SPACES OF SECTIONS OF EILENBERG-MAC LANE FIBRATIONS

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We show first that the space of sections of a fibration with an Eilenberg-Mac Lane space as fibre has the weak homotopy type of a product of Eilenberg-Mac Lane spaces. Secondly, mapping spaces with twisted Eilenberg-Mac Lane spaces as targets are shown to be generalized twisted Eilenberg-Mac Lane spaces.

1. Introduction. Let \( p: Y \to B \) be a (Serre) fibration, \( i: A \to X \) a cofibration and \( u: X \to Y \) a (continuous) map. Using Switzer's notation from [14], let

\[
F_u(X, A; Y, B)
\]

be the space of all maps \( f: X \to Y \) such that \( f \circ i = u \circ i \) and \( p \circ f = p \circ u \). In other words, \( F_u(X, A; Y, B) \) is the solution space for the lifting extension problem

\[
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow & & \downarrow \text{pu} \\
Y & \xrightarrow{u} & B
\end{array}
\]

with data \( u|A: A \to Y \) and \( \text{pu}: X \to B \).

We shall be concerned with decompositions of \( F_u(X, A; Y, B) \) when \( p: Y \to B \) has an Eilenberg-Mac Lane space as fibre. Suppose for instance that \( p: K(G, n) \to * \) is the trivial fibration mapping an Eilenberg-Mac Lane space onto a point. Then

\[
F_u(X, \emptyset; K(G, n), *) = \prod_{i=0}^{n} K(H^{n-i}(X; G), i)
\]

by Haefliger's sharpened version [7] of a theorem of Thom [15] and independently Federer [4]. The main purpose of this paper is to establish a twisted version of (*).

2. Preliminaries. We shall work in the category of compactly generated spaces. For any two compactly generated spaces \( X \) and \( Y \), we let \( X \times Y \) and \( F(X; Y) \) denote the compactly generated spaces associated to
the Cartesian product of $X$ and $Y$ and the space of maps of $X$ into $Y$ with the compact-open topology, respectively. These constructions assure the continuity of the evaluation map $e: F(X; Y) \times X \to Y$ and the validity of the Exponential Law ([16], pp. 17–21) and thus eliminate the difficulties with the topology of function spaces as pointed out by Thom in the first paragraphs of [15].

Throughout this paper we let $(X, A)$ denote an NDR-pair ([16], p. 22) with $X$ 0-connected and $p: Y \to B$ a fibration with 0-connected base space $B$. Then $F_u(X, A; Y, B)$ is a closed subset of $F(X; Y)$ and thus compactly generated in the (usual) subspace topology.

Composition with maps from the right or from the left defines maps of function spaces. If for instance $A \subset X' \subset X$ is a nested sequence of NDR-pairs and $j: X' \to X$ the inclusion, then the induced map

$$\tilde{j}: F_u(X, A; Y, B) \to F_u(X', A; Y, B)$$

is a fibration with $F_u(X, X'; Y, B)$ as fibre. Similarly, if $Y \to Y' \to B$ is a sequence of fibrations and $q: Y \to Y'$ the projection, then the induced map

$$q: F_u(X, A; Y, B) \to F_{uq}(X, A; Y', B)$$

is a fibration with $F_u(X, A; Y, Y')$ as fibre ([14], Proposition, p. 528).

Let $\pi$ be an abelian group. We shall be particularly interested in the $K(\pi, 1)$-sectioned spaces [10] that arise in the following way. Suppose that $G$ is a system of local coefficients in the Eilenberg-Mac Lane space $K(\pi, 1)$ given by a homomorphism $\varphi: \pi_1(K(\pi, 1)) = \pi \to \text{Aut}(G_0)$ of $\pi$ into the automorphism group of a typical group $G_0$ of $G$. For any integer $n > 0$, $G$ may be realized, see ([5], Ch. III) or ([10], p. 7), as the system of local coefficients defined by the $n$-dimensional homotopy groups of the fibres of a sectioned fibration

$$K(G_0, n) \to K(G_0, n; \varphi) \xrightarrow{k} K(\pi, 1)$$

over $K(\pi, 1)$. This fibration, which we shall denote by $\kappa(G, n)$, classifies cohomology with local coefficients in the sense that by the Classification Theorem ([16], Theorem 6.13, p. 302), ([13], Theorem 3.6), ([12], Theorem II),

$$\pi_0\left( F_u(X, A; K(G_0, n, \varphi), K(\pi, 1)) \right) = H^n(X, A; u_1^*G)$$
for any map \( u: X \to K(G_0, n; \varphi) \) with \( u_1 = ku \). Via pull-back of the path-space fibration in the category of \( K(\pi, 1) \)-sectioned spaces [10],

\[
K(G_0, n - 1) \to \bar{P}K(G_0, n; \varphi) \to K(G_0, n; \varphi),
\]

this equality may be interpreted as a bijective correspondence between fibre homotopy equivalence classes of \( K(G_0, n - 1) \)-fibrations over \( X \) with \( u^*G \) as associated system of local coefficients and the cohomology group \( H^n(X; u^*G) \).

As a final subject of this mixed section we shall now discuss Künneth theorems for cohomology with local coefficients. First an algebraic lemma ([1], Theorem 2.8).

**Lemma 2.2.** Let \( P \) be a free positive and \( N \) a negative chain complex over \( \mathbb{Z} \). Then there is an isomorphism

\[
\Phi_N: H(\text{Hom}(P, N)) \to H(\text{Hom}(P, H(N)))
\]

which is natural in the first variable.

