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In this paper we consider the existence of eigenfunctions of the boundary value problem for the nonlinear equation mentioned in the title with vanishing boundary values on bounded planar domains.

Let Ω be a bounded domain in R^2. In this paper we consider the existence of eigenfunctions of the boundary value problem

$$\begin{cases} \Delta u + \nu f(x, u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where f is a continuous function in both x and u variables for all $(x, u) \in \Omega \times R$. We assume that f satisfies the growth condition

$$\begin{cases} f(x,0) = 0 \\ |f(x,u)| \leq A + B|u|^m e^{\alpha u^2} \text{ uniformly in } x \end{cases} \tag{0.2}$$

for some nonnegative constants A, B, m and $\alpha > 0$. We note that $u \equiv 0$ is a trivial solution for (0.1). Let $H_0^1(\Omega)$ denote the completion of the space of compactly supported C^1 functions on Ω under the norm

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \right)^{1/2}.$$

We set $F(x, u) = \int_0^u f(x, s)\, ds$. Our main results are the following:

THEOREM 1. *Let $f(x, u)$ be a continuous function in $(x, u) \in \Omega \times R$ and f satisfies condition* (0.2). *For any $\mu > 0$ such that there exists a $v \in H_0^1(\Omega)$ with $\int_\Omega |\nabla v|^2 = \gamma < (4\pi/\alpha)$ and $\int_\Omega F(x, v) = \mu$, the eigenvalue problem* (0.1) *has a nontrivial eigenfunction u satisfying $\int_\Omega F(x, u) = \mu$.*

If we are interested in positive solutions, a similar theorem applies.

THEOREM 2. *Let $f(x, u)$ be a continuous function in $(x, u) \in \Omega \times R$ that satisfies condition* (0.2) *and the condition*

$$f(x, u) > 0 \quad \text{if } u > 0. \tag{0.2'}$$

For every $\mu > 0$ such that there exists a $v \in H_0^1(\Omega)$ with $\int_\Omega |\nabla v|^2 = \gamma < 4\pi/\alpha$ and $\int_\Omega F(x, |v|) = \mu$, the eigenvalue problem (0.1) *has a positive eigenfunction u, i.e. $u(x) > 0$ for all $x \in \Omega$, and $\int F(x, u) = \mu$.*

Theorems 1 and 2 are related to a question raised in Hempel-Morris-Trudinger [4]. Our approach in proving Theorem 1 is to look for critical points of the functional $\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$ on the surface $S_\mu = \{u \in H_0^1(\Omega) \mid \int_\Omega F(x, u) = \mu\}$. We shall prove that

$$\inf_{u \in S_\mu} \Phi(u) \text{ is achieved} \tag{0.3}$$

if μ satisfies the assumption in the theorem. The major difficulty in the proof is to show that the functional $u \to I(u) = \int_\Omega e^{\alpha u^2}$ is continuous on the subset $G_\varepsilon = \{u \in H_0^1(\Omega) \mid \|u\|^2_{H_0^1(\Omega)} < 4\pi/\alpha - \varepsilon\}$ under weak convergence in $H_0^1(\Omega)$ for small $\varepsilon > 0$. We note that $I(u)$ is not continuous on the whole space $H_0^1(\Omega)$ under weak convergence for any $\alpha > 0$ (see Lemma 2 and the discussion after that). This is in sharp contrast with the case when f satisfies the stronger condition

$$|f(u)| < A + B|u|^m e^{\alpha|u|^p} \tag{0.4}$$

for some constants A, B, m and $p < 2$. In this case (0.3) can be proved easily for all $\mu > 0$ such that $S_\mu \neq \varnothing$, since $H_0^1(\Omega)$ can be embedded into the Orlicz space $L_{\phi^*}(\Omega)$ where $\phi(t) = e^{|t|^p} - 1$, $p \leq 2$, and the embedding is compact if $p < 2$. (See Trudinger [11] or Pohozaev [8] and for Orlicz spaces see Krasnoselskii-Rutickii [6]). Thus if f satisfies (0.4), the functional Φ is C^1 and satisfies the (P.S.) condition. It follows from standard variational method that (0.3) holds. However, when f only satisfies (0.2), Φ does not satisfy the (P.S.) condition. We overcome this difficulty by applying the method of symmetrization used by Moser [7] for proving the sharp form of Trudinger's inequality [11]. The constant $4\pi/\alpha$ stated in the theorems corresponds to the sharp constant in Moser's paper. Some open questions are stated in Remark 2 at the end of this paper.

