CONTINUATION OF BOUNDED HOLOMORPHIC
FUNCTIONS FROM CERTAIN SUBVARIETIES
TO WEAKLY PSEUDOCONVEX DOMAINS

KENZÔ ADACHI

Let \( D \) be a weakly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^\infty \)-boundary
and \( V \) be a subvariety in \( D \) which intersects \( \partial D \) transversally. If \( \partial V \) is
nonsingular and consists of strictly pseudoconvex boundary points of \( D \),
then any bounded holomorphic function in \( V \) can be extended to a
bounded holomorphic function in \( D \).

1. Introduction. Let \( \Omega \) be an open set in some complex manifold. We
denote by \( H^\infty (\Omega) \) the space of all bounded holomorphic functions in \( \Omega \)
and by \( A(\Omega) \) the space of all holomorphic functions in \( \Omega \) which are
continuous in \( \overline{\Omega} \). Let \( G \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \)
with \( C^2 \)-boundary and \( M \) be a submanifold in a neighborhood of \( \overline{G} \)
which intersects \( \partial G \) transversally. Let \( M = \overline{M} \cap G \). Then Henkin [5]
proved the following.

**Fundamental Theorem.** There exists a continuous linear operator
\[ E : H^\infty (M) \rightarrow H^\infty (G) \]
satisfying \( Ef \mid _M = f \).
Moreover \( Ef \in A(G) \) if \( f \in A(M) \).

In the present paper we shall extend the above results to the weakly
pseudoconvex case. Let \( D \) be a bounded weakly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^\infty \)-boundary. Let \( \tilde{V} \) be a subvariety in a neighborhood \( \tilde{D} \) of \( \overline{D} \)
which intersects \( \partial D \) transversally. Let \( V = \tilde{V} \cap D \) and \( D = \{ z \in \tilde{D} : \rho (z) < 0 \} \). Suppose that \( \tilde{V} \) is written in the form
\[ \tilde{V} = \{ z \in \tilde{D} : h_1(z) = \cdots = h_p(z) = 0 \}, \]
where \( h_1, \ldots, h_p \) are holomorphic in \( \tilde{D} \) and \( \partial h_1 \wedge \cdots \wedge \partial h_p \neq 0 \) on
\( \partial D \cap \tilde{V} \). In addition, we assume that \( \partial V \) consists of strictly pseudoconvex
boundary points of \( D \). In this setting we shall show the following:

**Theorem 1.** There exists a continuous linear operator
\[ E : H^\infty (V) \rightarrow H^\infty (D) \]
satisfying \( Ef \mid _V = f \).
Moreover \( Ef \in A(D) \) if \( f \in A(V) \).
In the case when \( p = 1 \), the above theorem is nothing but the result of Adachi [1].

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2. Some results. Let

\[ s(\xi, z) = (s_1(\xi, z), \ldots, s_n(\xi, z)) : \partial D \times D \to \mathbb{C}^n \]

be a \( C^\infty \) function that satisfies

\[ \langle s, \xi - z \rangle = \sum_{j=1}^{n} s_j(\xi_j - z_j) \neq 0 \quad \text{for} \quad (\xi, z) \in \partial D \times D. \]

Then Hatziafratis [3] proved the following theorem.

**Theorem 2.** For \( f \in A(V) \) and \( z \in V \) we have the integral formula

\[ f(z) = \int_{\partial V} f(\xi) \frac{K(\xi, z)}{\langle s, \xi - z \rangle^{n-p}} \]

where \( K(\xi, z) \) is a \( C^\infty(n-p, n-p-1) \)-form on \( \partial D \times D \). Moreover, if \( s_1(\xi, z), \ldots, s_n(\xi, z) \) are holomorphic in \( z \), then \( K(\xi, z) \) is also holomorphic in \( z \).