**Proof.** Choose a free negative complex \( N' \) and chain maps

\[
H(N) \xleftarrow{\beta} N' \xrightarrow{\alpha} N
\]

such that \( \alpha \) is a quasi-isomorphism and \( \beta_* = \alpha_*: H(N') \to H(N) \); cf. ([3], p. 169). Since \( P \) is free (projective), the induced chain maps

\[
\text{Hom}(1, \alpha): \text{Hom}(P, N') \to \text{Hom}(P, N),
\]

\[
\text{Hom}(1, \beta): \text{Hom}(P, N') \to \text{Hom}(P, H(N))
\]

are again quasi-isomorphisms. Thus

\[
\Phi_N = \text{Hom}(1, \beta)_* \circ \text{Hom}(1, \alpha)^{-1}: H(\text{Hom}(P, N)) \to H(\text{Hom}(P, H(N)))
\]

is an isomorphism. \( \Phi_N \) is easily seen to commute with \( \text{Hom}(\gamma, 1)_* \) for any chain map \( \gamma: P \to P' \) between free positive chain complexes. \( \square \)

Note that since the complex \( H(N) \) has trivial differentiation,

\[
H_n(\text{Hom}(P, H(N))) = \coprod_{p+q=n} H_p(\text{Hom}(P, H_q(N)))
\]

where \( H_q(N) \) is considered as a complex concentrated in degree 0.

As to cohomology of spaces, Lemma 2.2 has the following reformulation.
Lemma 2.3. Let \((Z, C)\) and \((X, A)\) be NDR-pairs, \(G\) a system of local coefficients in \(X\), and \(\text{pr}_2: Z \times X \to X\) the projection onto the second factor. Then there is an isomorphism

\[
\Phi_{(X,A)}: H^n((Z,C) \times (X,A); \text{pr}_2^*G) \to \coprod_{p+q=n} H^p(Z, C; H^q(X, A; G))
\]

which is natural in the first factor.

Proof. We may assume that \(Z\) and \(X\) are 0-connected spaces and that \((Z, C)\) and \((X, A)\) are CW-pairs. Let \((\tilde{Z}, \tilde{C}) \to (Z, C)\) and \((\tilde{X}, \tilde{A}) \to (X, A)\) be the universal covering spaces so that ([16], Theorem 4.9, p. 288)

\[
\Gamma^*((Z, C) \times (X, A); \text{pr}_2^*G) \cong \text{Hom}_R(\Gamma(\tilde{Z}, \tilde{C}) \otimes \Gamma(\tilde{X}, \tilde{Z}), G_0)
\]

where \(R = \mathbb{Z}(\pi_1(Z)) \otimes \mathbb{Z}(\pi_1(X))\) acts on the typical group \(G_0\) by \((\xi \otimes \eta)g = \eta g\) for \(\xi \in \pi_1(Z)\), \(\eta \in \pi_1(X)\) and \(g \in G_0\). We use \((\Gamma^*)\Gamma\) to denote cellular (co-)chain complexes. Since

\[
\text{Hom}_R(\Gamma(\tilde{Z}, \tilde{C}) \otimes \Gamma(\tilde{X}, \tilde{A}), G_0)
\]

\[
= \text{Hom}_{\pi_1(Z)}(\Gamma(\tilde{Z}, \tilde{C}), \text{Hom}_{\pi_1(X)}(\Gamma(\tilde{X}, \tilde{A}), G_0))
\]

\[
= \text{Hom}_{\pi_1(Z)}(\Gamma(\tilde{Z}, \tilde{C}), \Gamma^*(X, A; G))
\]

\[
= \text{Hom}(\Gamma(Z, C), \Gamma^*(X, A; G)),
\]

Lemma 2.3 follows from Lemma 2.2. \(\square\)

The isomorphisms of the last two lemmas are not uniquely defined.

3. Spaces of lifts in \(K(G_0, n)\)-fibrations. In this section we assume that \(p: Y \to B\) is a fibration with an Eilenberg-Mac Lane space \(K(G_0, n)\), where \(G_0\) is an abelian group, as fibre. Let \(u: X \to Y\) be any map and put \(u_1 = pu: X \to B\).

First assume that \(p: Y \to B\) is a principal \(K(G_0, n)\)-fibration. Then the pull-back \(u_1^*(p)\) is a fibre homotopically trivial fibration ([15], II). Hence

\[
F_u(X, A; Y, B) = F_{u'}(X, A; K(G_0, n), *)
\]

for some map \(u': X \to K(G_0, n)\), for \(F_u(X, A; Y, B)\) may be interpreted as a space of sections of \(u_1^*(p)\). The (relative version of the) theorem of Thom ([15], Théorème 3), ([7], Proposition, p. 609), ([8], Theorem 1) thus asserts that

\[
F_u(X, A; Y, B) = \prod_{i=0}^{n} K(H^{n-i}(X, A; G_0), i)
\]

up to weak homotopy type.
Now consider the general case of a not necessarily principal $K(G_0, n)$-fibration $p: Y \to B$. Let $G$ denote the system of local coefficients in $B$ defined by the $n$-dimensional homotopy groups of the fibres of $p$. Following the proof of Thom's theorem as it appears in [7], we consider the evaluation map

