_f(x) \cdot x$, where $\theta_f(x) \cdot x$ means $\Theta(\theta_f(x), x)$.

**Proposition 2.3.** Given an orientable fibration and a map $f$ then $\text{Fix}(f) = \theta_f^{-1}(1)$ where $1 \in S^1$.

**Proof.** Obvious.

Let $i(f)$ denote the number of elements of the group

$$\Pi_1(S^1)/\theta_f\#(\Pi_1(M)).$$

**Proposition 2.4.** If $i(f) = \infty$ then $f$ can be deformed over $B$ to a fixed point free map.

**Proof.** If $i(f) = \infty$ then $\theta_f\#$ is the constant map. Therefore $\theta_f$ is homotopic to the constant map equal to $-1 \in S^1$. Therefore $f$ is homotopic over $B$ to the antipodal map $A$.

**Theorem 2.5.** Let $i(f) = r < \infty$. If $g$ is homotopic to $f$ over $B$ then there exist at least $r$ Nielsen classes $F_1, \ldots, F_r$ such that $\tilde{H}^{n-1}(F_i, K) \neq 0$ (Čech cohomology) where $K$ is $Z$ or $Z_2$ depending on whether $M$ is an orientable or a non-orientable $n$-dimensional compact manifold.

**Proof.** Let us first assume that $r = 1$. Then we have the following commutative diagram

$$
\begin{array}{cccc}
H_1(M) & \rightarrow & H_1(M, M - \text{Fix}(f)) & \rightarrow & H_0(M - \text{Fix}(f)) \\
\downarrow & & \downarrow & & \downarrow \\
H_1(S^1) & \rightarrow & H_1(S^1, S^1 - \{1\}) & \rightarrow & H_0(S^1 - \{1\}) & \rightarrow & \tilde{H}_0(S^1)
\end{array}
$$

where the coefficients are in $Z$ or $Z_2$. Since $H_1(M) \rightarrow H_1(S^1)$ is surjective then $H_1(M, M - \text{Fix}(f)) \neq 0$. So by Poincaré Duality it follows that $\tilde{H}^{n-1}(\text{Fix}(f)) \neq 0$ and the result follows.
Now let \( i(f) = r \) and \( S^1 \to S^1 \) be the \( r \)-fold covering map. There is a lifting \( \overline{\Theta}_f: M \to S^1 \) where

\[
\text{Fix}(f) = \bigcup_{k=0}^{r-1} \Theta_f^{-1}(e^{2\pi ik/r})
\]

and we have that \( \overline{\Theta}_f*: \Pi_1(M) \to \Pi_1(S^1) \) is surjective. By the case \( r = 1 \) we have that

\[
\check{H}^{n-1}(\overline{\Theta}_f^{-1}(e^{2\pi ik'/r})) \neq 0.
\]

It is easy to see that

\[
x \in \overline{\Theta}_f^{-1}(e^{2\pi ik/r}), \quad y \in \Theta_f^{-1}(e^{2\pi ik'/r})
\]

and \( K \neq K' \) then \( x \) and \( y \) do not belong to the same Nielsen class. Therefore the result follows.

**Remark.** (1) The Proposition 2.2. and Theorem 2.5. suggest what should be a function \( g \in [f] \) over \( B \) s.t. \( \text{Fix}(g) \) is minimal. The definition will be given in Part III.

(2) I have not been able to extend the definition of this lower bound for non-orientable fibrations.

**Part III. The realization problem.** From now on let us assume that \( S^1 \to M \to B \) is a principal \( S^1 \)-fibration and \( M \) is a compact orientable \( n \)-manifold. Let \( f: M \to M \) be a fibre-preserving map.

**Definition 3.1.** We say that \( \text{Fix}(g) \) is minimal, where \( g \) is homotopic to \( f \) over \( B \), if \( \text{Fix}(g) \) is an \( n-1 \)-submanifold with \( i(f) \) connected components.

**Proposition 3.2.** Given \( f: M \to M \) we can find \( g \) homotopic to \( f \) over \( B \) such that \( \text{Fix}(g) \) is an \( n-1 \)-submanifold.

**Proof.** Let \( \theta_f: M \to S^1 \) be as defined in Part II. Now we can deform \( \theta_f \) to a map \( \overline{\Theta}: M \to S^1 \) such that \( 1 \in S^1 \) is a regular value. Therefore \( g(x) = \overline{\Theta}(x) \). \( x \) is homotopic to \( f \) over \( B \) and \( \text{Fix}(g) = \overline{\Theta}^{-1}(1) \) is an \( n-1 \)-submanifold.