Let \( G \) be a bounded strictly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^\infty \) boundary. According to the construction of Henkin [4], there exist a neighborhood \( U \) of \( \bar{G} \), a neighborhood \( V \) of \( \partial G \), and a \( C^\infty \) function \( \Phi : V \times U \to \mathbb{C} \) such that for each \( \xi \in V \), \( \Phi(\xi, z) \) is holomorphic in \( U \) and such that \( \Phi(\xi, z) = 0 \) implies \( \xi = z \). Moreover, \( \Phi \) admits a division

\[ \Phi(\xi, z) = \sum_{j=1}^{n} P_j(\xi, z)(\xi_j - z_j) \]

where \( P_j : V \times U \to \mathbb{C} \) of class \( C^\infty \) and holomorphic in the second variable. In addition, if we set

\[ T(\xi, z) = 2 \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(\xi)(z_i - \xi_i) \]

\[ + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(\xi)(z_i - \xi_i)(z_j - \xi_j) \]
then there exists a positive constant $r$ such that

$$\Phi(\xi, z) = T(\xi, z)G(\xi, z) \quad \text{for } \{(\xi, z) \in V \times U : |\xi - z| < r\} = S_r,$$

where $G(\xi, z)$ is a non-vanishing $C^\infty$ function in $S_r$.

Now we have the following proposition using the techniques of the proof of Fornaess Imbedding theorem [2].

**Proposition 1.** Let $D$ be a bounded weakly pseudoconvex domain in $C^n$ with $C^\infty$ boundary. Let $K$ be a compact subset of $\partial D$ and consist of strictly pseudoconvex boundary points of $D$. Then there exists a strictly pseudoconvex domain $\hat{D}$ in $C^n$ with $C^\infty$ boundary such that $\hat{D} \supset D$ and $\partial \hat{D}$ coincides with $\partial D$ near $K$.

In view of Proposition 1, if we can get the extension $F$ to $\hat{D}$, then $F|_D$ is the required function. Therefore we may assume that $D$ is a strictly pseudoconvex domain. Let $\{\varepsilon_n\}$ be a sequence of positive numbers which converges to 0. Let $D_\varepsilon = \{z \in D : \rho(z) < -\varepsilon\}$, $V_\varepsilon = V \cap D_\varepsilon$, and $n - p = k$. If $f \in H^\infty(V)$, then by Hatziafratis [3], we have, for large $\nu$ and $z \in V_\nu$,

$$f(z) = \int_{\partial V_\nu} f(\xi) \frac{K(\xi, z)}{\Phi(\xi, z)}$$

where $K(\xi, z)$ is a $C^\infty(k, k - 1)$-form depending holomorphically on $z$. We set for $z \in D_\varepsilon$

$$H_\nu(z) = \int_{\partial V_\nu} f(\xi) \frac{K(\xi, z)}{\Phi(\xi, z)^k}.$$

Then we have the following proposition which is proved by the same argument as the proof of lemma 1 in Adachi [1].

**Proposition 2.** For $z \in \overline{D} \setminus \partial V$, $H(z) = \lim_{\nu \to \infty} H_\nu(z)$ exists. $H(z)$ is holomorphic in $D$ and $H(z) = f(z)$ for $z \in V$.

Let $z^0 \in \partial V$ and $S_{z^0, \sigma} = \{z : |z - z^0| < \sigma\}$. Then there exist a constant $\sigma_1 > 0$ and a biholomorphic change of coordinates on a neighborhood of $z^0$ such that $\rho$ is strictly convex in a neighborhood of

$$\overline{D} \cap S_{z^0, \sigma_1}, \quad V \cap S_{z^0, \sigma_1} = \{z \in S_{z^0, \sigma_1} : z_{p+1} = \cdots = z_n = 0\},$$
and \((\partial \rho / \partial z_i)(z^0) \neq 0\) for some \(i\) \((1 \leq i \leq p)\). Without loss of generality we may assume that \((\partial \rho / \partial z_1)(z^0) \neq 0\). Let \(z \in S_{z_0, \sigma_1}\). We consider the system of equations for \(\xi^0 = (\xi^0_1, \ldots, \xi^0_n)\) of the following form:

\[
\begin{cases}
\sum_{i=1}^{n} \frac{\partial \rho}{\partial \xi_i}(\xi^0)(\xi^0_i - z_i) = 0, \\
(\xi^0_1 = z_1, \quad i = 2, \ldots, p), \quad \xi^0_{p+1} = \cdots = \xi^0_n = 0.
\end{cases}
\]