$$e: F_u(X, A; Y, B) \times X \to Y$$
given by $e(f, x) = f(x)$. Note that

$$e \in F_{u \ast \text{pr}_2}((F_u(X, A; Y, B), u) \times (X, A); Y, B).$$

For $0 \leq i \leq n$, choose maps

$$e^i: (F_u(X, A, Y, B), u) \to \left(K\left(H^{n-i}(X, A; u_1^*G), i\right), *\right)$$
such that the array of homotopy classes $([e^0], [e^1], \ldots, [e^n])$ corresponds to the (vertical and relative) homotopy class $[e]$ of $e$ under the composite bijection

$$\pi_0(F_{u \ast \text{pr}_2}((F_u(X, A; Y, B), u) \times (X, A); Y, B))$$

$$= H^n((F_u(X, A; Y, B), u) \times (X, A); \text{pr}_2 u_1^*G)$$

$$\Phi^{(X, A)} \to \prod_{0 \leq i \leq n} H^i(F_u(X, A; Y, B), u; H^{n-i}(X, A; u_1^*G)).$$

The main result of this section is the following generalization of Thom's theorem ([15], I) and the Classification Theorem ([12], Theorem II).

**Theorem 3.1.** The map

$$(e^0, e^1, \ldots, e^n): F_u(X, A; Y, B) \to \prod_{i=0}^n K(H^{n-i}(X, A; u_1^*G), i)$$
is a weak homotopy equivalence.

**Proof.** For $i \geq 0$, the Exponential Law

$$F_u\left(S^i, *; F_u(X, A; Y, B), u\right) = F_{u \ast \text{pr}_2}((S^i, *) \times (X, A); Y, B)$$

$$\alpha \to e \circ (\alpha \times 1)$$

induces a bijection

$$\psi^i: \pi_i(F_u(X, A; Y, B), u) \to H^n((S^i, *) \times (X, A); \text{pr}_2 u_1^*G)$$

$$[\alpha] \to (\alpha \times 1)[e]$$
between path-components. According to Lemma 2.3 there is a commutative diagram (with $F_u = F_u(X, A; Y, B)$)

$$H^n((F_u, u) \times (X, A); \text{pr}_2^* u_1^* G) \rightarrow \coprod_{0 \leq j \leq n} H^i(F_u, u; H^{n-j}(X, A; u_1^* G))$$

$$(\alpha \times 1)^* \downarrow \downarrow \alpha^* \circ \text{pr}_i$$

$$H^n((S^i, \ast) \times (X, A); \text{pr}_2^* u_1^* G) \rightarrow H^i(S^i, \ast; H^{n-i}(X, A; u_1^* G))$$

showing that

$$\Phi_{(X, A)}\psi^i([\alpha]) = \Phi_{(X, A)}(\alpha \times 1)^*[e]$$

$$= \alpha^* \circ \text{pr}_i \circ \Phi_{(X, A)}([e]) = \alpha^*([e^i]).$$

In other words, the bijection

$$\Phi_{(X, A)}\psi^i: \pi_i(F_u, u) \rightarrow H^i(S^i, \ast; H^{n-i}(X, A; u_1^* G)) = H^{n-i}(X, A; u_1^* G)$$

equals the homomorphism

$$(e_i)_*: \pi_i(F_u, u) \rightarrow \pi_i(K(H^{n-i}(X, A; u_1^* G), i), \ast) = H^{n-i}(X, A; u_1^* G)$$

induced by $e_i$. Hence $(e_i)_*$ is an isomorphism (for $i \geq 1$) of homotopy groups. \hfill \Box

**Remark 3.2.** Let $(Z, C)$ be an NDR-pair and $\alpha: (Z, C) \rightarrow (F_u(X, A; Y, B), u)$ a map. Then

$$[e \circ (\alpha \times 1)] \in H^n((Z, C) \times (X, A); \text{pr}_2^* u_1^* G)$$

and $e^i \circ \alpha: (Z, C) \rightarrow (K(H^{n-i}(X, A; u_1^* G), i), \ast)$ represents

$$\text{pr}_i(\Phi_{(X, A)}([e \circ (\alpha \times 1)])) \in H^i(Z, C; H^{n-i}(X, A; u_1^* G)).$$

An application of Theorem 3.1 to the classifying fibration $\kappa(G, n)$ over $K(\pi, 1)$ yields

**Corollary 3.3.** The space $\Gamma(\kappa(G, n))$ of sections of $\kappa(G, n)$ has the weak homotopy type of the product

$$\prod_{i=0}^n K(\text{Ext}_\pi^{n-i}(Z, G_0), i)$$

where $Z$ is considered as a trivial $\pi$-module.

**Proof.** $H^{n-i}(K(\pi, 1); G) = \text{Ext}_\pi^{n-i}(Z, G_0)$ by a theorem of Eilenberg ([16], Theorem 3.5*, p. 281). \hfill \Box
Note that the additive structure of $H^*(X, A; G)$ suffices to determine the weak homotopy type of $F_u(X, A; Y, B)$ when $p: Y \to B$ is a $K(G_0, n)$-fibration; cf. ([15], I). This is not true in general.

4. Change of base point. Let $p: Y \to B$ be the $K(G_0, n)$-fibration of the previous section and let $u, v: X \to Y$ be two maps such that $u|A = v|A$ and $pu = pv$. Then $F_u(X, A; Y, B) = F_v(X, A; Y, B)$ as free spaces. The purpose of this section is to discuss the relation between the pointed spaces $(F_u(X, A; Y, B), u)$ and $(F_v(X, A; Y, B), v)$.