The lack of (P.S.) condition has also been the major difficulty in many recent works on the solutions of nonlinear elliptic equations involving critical Sobolev exponents (see e.g. Aubin [1], Brezis-Nirenberg [2]). Instead of the approach taken here, these authors used the method of best constants in Sobolev inequality (as in [2]) or best constant in certain isoperimetric inequalities (as in [1]). In fact, our theorems can be viewed as two-dimensional extensions of results for R^n, $n \geq 3$, in [2].

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I. Proof of the theorems. For any functions $u(x) \geq 0$, we associate a symmetrization u^* of u by the requirement

$$m\{x \mid u^* > \rho\} = m\{x \in \Omega \mid u > \rho\} \quad \text{for all } \rho > 0.$$

u^* depends on $|x|$ only and is nonincreasing as a function of $|x|$. Let R be the radius of a ball whose volume is $m(\Omega)$, the Lebesgue measure of the set Ω, i.e.

$$m(\Omega) = \int_{|x| \leq R} dx.$$

We define $\Omega^* = \{x \mid |x| \leq R\}$. Clearly u^* vanishes outside of Ω^*.

The basic results on symmetrization are the following:

$$\int_{\Omega^*} |u^*|^p \, dx = \int_\Omega |u|^p \, dx, \qquad 1 \leq p < \infty, \tag{1.1}$$

$$\int_{\Omega^*} e^{\alpha u^{*2}} \, dx = \int_\Omega e^{\alpha u^2} \, dx, \tag{1.2}$$

$$\int_{\Omega^*} |\nabla u^*|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx. \tag{1.3}$$

Properties (1.1) and (1.2) are trivial while the proof of inequality (1.3) can be found in many articles (e.g. Polya & Szego [**9**], Hilden [**5**]).

To prove our theorems we need the following lemma (see Coron [**3**] also).

LEMMA 1. *If $u_j \to u$ strongly in $L^p(\Omega)$, $1 \leq p < \infty$, then $|u_j|^* \to |u|^*$ a.e. Furthermore, we have $|u_j|^* \to |u|^*$ strongly in $L^p(\Omega^*)$.*

Proof. Let λ_j and λ be the distribution functions for u_j and u respectively, i.e.,

$$\lambda_j(s) = m\{x \in \Omega \mid |u_j(x)| > s\},$$
$$\lambda(s) = m\{x \in \Omega \mid |u(x)| > s\},$$

where m denotes the Lebesgue measure. Then each λ_j and λ are functions continuous on the right (see Lemma 3.4 on p. 189 in Stein-Weiss [**10**]). We shall prove that $\lambda_j(s) \to \lambda(s)$ for each $s > 0$. For any $\varepsilon > 0$, it is easy to see

$$\{x \mid |u_j(x)| > s\} \supset \{x \mid |u(x)| > s + \varepsilon\} \cap \{x \mid |u(x) - u_j(x)| \leq \varepsilon\}.$$

Thus

$$m\{x \mid |u_j(x)| > s\} \geq m\{x \mid |u(x)| > s + \varepsilon\} - m\{m \mid |u(x) - u_j(x)| > \varepsilon\}. \tag{1.4}$$

Since $u_j \to u$ strongly in $L^p(\Omega)$, for any $\delta > 0$, we can choose j sufficiently large such that $m\{x \in \Omega \,|\, |u(x) - u_j(x)| > \varepsilon\} < \delta$. It follows from (1.4) that

$$\lambda_j(s) \geq \lambda(s + \varepsilon) - \delta.$$

Letting $\delta \to 0$, we have

$$\lim_{j\to\infty} \lambda_j(s) \geq \lambda(s + \varepsilon) \quad \text{for every } \varepsilon > 0.$$

By the fact that λ is continuous on the right, we have $\lim_{j\to\infty} \lambda_j(s) \geq \lambda(s)$. Reversing the role of λ_j and λ and repeating the argument above, we have $\lambda(s) \geq \lim_{j\to\infty} \lambda_j(s)$ for any $s > 0$. Therefore we have $\lim \lambda_j(s) = \lambda(s)$ for all $s > 0$. It follows from the definition of symmetrization that $|u_j|^*(x) \to |u|^*(x)$ for every x. From (1.1) we also have

$$\int_{\Omega^*} \left(|u_j|^*\right)^p dx \to \int_{\Omega^*} (|u|^*)^p \, dx.$$

Thus by the Dominated Convergence Theorem, we have $|u_j|^* \to |u|^*$ strongly in $L^p(\Omega^*)$ and the lemma is proved.