**Lemma 1.** There exist positive constants \(\sigma_2\) \((< \sigma_1)\), \(\gamma_1\) and \(\gamma_2\), depending only on \(D\) and \(V\), such that for any \(\sigma \leq \sigma_2\) and any \(z \in S_{z_0, \sigma_2}\) there exists a unique solution \(\xi^0 = \xi^0(z)\) of the system (1) which belongs to the set \(S_{z_0, \sigma} \cap \tilde{V}\). Here the point \(\xi^0 = \xi^0(z)\) has the following properties:

1. \(|z - \xi^0|^2 \leq (\rho(z) - \rho(\xi^0))/\gamma_1\)
2. \(|z - \xi^0|^2 \geq |z_{p+1}|^2 + \cdots + |z_n|^2 \geq \gamma_2[\rho(z) - \rho(\xi^0)].

**Proof.** From the system (1), we have

\[
\xi_1 = z_1 - \sum_{i=p+1}^{n} \frac{\partial \rho}{\partial z_i}(\xi) \left(\frac{\partial \rho(\xi)}{\partial z_1}\right)^{-1}.
\]

We set

\[
a_i(\xi) = -\frac{\partial \rho}{\partial z_i}(\xi) \left(\frac{\partial \rho(\xi)}{\partial z_1}\right)^{-1},
\]

then \(a_i(\xi)\) is \(C^\infty\) in a neighborhood of \(z^0\). We set by recurrence that

\[
\xi^{(1)}_1 = z_1, \quad \xi^{(j)} = (\xi^{(j)}_1, z_2, \ldots, z_p, 0, \ldots, 0)
\]

\[
\xi^{(j)}_1 = z_1 + \sum_{i=p+1}^{n} a_i(\xi^{(j-1)}) z_i.
\]

Then

\[
|\xi^{(j)}_1 - \xi^{(j-1)}_1| \leq \sum_{i=p+1}^{n} |\nabla a_i| |\xi^{(j-1)}_1 - \xi^{(j-2)}_1| |z_i|
\]

\[
\leq \frac{1}{2} |\xi^{(j-1)}_1 - \xi^{(j-2)}_1|.
\]

Therefore \(\{\xi^{(j)}\}\) converges. \(\lim_{j \to \infty} \xi^{(j)} = \xi^0\) is the solution of the system (1). The strict convexity of the function \(\rho\) and the equation (1) imply

1. \(\rho(\xi^0) - \rho(z) + \gamma_1|\xi^0 - z|^2 \leq 0,\)
2. \(\rho(\xi^0) - \rho(z) + \gamma_2|\xi^0 - z|^2 \geq 0.\)
From the inequality (4), we have the inequality (2). From the system (1), we have
\[ |\xi^0 - z|^2 = |z_{p+1}|^2 + \cdots + |z_n|^2 + |\xi^0 - z_1|^2 \]
\[ \leq |z_{p+1}|^2 + \cdots + |z_n|^2 + \left( \sum_{i=p+1}^n |a_i(\xi^0)| |z_i| \right)^2 \]
\[ \leq \gamma_2'' \left( |z_{p+1}|^2 + \cdots + |z_n|^2 \right). \]
Together with the inequality (5), we have
\[ |\xi^0 - z|^2 \geq |z_{p+1}|^2 + \cdots + |z_n|^2 \geq \gamma_2 (\rho(z) - \rho(\xi^0)). \]
This completes the proof of Lemma 1.