To clarify the role of the chosen base point, we now write $\psi^i_u$ for the homomorphism $\psi^i$ introduced in the proof of Theorem 3.1. Explicitly,

$$\psi^i_u: \pi_i(F_u(X, A; Y, B), u) \to H^n((S^i, *) \times (X, A); \text{pr}_2^*u_1^*G)$$

takes $[\alpha] \in \pi_i(F_u(X, A; Y, B), u)$ to the primary difference

$$\psi^i_u([\alpha]) = \delta^n(u \circ \text{pr}_2, e \circ (\alpha \times 1))$$

of $u \circ \text{pr}_2$ and the adjoint $e \circ (\alpha \times 1)$ of $\alpha$.

In order to compare $\psi^i_u$ and $\psi^i_v$, we introduce the set $[S^i, F_u(X, A; Y, B)]$ of free homotopy classes of free maps of $S^i$ into $F_u(X, A; Y, B)$. (Note in this connection that $F_u(X, A; Y, B)$ is a simple space by Theorem 3.1.) Also in this case we get a bijection

$$\psi^i_u: [S^i, F_u(X, A; Y, B)] \to H^n(S^i \times (X, A); \text{pr}_2^*u_1^*G)$$

by forming primary differences as above.

Let $j: \pi_i(F_u(X, A; Y, B), u) \to [S^i, F_u(X, A; Y, B)]$ be the inclusion induced by the inclusion $j: S^i \to (S^i, *)$. Then one easily proves:

**Lemma 4.1.** The deviation from commutativity of the diagram

$$\begin{align*}
\pi_i(F_u(X, A; Y, B), u) & \quad \xrightarrow{\psi^i_u} \quad H^n((S^i, *) \times (X, A); \text{pr}_2^*u_1^*G) \\
j \downarrow & \quad \downarrow (j \times 1)^* \\
[S^i, F_u(X, A; Y, B)] & \quad \xrightarrow{\psi^i_v} \quad H^n(S^i \times (X, A); \text{pr}_2^*u_1^*G)
\end{align*}$$

is given by

$$(j \times 1)^* \circ \psi^i_u - \psi^i_v \circ j = \text{pr}_2^*\delta^n(u, v)$$

where $\delta^n(u, v) \in H^n(X, A; u_1^*G)$ is the primary difference of $u$ and $v$. 


Now assume that $p_1: Y_1 \to B_1$ is another fibration with an Eilenberg-Mac Lane space $K(G'_0, q)$, $G'_0$ abelian, $q \geq 1$, as fibre and that
\[
\begin{array}{ccc}
Y & \to & Y_1 \\
p & \downarrow & \downarrow p_1 \\
B & \to & B_1 \\
\end{array}
\]
is a fibre map of $p$ into $p_1$. Let $G$ denote the local coefficient system in $B_1$ determined by $p_1$.

For any pair $(Z, C; f)$ over $Y$ and any integer $i \geq 0$, let $\sigma^i[k]_f$ denote the primary twisted cohomology operation that makes the diagram
\[
\begin{array}{ccc}
\pi_i(F_f(Z, C; Y, B), f) & \overset{k*}{\to} & \pi_i(F_{k*f}(Z, C; Y_1, B_1), k*f) \\
\Phi_{(Z,C)}\psi_f \downarrow \cong & & \cong \downarrow \Phi_{(Z,C)}\psi_{k*f} \\
H^{n-i}(Z, C; f_1^*G) & \overset{\sigma^i[k]_f}{\to} & H^{q-i}(Z, C; f_1^*k_1^*G_1) \\
\end{array}
\]
commute. The operation $[k]_f := \sigma^0[k]_f$ is given by $[k]_f \delta^n(f, g) = \delta^q(kf, kg)$ for any $g \in F_f(Z, C; Y, B)$.

In particular, $u: X \to Y$ determines operations
\[
\sigma^i[k]_u: H^{n-i}(X, A; u_1^*G) \to H^{q-i}(X, A; u_1^*k_1^*G_1), \quad i \geq 0,
\]
and the maps $u \circ \text{pr}_2: X \times S' \hookrightarrow Y$, $i \geq 0$, determine operations $[k]_{u \circ \text{pr}_2}$ such that
\[
[S^i, F_u(X, A; Y, B)] \overset{k*}{\to} [S^i, F_{k*u}(X, A; Y, B)]
\]
\[
\begin{array}{c}
\psi_u^i \downarrow \\
\Phi_{(X,A)} \downarrow \cong \\
H^n(S^i \times (X, A); \text{pr}_2^*u_1^*G) \overset{[k]_{u \circ \text{pr}_2}}{\to} H^q(S^i \times (X, A); \text{pr}_2^*u_1^*k_1^*G_1) \\
\end{array}
\]
commutes. If $s^i \times -$ denotes the homomorphism that renders
\[
H^n((S^i, *) \times (X, A); \text{pr}_2^*u_1^*G) \overset{(j \times 1)^*}{\to} H^n(S^i \times (X, A); \text{pr}_2^*u_1^*G)
\]
\[
\begin{array}{c}
\Phi_{(X,A)} \downarrow \\
\Phi_{(X,A)} \\
H^{n-i}(X, A; u_1^*G) \overset{s^i \times -}{\twoheadrightarrow}
\end{array}
\]
commutative, then the equation
\[
[k]_{u \circ \text{pr}_2}(s^i \times \chi) = s^i \times \sigma^i[k]_u(\chi), \quad \chi \in H^{n-i}(X, A; u_1^*G)
\]
shows the relation between $[k]_u$ and $[k]_{u \circ \text{pr}_2}$. 