LEMMA 2. *If $u_j \to u$ weakly in $H_0^1(\Omega)$ and there exists a constant γ_0 such that $\|u_j\|^2_{H_0^1(\Omega)} \leq \gamma_0 < 4\pi/\alpha$ for every j, then we have*

$$\int_\Omega e^{\alpha u_j^2} \to \int_\Omega e^{\alpha u^2}. \tag{1.5}$$

Proof. We may assume $u_j \geq 0$ and $u \geq 0$ (otherwise we replace u_j and u by $|u_j|$ and $|u|$). Since $u_j \to u$ weakly in $H_0^1(\Omega)$, $u_j \to u$ strongly in L^p for all $1 \leq p < \infty$ by the Sobolev embedding theorem in R^2. It follows from Lemma 1 that $u_j^* \to u^*$ strongly in $L^p(\Omega^*)$ and $u^* \in H_0^1(\Omega^*)$ by (1.3). To prove (1.5) it follows from eq. (1.2) that it suffices to show

$$\int_{\Omega^*} e^{\alpha(u_j^*)^2} \to \int_{\Omega^*} e^{\alpha(u^*)^2}.$$

Using a change of variable t introduced by Moser [7], we set

$$\frac{|x|^2}{R^2} = e^{-t}$$

and

$$w_j(t) = 2\sqrt{\pi}\, u_j^*(x), \qquad w(t) = 2\sqrt{\pi}\, u^*(x).$$

Then $w_j(t)$, $w(t)$ are monotone increasing on $[0, \infty)$ and $w_j(t) \to w(t)$ a.e. as $[0, \infty)$ since $u_j^* \to u^*$ a.e. We also have $w_j(0) = 0$, $\dot{w}_j(t) \geq 0$. Furthermore, we have

$$(1.7) \qquad \int_0^\infty \dot{w}_j^2\, dt = \int_{\Omega^*} |\nabla u_j^*|^2\, dx, \qquad \int_0^\infty \dot{w}^2\, dt = \int_{\Omega^*} |\nabla u^*|^2\, dx,$$

$$(1.8.1) \qquad \int_0^\infty e^{\beta w_j^2 - t}\, dt = \frac{1}{m(\Omega)} \int_{\Omega^*} e^{\alpha (u_j^*)^2}, \qquad \text{where } \beta = \frac{\alpha}{4\pi},$$

$$(1.8.2) \qquad \int_0^\infty e^{\beta w^2 - t}\, dt = \frac{1}{m(\Omega)} \int_{\Omega^*} e^{\alpha (u^*)^2}.$$

It follows from (1.3) and (1.7) that

$$(1.9.1) \qquad \int_0^\infty \dot{w}_j^2\, dt \leq \int_\Omega |\nabla u_j|^2\, dx \leq \gamma_0.$$

By the lower semi-continuity for the norm in $H_0^1(\Omega)$ under weak convergence, we have

$$(1.9.2) \qquad \int_0^\infty |\dot{w}|^2\, dt \leq \int_\Omega |\nabla u|^2 \leq \varliminf_{j \to \infty} \int |\nabla u_j|^2 \leq \gamma_0.$$

From (1.9.1) and Hölder's inequality,

$$w_j(t) = \int_0^t w_j\, dt \leq \sqrt{t} \left(\int_0^t \dot{w}_j^2\, dt \right)^{1/2} \leq \sqrt{\gamma_0 t}\,.$$

Since $w_j(t) \to w(t)$ a.e. we get

$$w(t) \leq \sqrt{\gamma_0 t}\,.$$

Thus we have

$$(1.10.1) \qquad e^{\beta w_j^2 - t} \leq e^{(\beta\gamma_0 - 1)t},$$

$$(1.10.2) \qquad e^{\beta w^2 - t} \leq e^{(\beta\gamma_0 - 1)t},$$

and $\int_0^\infty e^{(\beta\gamma_0 - 1)t}\, dt = 1/(1 - \beta\gamma_0) < \infty$ since $\beta\gamma_0 < 1$. From (1.10.1) and (1.10.2) and the Dominated Convergence Theorem, we have

$$\int_0^\infty e^{\beta w_j^2 - t}\, dt \to \int_0^\infty e^{\beta w^2 - t}\, dt.$$

Thus from (1.8.1) and (1.8.2) we have proved (1.6) and the lemma also.