3. Proof of Theorem 1. At first we prove that if \( f \in H^\infty(V) \), then \( H(z) \in H^\infty(D) \). Let \( z \in S_{\sigma, \sigma_1} \cap D_p \). We set
\[ \tilde{H}_\nu(z) = \int_{\partial V \cap S_{\sigma, \sigma_1}} \frac{f(\xi) K(\xi, z)}{\Phi(\xi, z)^k}. \]
It is sufficient to show that
\[ |\tilde{H}_\nu(z)| \leq \gamma_3 \sup_{\xi \in V} |f(\xi)|. \]

**Lemma 2.** Let \( f(z) \in H^\infty(V) \). Then for any point \( z^0 \in \partial V \) and any point \( z \in \partial(S_{\sigma, \sigma_1} \cap D_p) \cap \partial V, (\sigma < \sigma_2/2) \), we have
\[ \left| \frac{d\tilde{H}_\nu(\xi^0 + \lambda(z - \xi^0))}{d\lambda} \right|_{\lambda=1} \leq \gamma_4 \sup_{\xi \in V} |f(\xi)|. \]

**Proof of Lemma 2.** We set \( \varepsilon = (|z_{p+1}|^2 + \cdots + |z_n|^2)^{1/2} \), where \( z = (z_1, \ldots, z_n) \in \partial(S_{\sigma, \sigma_1} \cap D_p) \cap \partial V \). By Lemma 1, we have
\[ \varepsilon \leq |\xi^0 - z| \leq \left( \frac{\rho(z) - \rho(\xi^0)}{\gamma_1} \right)^{1/2} \leq \frac{\varepsilon}{(\gamma_1 \gamma_2)^{1/2}}. \]
Since
\[ \sum_{i=1}^n \frac{\partial \rho}{\partial \xi_i}(\xi^0)(\xi^0_i - z_i) = 0, \]
it follows that
\[
\left| \sum_{i=1}^{n} \frac{\partial \Phi}{\partial z_i}(\zeta, z)(\xi_i^0 - z_i) \right| \leq \gamma_5 \varepsilon(|\zeta - z| + \varepsilon).
\]

On the other hand, we have
\[
\frac{d\tilde{H}_\nu(z + \lambda w)}{d\lambda} \mid_{\lambda=0} = \int_{\partial \mathcal{V}_\nu \cap S_{\nu, \omega}} \frac{f(\zeta) \sum_{j=1}^{n} \frac{\partial K}{\partial z_j}(\zeta, z)w_j}{\Phi(\zeta, z)^k} \Phi(\zeta, z)^{k+1}
\]

Therefore we have
\[
\left| \frac{d\tilde{H}_\nu(\xi^0 + \lambda(z - \xi^0))}{d\lambda} \right| \mid_{\lambda=1} \leq \gamma_6 \int_{\partial \mathcal{V}_\nu \cap S_{\nu, \omega}} |f(\zeta)| d\lambda \frac{\varepsilon \left| f(\zeta) \right|}{\Phi(\zeta, z)^k}
\]
\[
+ \gamma_7 \int_{\partial \mathcal{V}_\nu \cap S_{\nu, \omega}} \frac{|f(\zeta)| \varepsilon(|\zeta - z| + \varepsilon)}{\Phi(\zeta, z)^{k+1}} d\lambda.
\]

where \(d\lambda\) is surface measure on \(\partial \mathcal{V}_\nu\). We can choose coordinates \((\eta_1(\zeta), \ldots, \eta_n(\zeta))\) in \(S_{\nu, \omega}\) such that
\[
\eta_1(\zeta) = \rho(\zeta) - \rho(z) + i \text{Im} \Phi(\zeta, z).
\]

Then we have
\[
|\Phi(\zeta, z)| \geq \gamma_8 \left[ (t_1 + |\zeta - z|^2 + t_2^2)^{1/2} \right].
\]

By the estimates of Henkin [5], we have
\[
\left| \frac{d\tilde{H}_\nu(\xi^0 + \lambda(z - \xi^0))}{d\lambda} \right| \mid_{\lambda=1} \leq \gamma_9 \sup_{\xi \in \mathcal{V}} |f(\zeta)|.
\]

This completes the proof of Lemma 2.

By the same method as the proof of Henkin [5] (cf. Adachi [1]), we can prove that
\[
\sup_{z \in D} |H(z)| \leq \gamma_{10} \sup_{\xi \in \mathcal{V}} |f(\zeta)|.
\]
The next step is to show that if \( f \in A(V) \), the \( H(z) \in A(D) \). In order to prove this statement, we need the following modified version of N. Kerzman [6]. In the Theorem 1.4.1' of Kerzman, \( V \) is a manifold. But the proof of the theorem is applicable in our case.