**Proposition 3.3.** The homology class \([\overline{\Theta}^{-1}(1)]\) represented by the submanifolds \( \overline{\Theta}^{-1}(1) \) is the Poincaré dual of the 1-dimensional cohomology class \( \overline{\Theta}^*(i_1) \) where \( \overline{\Theta}*: H^1(S^1, \mathbb{Z}) \to H^1(M, \mathbb{Z}) \) and \( i_1 \) is the generator of \( H^1(S^1, \mathbb{Z}) \).
Proof. See [6].

Recall that $H^1(M, \mathbb{Z})$ is a free abelian group and $H_{n-1}(M, \mathbb{Z})$ is also a free abelian group by Poincaré Duality.

**Proposition 3.4.** If $\overline{\Theta}_*: \pi_1(M) \to \pi_1(S^1)$ is surjective then $\overline{\Theta}^*(i_1)$ is indivisible.

*Proof.* Let $\overline{\Theta}^*(i_1) = \lambda \alpha$, $\lambda \in \mathbb{R}$, $\alpha \in H^1(M, \mathbb{Z})$.

So
\[
\langle i_1, h\overline{\Theta}_#(x) \rangle = \langle \overline{\Theta}^*(i_1), h(x) \rangle = \langle \lambda \alpha, h(x) \rangle = \lambda \langle \alpha, h(x) \rangle
\]

where $h$ is the Hurewicz homomorphism, $x \in \Pi_1(M)$ and $\langle , \rangle$ is the evaluation. Therefore $\text{im} \overline{\Theta}_# \subset \lambda \cdot \mathbb{Z}$. Since $\overline{\Theta}_#$ is surjective we have $\lambda = 1$ and the result follows.

**Proposition 3.5 (D. Sullivan).** Given an indivisible homology class of $H_{n-1}(M, \mathbb{Z})$ then it can be represented by a connected $n - 1$-submanifold.

*Proof.* See [5] or the appendix.

**Theorem 3.6.** If $\theta_f: \Pi_1(M) \to \Pi_1(S^1)$ is surjective then $f$ can be deformed over $B$ to a map $g$ s.t. $\text{Fix}(g)$ is a connected $n - 1$-submanifold.

*Proof.* Since $\theta_f: \pi_1(M) \to \pi_1(S^1)$ is surjective, by Proposition 3.4. $\theta_f$ defines an $n - 1$-homology class of $M$ which is indivisible. By Proposition 3.5 there is a connected $n - 1$-submanifold $N$ which represents this class. Now let us take a tubular neighborhood of this submanifold. This neighborhood is homeomorphic to $N \times (-\varepsilon, \varepsilon)$. Then we define $\overline{\Theta}: M \to S^1$ such that $\overline{\Theta}^{-1}(1) = N$ and 1 is a regular value of $\overline{\Theta}$. By Proposition 3.3 $\overline{\Theta}$ is homotopic to $\theta_f$ and $g(x) = \overline{\Theta}(x) \cdot x$ is a function such as we are looking for.

Finally the main result.

**Theorem 3.7.** Given $f: M \to M$ there is a map $g$ homotopic to $f$ over $B$ such that $\text{Fix}(f)$ is minimal.

*Proof.* Let $\tilde{\Theta}_f: M \to S^1$ be a lifting of $\theta_f$ i.e. $p_r \circ \tilde{\Theta}_f = \theta_f$ where $p_r$ is the $r$-fold cover of $S^1$. By Theorem 3.6 $\tilde{\Theta}_f$ is homotopic to a map $\tilde{\theta}: M \to S^1$ such that $\tilde{\theta}^{-1}(1)$ is a connected $n - 1$-submanifold. Let $\phi: S^1 \to S^1$ be a diffeomorphism homotopic to the identity which sends the
set \( \{ e^{2\pi iK/r} | K = 0,1,\ldots, r - 1 \} \) into a small neighbourhood of 1 whose points are regular values of \( \tilde{\Theta} \). Let \( \tilde{\Theta} = p_r \circ \phi^{-1} \circ \tilde{\Theta} \). Then \( g(x) = \tilde{\Theta}(x) \cdot x \) is a function such as we are looking for.