We note that by modifying an example of Moser [7], one can construct a sequence u_j, $\|u_j\|^2_{H_0^1(\Omega)} = C > 4\pi/\alpha$, $u_j \to u$ weakly in $H_0^1(\Omega)$ but $\int_\Omega e^{\alpha u_j^2}$ does not converge to $\int_\Omega e^{\alpha u^2}$. In fact, let $g(s) = \min(s, 1)$ and $w_n = \sqrt{Cn}\, g(t/n)$, then $\|u_n\|^2_{H_0^1(\Omega)} = \int_0^\infty \dot{w}_n^2\, dt = C$ and

$$\int_\Omega e^{\alpha u_n^2} = \int_0^\infty e^{\beta w_n^2 - t} > \int_0^\infty e^{C\beta n - t}\, dt = e^{(C\beta - 1)n}.$$

Thus $\int_\Omega e^{\alpha u_n^2} \to \infty$. But u_n has a subsequence which converges to some function $u \in H_0^1(\Omega)$ weakly and $\int_\Omega e^{\alpha u^2} < \infty$. It is unknown if one can improve Lemma 2 to include the case when $\gamma_0 = 4\pi/\alpha$.

LEMMA 3. *Let $\Psi(u) = \int_\Omega F(x,u)$. If $u_j \to u$ weakly in $H_0^1(\Omega)$ and there exists a constant γ_0 such that $\|u_j\|^2_{H_0^1(\Omega)} \le \gamma_0 < 4\pi/\alpha$ for every j, then we have*

$$\Psi(u_j) \to \Psi(u), \tag{1.10}$$

$$\Psi'(u_j) \to \Psi'(u) \quad \text{in } H_0^1(\Omega)^*, \tag{1.11}$$

i.e., for every $v \in H_0^1(\Omega)$, we have

$$\int_\Omega f(x,u_j)v \to \int_\Omega f(x,u)v. \tag{1.12}$$

Proof. By (0.2), we have

$$\begin{aligned}|F(x,u)| &= \left|\int_0^u f(x,s)\,dx\right| \\ &\le A|u| + B\int_0^u s^m\left(1 + \alpha s^2 + \frac{(\alpha s^2)^2}{2!} + \frac{(\alpha s^2)^3}{3!} + \cdots\right) ds \\ &\le A|u| + B|u|^{m+1}e^{\alpha u^2}.\end{aligned}$$

For any $\varepsilon > 0$, there exists a constant C such that $|u|^{m+1} \le C^1 e^{\varepsilon u^2}$ for all u. Thus $|F(x,u)| \le C_1 e^{(\alpha+\varepsilon)u^2}$ for some $C_1 > 0$. Choosing ε so small such that $\gamma_0 < 4\pi/(\alpha+\varepsilon)$, then from Lemma 2, we have

$$\int_\Omega e^{(\alpha+\varepsilon)u_j^2} \to \int_\Omega e^{(\alpha+\varepsilon)u^2}.$$

By the Dominated Convergence Theorem, we have proved (1.10).

To prove (1.11), we note that for any nonnegative numbers α, β and q, $p > 1$ such that $1/q + 1/p = 1$, we have $\alpha\beta \le (1/q)\alpha^q + (1/p)\beta^p$. Thus

$$|f(x,u_j)v| \le \frac{1}{q}|f(x,u_j)|^q + \frac{1}{p}|v|^p.$$

From (0.2), for any $\varepsilon = 0$, there exists a constant C_2 such that

$$|f(x,s)|^q < C_2 e^{q(\alpha+\varepsilon)s^2}.$$

We choose q close to one and ε so small such that $\gamma_0 < 4\pi/q(\alpha+\varepsilon)$. It follows from Lemma 2 and the Dominated Convergence Theorem

$$\int_\Omega |f(x,u_j)|^q \to \int_\Omega |f(x,u)|^q.$$

Since $\int_\Omega |v|^p < \infty$ for all $p < \infty$ by the Sobolev embedding theorem in R^2, (1.11) follows from (1.12) and the Dominated Convergence Theorem and the lemma is proved.