**Proposition 3.** Let \( f \in A(V) \). Then there exists a sequence \( \{ f_k \} \) of holomorphic functions in a neighborhood of \( \bar{V} \) in \( \hat{V} \) such that \( \| f_k - f \|_V \to 0 \) when \( k \to \infty \).

From Proposition 3 we can suppose that \( f \) is holomorphic in \( \bar{V}' \) (\( V \subset V' \subset \bar{V}' \subset \hat{V} \)). Let \( z^0 \in \partial V \) and let \( z \in S_{z^0, \sigma/2} \cap (\overline{D}_{\sigma} \mid \partial V_{\sigma}) \). By using Stokes' formula, we have

\[
H_{\sigma}(z) = \int_{\partial V} \frac{f(\xi)K(\xi, z)}{\Phi(\xi, z)^k} \, d\xi - \int_{(V' - V) \cap S_{\sigma, \sigma/2}} f(\xi) \frac{\partial}{\partial z} \left( \frac{K(\xi, z)}{\Phi(\xi, z)^k} \right) \, d\xi - \int_{(V' - V) \cap S_{\sigma, \sigma/2}} f(\xi) \frac{\partial}{\partial z} \left( \frac{K(\xi, z)}{\Phi(\xi, z)^k} \right) \, d\sigma.
\]

Therefore it is sufficient to show that

\[
F_{\sigma}(z) = \int_{(V' - V) \cap S_{\sigma, \sigma/2}} f(\xi) \frac{\partial}{\partial z} \left( \frac{K(\xi, z)}{\Phi(\xi, z)^k} \right) \, d\sigma
\]

is continuous at \( z^0 \). In order to prove this fact, we need the following.

**Lemma 3.** Let \( z \in S_{z^0, \sigma/2} \cap (\overline{D}_{\sigma} \mid \partial V_{\sigma}) \). Then it follows that

\[
\left| \frac{dF_{\sigma}(\xi^0 + \lambda(z - \xi^0))}{d\lambda} \right|_{\lambda=1} \leq C_{11} |\log \xi| \sup_{\xi \in V} |f(\xi)|,
\]

where \( \xi^0 = \xi^0(z) \) is the solution of the system (1).

**Proof of Lemma 3.** We can write

\[
F_{\sigma}(z) = \int_{(V' - V) \cap S_{\sigma, \sigma/2}} f(\xi) \frac{A(\xi, z)}{\Phi(\xi, z)^k}
\]

\[
+ \int_{(V' - V) \cap S_{\sigma, \sigma/2}} f(\xi) \sum_{j=1}^{n} (\xi_j - z_j) B_j(\xi, z) \frac{\Phi(\xi, z)^{k+1}}{\Phi(\xi, z)^{k+1}}
\]
where \( A(\xi, z) \) and \( B_j(\xi, z) \) are \((k, k)\)-forms which are smooth in \((\xi, z)\) and holomorphic in \(z\). Therefore
\[
\left| \frac{dF_v(\xi^0 + \lambda(z - \xi^0))}{d\lambda} \right|_{\lambda = 1}
\leq \gamma_{12} \int_{(V' - V_0) \cap S^0_{\delta, 2\sigma}} |f(\xi)| \frac{\varepsilon d\lambda}{|\Phi(\xi, z)|^{k+1}}
+ \gamma_{13} \int_{(V' - V_0) \cap S^0_{\delta, 2\sigma}} |f(\xi)| \frac{|\xi - z| \varepsilon (|\xi - z| + \varepsilon) d\lambda}{|\Phi(\xi, z)|^{k+2}}.
\]

By applying the estimates of Henkin [5], we have the inequality (6). This completes the proof of Lemma 3.

Using the method of Henkin [5], we can prove
\[
|F_v(z) - F_v(z^0)| \leq \gamma_{14} |\log \sigma| \sup_{\xi \in V'} |f(\xi)| + \sigma \sup_{\xi \in V'} |\text{grad} f(\xi)|.
\]
Therefore \( F_v(z) \) is continuous at \( z^0 \). This completes the proof of Theorem 1.

**References**


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