The object of the next theorem is to compare the operations \( \sigma^i[k]_u \) and \( \sigma^i[k]_v \) induced by two different maps \( u \) and \( v \).

**Theorem 4.2.** For any \( \chi \in H^{n-i}(X, A; u_1^*G) \), \( i > 0 \), the equality

\[
[k]_{u \circ pr_2} \left( s^i \times \chi + pr_2^* \delta^n(u, v) \right) = s^i \times \sigma^i[k]_v(\chi) + pr_2^*([k]_u \delta^n(u, v))
\]

holds in

\[
H^q \left( S^i \times (X, A) \right;  pr_2^*u_1^*k_1^*G_1) = H^{q-i}(X, A; u_1^*k_1^*G_1) \oplus H^q(X, A; u_1^*k_1^*G_1).
\]

**Proof.** Some of the introduced maps are related by the following commutative diagram

\[
\begin{array}{ccc}
\pi_1(F_u, \nu) & \xrightarrow{k_\ast} & \pi_1(F_{ku}, \nu) \\
\downarrow \psi_u & & \downarrow \psi_{ku} \\
H^*(S^i \times (X, A)) & \xrightarrow{[k]_{u \circ pr_2}} & H^*(S^i \times (X, A)) \\
\downarrow (j \times 1)^\ast & & \downarrow (j \times 1)^\ast \\
H^*((S', \ast \times (X, A)) & \xrightarrow{(j \times 1)^\ast} & H^*((S', \ast \times (X, A))
\end{array}
\]

in which some self explanatory abbreviations occur. In particular

\[
(1) \quad [k]_{u \circ pr_2} \left( \psi_u^i j_\ast [\alpha] \right) = (j \times 1)^\ast \psi_u^i k_\ast [\alpha]
\]

for any homotopy class \( [\alpha] \in \pi_1(F_u(X, A; Y, B), \nu) \). If \( \psi_u^i j_\ast [\alpha] = \chi \), then by Lemma 4.1,

\[
\psi_u^i j_\ast [\alpha] = (j \times 1)^\ast \psi_u^i [\alpha] + pr_2^* \delta^n(u, v) = s^i \times \chi + pr_2^* \delta^n(u, v),
\]

so the left hand side of (1) becomes

\[
[k]_{u \circ pr_2} \left( \psi_u^i j_\ast [\alpha] \right) = [k]_{u \circ pr_2} \left( s^i \times \chi + pr_2^* \delta^n(u, v) \right).
\]

The right hand side of (1) can be rewritten, using Lemma 4.1 for the first equality, as follows

\[
(j \times 1)^\ast \psi_u^i k_\ast [\alpha] = \psi_u^i j_\ast k_\ast [\alpha] + pr_2^* \delta^n(ku, kv)
\]

\[
= [k]_{v \circ pr_2} ((j \times 1)^\ast \psi_u^i [\alpha]) + pr_2^* [k]_u \delta^n(u, v)
\]

\[
= [k]_{v \circ pr_2} (s^i \times \chi) + pr_2^* [k]_u \delta^n(u, v)
\]

\[
= s^i \times \sigma^i[k]_v(\chi) + pr_2^* [k]_u \delta^n(u, v).
\]

\( \square \)
Consequently, $[k]_u = [k]_v$ if $[k]_{u \circ \text{pr}_2}$ happens to be an additive operation. On the other hand, examples do occur, see e.g. [11], where $[k]_u \neq [k]_v$.

5. Spaces of maps into twisted Eilenberg-Mac Lane spaces. Suppose that both $\pi$ and $G_0$ are abelian groups, $\varphi : \pi \to \text{Aut}(G_0)$ an action of $\pi$ on $G_0$, and

$$K(G_0, n) \to K(G_0, n; \varphi) \xrightarrow{\kappa} K(\pi, 1)$$

the associated classifying fibration $\kappa(G, n)$. The purpose of this section is to describe mapping spaces with the total space $K(G_0, n; \varphi)$ as target.

The classifying fibration $\kappa(G, n)$ can be constructed more explicitly as follows. The Eilenberg-Mac Lane space $K(G_0, n)$ can be made into a left $\pi$-space in such a way that each $\xi \in \pi$ acts as a base-point preserving homeomorphism with the induced map

$$\xi_* : \pi_n(K(G_0, n), *) \to \pi_n(K(G_0, n), *)$$

equal to $\xi : G_0 \to G_0$ under some fixed isomorphism $\pi_n(K(G_0, n), *) \cong G_0$. The fibre bundle

$$K(G_0, n) \to E\pi \times_{\pi} K(G_0, n) \xrightarrow{\kappa} B\pi$$

associated to the universal principal $\pi$-bundle $\omega : E\pi \to B\pi$ is then a $\kappa(G, n)$.

Let $u : X \to K(G_0, n; \varphi) = E\pi \times_{\pi} K(G_0, n)$ be any map into the total space of $\kappa(G, n)$. Put $u_1 = \hat{k}u$. Consider the fibration of function spaces

$$F_u(X; K(G_0, n; \varphi), B\pi) \to F_u(X; K(G_0, n; \varphi), *) \xrightarrow{\kappa} F_{u_1}(X; B\pi, *)$$

induced by the projection $\hat{k}$. The base space $F_u(X; B\pi, *) = H^1(X; \pi) \times K(\pi, 1)$ is disconnected (in general), so we let $F^0_{u_1}(X; B\pi, *) = K(\pi, 1)$ denote the path-component of $F_u(X; B\pi, *)$ containing $u_1$ and concentrate our attention on the pre-image $F_u(X; K(G_0, n; \varphi), *) | u_1 = \hat{k}^{-1}(F^0_{u_1}(X; B\pi, *))$. By restriction of $\hat{k}$ we then get the fibration

$$\prod_{i=0}^n K(H^{n-i}(X; u_1^*G), i) \to F_u(X; K(G_0, n; \varphi), *) | u_1 \to K(\pi, 1)$$

where Theorem 3.1 has been used to identify the fibre.