Proof of Theorem 1. We can finish the proof of the theorem easily. Let

$$c_0 = \inf_{u \in S_\mu} \frac{1}{2} \int_\Omega |\nabla u|^2.$$

By assumption $S_\mu \neq \varnothing$ and $2c_0 < 4\pi/\alpha$. There exists a minimizing sequence $u_j \in H_0^1(\Omega)$ such that $\int |\nabla u_j|^2 \to 2c_0$. We may assume there exists a constant γ_0, $2c_0 < \gamma_0 < 4\pi/\alpha$ such that $\|u_j\|^2_{H_0^1(\Omega)} = \int |\nabla u_j|^2 < \gamma_0$ for all j. Thus we can find a $\tilde{u} \in H_0^1(\Omega)$ such that a subsequence of u_j, still denoted by u_j, converges to $\tilde{u}$ weakly in $H_0^1(\Omega)$. From Lemma 2 we have $\int_\Omega F(x, u_j) \to \int_\Omega F(x, \tilde{u})$. Thus $\tilde{u} \in S_\mu$ and $\tilde{u} \not\equiv 0$ since $f(x,0) = 0$. From the lower semicontinuity of Φ under weak convergence in $H_0^1(\Omega)$, we have $\Phi(\tilde{u}) \leq \underline{\lim}_{j \to \infty} \Phi(u_j) = c_0$, which implies $\Phi(\tilde{u}) = c_0$ and (0.3) is established. By Lemma 3 we have that Φ is a C^1 function on the subset

$$G = \left\{ u \in H_0^1(\Omega) \,|\, \|u\|^2_{H_0^1(\Omega)} < \frac{4\pi}{\alpha} \right\}$$

under the strong convergence in $H_0^1(\Omega)$. It is trivial to check that Φ is C^1 on $H_0^1(\Omega)$. Thus $\tilde{u}$ is a critical point of functional Φ under the constraint S_μ. It follows from standard variational argument that one can find a Lagrange multiplier ν such that $\tilde{u}$ satisfies the equation

$$\Delta \tilde{u} + \nu f(x, \tilde{u}) = 0$$

in the weak sense and Theorem 1 is proved.

Proof of Theorem 2. Let $u^+ = \max(u, 0)$ and consider the set $\tilde{S}_\mu = \{u \in H_0^1(\Omega) \,|\, \int_\Omega F(x, u^+) = \mu\}$. Then by assumption in Theorem 2, $\tilde{S}_\mu \neq \varnothing$. Considering the functional $\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$ restricted to the set $\tilde{S}_\mu$. Then the same arguments as in the proof of Theorem 1 will show that $\inf_{u \in \tilde{S}_\mu} \Phi(u)$ is achieved by a function $\tilde{u} \in \tilde{S}_\mu$ and $\tilde{u} \not\equiv 0$. There exists a $\nu \neq 0$ such that $\tilde{u}$ satisfies

$$\Delta \tilde{u} + \nu f(x, \tilde{u}^+) = 0. \tag{1.13}$$

Integrating (1.13) with $\tilde{u}$, we have from (0.2)

$$\frac{1}{\nu} \int_\Omega |\nabla \tilde{u}|^2 = \int_\Omega f(x, \tilde{u}^+)\tilde{u}^+ > 0.$$

Thus $\nu > 0$ and $\Delta \tilde{u} \leq 0$. From the maximum principle, we have $\tilde{u} \geq 0$ on Ω and $\tilde{u}$ satisfies

$$\Delta \tilde{u} + \nu f(x, \tilde{u}) = 0.$$

Since $\tilde{u} \not\equiv 0$, by the strong maximum principle, $\tilde{u} > 0$ on Ω and the Theorem is proved.

REMARK 1. Regularity of the solution: the function $\tilde{u}$ obtained above is in $H_0^1(\Omega)$, but by a standard boot strap argument one shows that it is smooth in the interior and up to the boundary (as $\partial\Omega$ permits).

REMARK 2. Open questions: since Moser's sharp result includes the constant $\gamma = 4\pi/\alpha$, it may be possible to improve Theorems 1 and 2 to include the case when $\gamma = 4\pi/\alpha$. Trudinger [**11**] has also pointed out that a nonexistence result similar to the Pohozaev theorem may also hold for f which does not satisfy condition (0.2).

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