**Appendix.** Now let us sketch the proof of Proposition 3.5. (This is due to Prof. D. Sullivan.)

**Proof.** Let \( M \) be a compact orientable manifold of dimension \( n \) and \( N \subset M \) an \( n - 1 \)-compact embedded submanifold. Suppose \( n \geq 3 \) and \( N \) has more than 1 connected component. Call \( N_1, N_2 \) two components. Given \( p \in N_1, q \in N_2 \) there is a path \( \lambda \) in \( M \) such that \( \lambda(0) = p, \lambda(1) = q \) since \( M \) is connected. We can assume that \( \lambda[0,1] \cap N \) is a finite set \( \{ a_1, \ldots, a_t \} \) and \( a_1 = p, a_t = q \). Let \( \lambda \) have the natural orientation. At each point \( a_i \) we have the intersection number of \( \lambda \) and \( N \) which is \( +1, -1 \) or 0. We can assume that the intersection number of \( a_i \) is either \( +1 \) or \( -1 \), otherwise we deform \( \lambda \) in such a way that \( a_i \) is not in the intersection. Now let us suppose that the total intersection number of \( \lambda \) and \( N \) is equal to zero. Then we can find 2 consecutive points, \( a_i, a_{i+1} \) such that one has intersection number \( +1 \) and the other has intersection number \( -1 \). Now we apply surgery, replacing two small discs around \( a_i, a_{i+1} \) by a tube around the arc from \( a_i \) to \( a_{i+1} \). The new manifold represents the same homology class. Since \( m \geq 3 \) the following fact is true: if \( a_i, a_{i+1} \) belong to the same component of \( N \) then the new submanifold has the same number of components as \( N \), otherwise the number of components decreases by one. Because the total intersection number is zero we can continue this process and end up with a submanifold \( N' \) with less components than \( N \). If \( N' \) is not connected we apply the above procedure again until we get a connected submanifold.

Now let me show that it is always possible to connect one point of \( N_1 \) to a point of \( N_2 \) by a curve which has total intersection number zero. Let \( p \in N_1, q \in N_2 \) and \( \lambda \) a curve from \( p \) to \( q \) such that \( \lambda[0,1] \cap N \) is finite. Call \( r \) the intersection number of \( \lambda \) and the submanifold \( N \). Since \( [N] \in H_{m-1}(M,\mathbb{Z}) \) is indivisible by Poincaré Duality we can pass by \( g \) an embedded circle \( \phi: [0,1] \to M \phi(0) = \phi(1) = g \) which has total intersection number \( +1 \) with \( N \). Given a number \( s \) let \( s \cdot \phi = \phi \ast \cdots \ast \phi \) where \( \ast \) is the composition of paths. Let \( T \) be a tubular neighborhood of \( \phi[0,1] \). Since \( M \) is orientable then \( T \cong D^{n-1} \times S^1 \) where \( D^{n-1} \) is the \( n - 1 \)-disc. Now we can deform \( s \phi \) to \( \phi_s \) in such a way that \( \phi_s([0,1]) \) is an embedded circle. Finally let \( \phi'_s \) be a small deformation of \( \phi_s \) such that \( \phi'_s(1) = g' \neq g \) and \( g' \in N_2 \) and near \( g \). Now consider the following curve \( \lambda \ast \phi'_s \). Call \( I_{\lambda}(g) \) and \( I_{\phi_s}(g) \) the intersection numbers of \( g \) as points of \( \lambda \) and \( \phi'_s \).
respectively. If $I_\lambda(g) = -I_{\phi'}(g)$ then let $s = r - I_\lambda(g)$. If $I_\lambda(g) = I_{\phi'}(g)$ let $s = r$. Then we have that the total intersection number of $\lambda \ast \phi'$ is zero.

Now let $m = 2$. The fact that, in this case, an indivisible homology class can be represented by an embedded circle is classical and was known by Poincaré.

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Received April 10, 1985 and in revised form July 31, 1986. Supported by IME-USP and CNPq.

UNIVERSIDADE DE SAO PAULO
SAO PAULO, BRAZIL
Pere Ara, Matrix rings over ∗-regular rings and pseudo-rank functions .... 209
Lindsay Nathan Childs, Representing classes in the Brauer group of quadratic number rings as smash products ......................... 243
Dicesar Lass Fernandez, Vector-valued singular integral operators on $L^p$-spaces with mixed norms and applications ....................... 257
Louis M. Friedler, Harold W. Martin and Scott Warner Williams, Paracompact $C$-scattered spaces ...................................... 277
Daciberg Lima Gonçalves, Fixed points of $S^1$-fibrations ................. 297
Adolf J. Hildebrand, The divisor function at consecutive integers ......... 307
George Alan Jennings, Lines having contact four with a projective hypersurface .......................................................... 321
Tze-Beng Ng, 4-fields on $(4k + 2)$-dimensional manifolds .................. 337
Mei-Chi Shaw, Eigenfunctions of the nonlinear equation $\Delta u + vf(x, u) = 0$ in $R^2$ ........................................................................ 349
Roman Svirsky, Maximally resonant potentials subject to $p$-norm constraints ................................................................. 357
Lowell G. Sweet and James A. MacDougall, Four-dimensional homogeneous algebras .......................................................... 375
William Douglas Withers, Analysis of invariant measures in dynamical systems by Hausdorff measure ..................................... 385