Since $\pi$ is abelian, $\xi : G_0 \to G_0$, $\xi \in \pi$, is an operator automorphism, i.e. an automorphism of the local coefficient system $G$ in $K(\pi, 1)$, and hence $\xi$ induces a coefficient group automorphism $\xi_*$ of $H^{n-i}(X; u_1^*G)$, $0 \leq i \leq n$. 

After these preliminaries we can now state

**THEOREM 5.1.** There is a weak (fibre) homotopy equivalence

$$F_u(X; K(G_0, n; \varphi), *) | u_1 \rightarrow E\pi \times_\pi \left( \prod_{i=0}^n K(H^{-i}(X; u_1^*G), i) \right),$$

where \( \pi \) acts on \( H^{-i}(X; u_1^*G), 0 \leq i \leq n \), through coefficient group automorphisms.

**Proof.** The cohomology operation \( \xi_* \) can be realized geometrically as in \( \S 4 \). For the based automorphism \( \xi \) of \( K(G_0, n) \) is a \( \pi \)-map, and hence it extends to a homeomorphism \( \xi: K \rightarrow K \) over and under \( B\pi \). (Here, and in the following, \( K = K(G_0, n; \varphi) = E\pi \times_\pi K(G_0, n) \).) As is easily seen, the \( i \)-fold suspension \( \sigma^i[\xi]_u \) of the corresponding cohomology operation \( [\xi]_u \) is the coefficient group automorphism \( \xi_*: H^{-i}(X; u_1^*G) \rightarrow H^{-i}(X; u_1^*G), 0 \leq i \leq n \).

Since \( \pi \) is abelian, there exist \( H \)-space structures \( \tilde{\mu}: E\pi \times E\pi \rightarrow E\pi, \mu: B\pi \times B\pi \rightarrow B\pi \) with strict units \( e_0 \in E\pi, b_0 = \omega(e_0) \in B\pi \) such that \( \mu \circ (\omega \times \omega) = \omega \circ \tilde{\mu} \). The unique path lifting property implies that \( \tilde{\mu}(e_1, e_2) = \tilde{\mu}(e_1, e_2) \xi = \tilde{\mu}(e_1, e_2) \xi \) for all \( e_1, e_2 \in E\pi, \xi \in \pi \).

The space \( F'(X; K, B\pi) \) of lifts of \( u_1 \) is a left \( \pi \)-space under composition with the fibre maps \( \xi: K \rightarrow K, \xi \in \pi \). Let

\[
\tilde{\psi}: E\pi \times_\pi F_u(X; K, B\pi) \rightarrow F_u(X; K, *)
\]

be the map given by

\[
\tilde{\psi}((e, v) \pi)(x) = (\tilde{\mu}(e, \tilde{u}_1(x)), \tilde{\nu}(x)) \pi
\]

where \( e \in E\pi, v \in F_u(X; K, B\pi), x \in X, \tilde{u}_1(x) \in E\pi \) is any lift of \( u_1(x) \in B\pi, v(x) \in K \) and \( \tilde{\nu}(x) \in K(G_0, n) \) are related by the formula \( v(x) = (\tilde{u}_1(x), \tilde{\nu}(x)) \). Note that \( \tilde{\psi} \) is a fibre map which restricts to the identity on the fibre. The induced map \( \psi: B\pi \rightarrow F_{u_1}(X; B\pi, *) \) between the base spaces satisfies \( \psi(b, x) = \mu(b, u_1(x)), b \in B\pi, x \in X \). This means that \( \psi \) is a homotopy equivalence between \( B\pi \) and \( F_{u_1}^0(X; B\pi, *) \). Hence \( \tilde{\psi} \) is a fibre homotopy equivalence from \( E\pi \times_\pi F_u(X; K, B\pi) \) to \( F_u(X; K, *) \) by Dold [2].

The proof is now completed by noting that the weak homotopy equivalence of \( F_u(X; K, B\pi) \) into \( \prod_{i=0}^n K(H^{-i}(X; u_1^*G), i) \) from Theorem 3.1 is a \( \pi \)-map enabling us to construct a weak homotopy equivalence

\[
E\pi \times_\pi F_u(X; K, B\pi) \rightarrow E\pi \times_\pi \prod_{i=0}^n K(H^{-i}(X; u_1^*G), i)
\]
as claimed. \( \square \)
REMARK 5.2. During the proof of Theorem 5.1 we actually established the identity

\[ F_u(X; E\pi \times_\pi F, *) \mid u_1 = E\pi \times_\pi F_u(X; E\pi \times_\pi F, B\pi) \]

for any left \(\pi\)-space \(F\) and any map \(u : X \rightarrow E\pi \times_\pi F\).

EXAMPLE 5.3. The classifying space \(BO(2)\) for the orthogonal group \(O(2)\) is the twisted Eilenberg-Mac Lane space \(K(\mathbb{Z}, 2; \varphi)\) where \(\varphi : \mathbb{Z}/2 \rightarrow \text{Aut}(\mathbb{Z})\) is the non-trivial action.

Let \(u : BO(1) \rightarrow BO(2)\) be any map. Then up to homotopy, \(u_1 = 0\) or \(u_1 = w_1\), the first Stiefel-Whitney class. An application of Theorem 5.1 yields

\[ F_u(BO(1); BO(2), *) \mid 0 = BO(2) + BO(2), \]
\[ F_u(BO(1); BO(2), *) \mid w_1 = BO(1) \times BO(1) \]

where \(+\) denotes disjoint union.

6. Spaces of lifts in \(K(G, 1)\)-fibrations. In this section we let \(p : Y \rightarrow B\) denote a fibration with an aspherical space \(F = K(G, 1)\) as fibre. \(G\) can be any, not necessarily abelian, group. We shall investigate the space \(F_u(X, A; Y, B)\).

The pull-back \(F \rightarrow Y' \rightarrow X\) of \(F \rightarrow Y \rightarrow B\) along \(u_1 = pu\) has a canonical section \(u' : X \rightarrow Y'\) induced from \(u\). Hence \(i'_* : \pi_1(F) \rightarrow \pi_1(Y')\) is a monomorphism and a homomorphism \(\varphi_u : \pi = \pi_1(X) \rightarrow \text{Aut}(G)\) is uniquely defined \(i'_*(xg) = u'_*(x)i'_*(g)u'_*(x)^{-1}, x \in \pi, g \in G\). We write \(xg\) for \(\varphi_u(x)g\). Let

\[ G^\pi = \{ g \in G \mid \pi g = g \} \]

denote the fixpoint set of this action and let

\[ Q(\pi, G) = \{ f : \pi \rightarrow G \mid \forall x, y \in \pi : f(xy) = f(x)xf(y) \} \]

denote the set of crossed homomorphisms of \(\pi\) into \(G\). There is an action

\[ Q(\pi, G) \times G \rightarrow Q(\pi, G) \]

of \(G\) on the set of crossed homomorphisms given by \((fg)(x) = g^{-1}f(x)xg, f \in Q(\pi, G), g \in G, x \in \pi. Q(\pi, G)/G\) denotes the set of orbits for this action.

Let \(x_0 \in X\) be the base point. To any based lift \(v \in F_u(X, x_0; Y, B)\) of \(u_1\), we can associate a crossed homomorphism \(f_v \in Q(\pi, Q)\) given by \(i'_*f_v(x) = v'_*(x)u'_*(x)^{-1}\), where \(v' : X \rightarrow Y'\) is the section of \(p'\) induced from \(v\). By some obvious modifications of the classification of based
homotopy classes of based maps into an aspherical space ([16], Theorem 4.3, p. 225) we get

**Lemma 6.1.** For any connected CW-complex $X$, the map $v \to f_v$ induces a bijective correspondence between $\pi_0 F_u(X, x_0; Y, B)$ and $Q(\pi, G)$.

Also the free vertical homotopy classes of free lifts of $u_1$ can be classified; cf. ([16], Corollary 4.4, p. 226).

**Lemma 6.2.** For any connected CW-complex $X$, there is a bijective correspondence between $\pi_0 F_u(X; Y, B)$ and $Q(\pi, G)/G$.

**Proof.** The sets $F_u(X, x_0; Y, B)$ and $F_u(X; Y, B)$ of based and free lifts of $u_1$ are related by the evaluation fibration

$$F_u(X, x_0; Y, B) \to F_u(X; Y, B) \to F_u(x_0; Y, B) = F.$$ 

This evaluation fibration determines an action $Q(\pi, G) \times G \to Q(\pi, G)$ of the fundamental group $G = \pi_1(F)$ of its base space on the set $\pi_0 F_u(X, x_0; Y, B) = Q(\pi, G)$ of path-components of its fibre. We must show that this action coincides with the one introduced above.

Since $X$ is connected, we may assume that the 1-skeleton $X_1$ is a wedge of circles. The inclusion map $i_1: X_1 \to X$ induces an injection $\hat{i}_1: Q(\pi, G) \to Q(\pi_1(X_1), G)$ which is compatible with the $G$-action. Therefore, we may assume that $X = X_1$ is 1-dimensional. Furthermore, since a crossed homomorphism of $\pi_1(X_1)$ into $G$ is uniquely determined by its value on a set of free generators, we can assume that $X = S^1$ consists of a single circle.

Let $h: (I, I) \to (S^1, x_0)$ be the usual proclusion representing the generator $i \in \pi_1(S^1, x_0)$. Choose a map $H: I \times F \to Y'$ such that the diagram

$$
\begin{array}{ccc}
I \times F & \xrightarrow{H} & Y' \\
pr_1 \downarrow & & \downarrow p' \\
I & \xrightarrow{h} & S^1
\end{array}
$$

commutes and such that $H(t, y_0) = u'(t)$, $y_0 = u(x_0)$, $t \in I$, and $H_0 = i': F \to Y'$. Then ([9], Theorem 1), $H_1 = i^{-1} \in \text{Aut } G$. 

Consider the following diagram of maps between fibrations induced by $h$ and $H$

$$
\begin{array}{ccc}
F_u(S^1, x_0; Y, B) & \xrightarrow{\overline{h}} & F_{uh}(I, \hat{I}; Y, B) & \leftarrow & F_{y_0}(I, \hat{I}; F) \\
\downarrow & & \downarrow & & \\
F_u(S^1; Y, B) & \rightarrow & F_{uh}(I; Y, B) & \leftarrow & F(I; F) \\
\downarrow & & \downarrow & & \\
F_u(x_0; Y, B) & \rightarrow & F_{uh}(\hat{I}; Y, B) & \leftarrow & F(I; F)
\end{array}
$$

The maps between the fibers are homeomorphisms ([14], p. 530) and the maps between the base spaces can be identified to

$$
F \xrightarrow{\Delta} F \times F \xleftarrow{1 \times H} F \times F
$$

where $\Delta$ is the diagonal map.

The fibre $F_{y_0}(I, \hat{I}; F)$ of the fibration to the right is the loop space $\Omega F$ of $F$ and the associated action of $\pi_1(F(\hat{I}; F), y_0) = G \times G$ on $\pi_0 F_{y_0}(I, \hat{I}; F) = \pi_0(\Omega F) = G$ is given by $g_1 \cdot (h_0, h_1) = h_0^{-1}g_1h_1$ for all $g_1, h_0, h_1 \in G$. Hence the corresponding action of $\pi_1(F_u(x_0; Y, B), y_0) = G$ on $\pi_0 F_u(S^1, x_0; Y, B) = Q(\pi_1(S^1), G) = G$ is given by $g_1 \cdot g = g^{-1}g_1$ for all $g \in G$. Taking into account the identifications made, this means that

$$(fg)(z) = g^{-1}f(z)gz$$

for all $f \in Q(\pi_1(S^1), G), g \in G, z \in \pi_1(S^1)$. \qed

Finally, we compute the higher homotopy groups of $F_u(X, x_0; Y, B)$ and $F_u(X; Y, B)$. More generally, let $(X, A)$ be a finite relative CW-complex where both $X$ and $A$ are 0-connected. Assume that $(X, A)$ has a CW-decomposition with 0-skeleton $X_0 = A$ if $A \neq \emptyset$ and $X_0 = \{x_0\}$ if $A = \emptyset$.

**Theorem 6.3.**

1. If $A \neq \emptyset$, each component of $F_u(X, A; Y, B)$ is weakly contractible.
2. If $A = \emptyset$, each component of $F_u(X; Y, B)$ is an aspherical space.

The fundamental group $\pi_1(F_u(X, Y, B), u)$ of the component containing $u$ is isomorphic to the fixpoint set $G^\pi$.

**Proof.** We proceed as in ([8], Theorem 2). Let $X_q$ be the $q$-skeleton of a CW-decomposition of $(X, A)$ such that $X_0 = A$ if $A \neq \emptyset$ and $X_0 = \{x_0\}$ if $A = \emptyset$. The inclusion maps $i_q : X_{q-1} \rightarrow X_q$ induce a tower of
The fibre \( F_u(X_q, A; Y, B) \) of \( i_q \) can be identified to a product of a number of copies of the \( q \)-fold loop space \( \Omega^q F \). The number of factors equals the number of \( q \)-cells in \( (X, A) \). Since \( F = K(G, 1) \) is aspherical, it follows that \( F_u(X, A; Y, B) \) and \( F_u(X_1, A; Y, B) \) are weakly homotopy equivalent. Moreover, if \( A \neq \emptyset \),

\[
F_u(X_1, A; Y, B) \simeq \Omega F \times \cdots \times \Omega F \simeq G \times \cdots \times G
\]

is just a discrete set of points.

If \( A = \emptyset \), we consider the evaluation fibration

\[
F_u(X, x_0; Y, B) \to F_u(X; Y, B) \to F_u(x_0; Y, B) = F
\]

with the discrete fibre \( F_u(X, x_0; Y, B) = F_u(x_1, x_0; Y, B) \). In the associated homotopy sequence

\[
1 \to \pi_1(F_u(X; Y, B), u) \to G \to Q(\pi, G) \to \pi_0 F_u(X; Y, B) \to *
\]

one has \( \partial g = 1g \) for all \( g \in G \). Hence

\[
\pi_1(F_u(X; Y, B), u) \cong \ker \partial = \{ g \in G | 1g = g \} = G^\pi.
\]

If \( p = \text{pr}_1: B \times K(G, 1) \to B \) is the trivial \( K(G, 1) \)-fibration over \( B \) and \( u = (b_0, u): X \to \{ b_0 \} \times K(G, 1) \subset B \times K(G, 1) \) a continuous map, the action of \( \pi \) on \( G \) is given by \( xg = u_*(x)gu_*(x)^{-1} \). Thus the fixpoint set \( G^\pi \) is the centralizer of \( u_*(\pi_1(X)) \) in \( G \). In this way we recover the theorem of Gottlieb [6].

If \( G \) is abelian, the fibration \( p: Y \to B \) determines a system of local coefficients, also denote by \( G \), in \( B \). The pull-back \( u_1^*G \) in \( X \) is given by \( \varphi_u: \pi \to \text{Aut}(G) \). Since \( Q(\pi, G) \cong H^1(X, x_0; u_1^*G) \), \( Q(\pi, G)/G \cong H^1(X; u_1^*G) \), and \( G^\pi = H^0(X; u_1^*G) \), 6.1–6.3 reduce to Theorem 3.1 for \( n = 1 \) in this case.

Although suppressed in the used notation, the group \( G^\pi \) in general depends on the choice of \( u \). Thus the components of \( F_u(X; Y, B) \) may represent more than just one (weak) homotopy type.
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