ON THE CONGRUENCE LATTICE OF A FRAME

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Recall that the Skula modification SkX of a topological space X is the space with the same underlying set as X whose topology is generated by the topology ΩX of X and the closed subsets of X. R. E. Hoffmann characterizes the spaces X for which SkX is compact Hausdorff as the noetherian sober spaces. The object of this note is to give a simple proof of the analogue of this characterization for frames and to show how our result for frames applies to the original one for spaces.

For basic information on frames and further references, see the second chapter of Johnstone [7]. Here we just recall the following:

A frame (also: locale) is a complete lattice in which \( x \land \lor x_i = \lor x \land x_i \) for binary meet (\( \land \)) and arbitrary join (\( \lor \)), and the terms subframe and frame homomorphism refer to finite meets and arbitrary joins. The category of frame and frame homomorphisms is called Frm. For any frame \( L \), 0 will be its zero (= bottom) and \( e \) its unit (= top). An element \( c \) of a frame \( L \) will be called compact (Johnstone [7]: finite) whenever \( c \leq \lor x_i \) implies that already \( c \leq x_{i_1} \lor \cdots \lor x_{i_n} \) for suitable \( i_1, \ldots, i_n \); if the unit \( e \in L \) is compact, one also calls \( L \) compact. A coherent frame is one in which (i) every element is a join of compact elements, and (ii) \( e \) is compact and finite meets of compact elements are compact.

For any frame \( L \), its congruence lattice \( CL \) consists of the congruences on \( L \), that is, the equivalence relations on \( L \) which are subframes of \( L \times L \), partially ordered by inclusion. The meet in \( CL \) is then intersection, so that \( CL \) is evidently a complete lattice. The more subtle and interesting fact that \( CL \) is again a frame (this was observed by Funayama and Nakayama for the congruence lattice of a distributive lattice see Birkhoff VI, 4 [2]). Dowker and Papert [4] used the isomorphism of \( CL \) with the lattice of quotient frames of \( L \) to investigate the latter. That \( CL \) is a frame can also be seen from the fact that it is isomorphic to the frame \( NL \) of nuclei on \( L \), the latter being the \( \land \)-preserving closure operators on \( L \) (Johnstone, [7]), by the map \( CL \to NL \) taking each congruence \( \theta \) to its associated nucleus defined by \( k(a) = \lor \{ x \mid (x, a) \in \theta \} \).

Particular congruences on \( L \) associated with each \( a \in L \) are \( \nabla_a = \{ (x, y) \mid x \lor a = y \lor a \} \) and \( \Delta_a = \{ (x, y) \mid x \land a = y \land a \} \), also characterized as the congruences generated by \( (0, a) \) and \( (a, e) \), respectively.
Similarly, for any \( a < b \), the congruence generated by \((a, b)\) is \( \Delta_a \cap \nabla_b \), which shows that each \( \theta \in CL \) is the join of such \( \Delta_a \cap \nabla_b \). Further, the map \( a \mapsto \nabla_a \ (a \in L) \) is a frame embedding \( \nu_L : L \to CL \), natural in \( L \), and \( \nabla_a \) and \( \Delta_a \) are complementary to each other, that is \( \nabla_a \cap \Delta_a = \Delta = \{(x, x) \mid x \in L\} \), the zero (= bottom) of \( CL \) and \( \nabla_a \lor \Delta_a = \nabla = L \times L \), the unit (= top) of \( CL \). The latter implies that \( \nu_L : L \to CL \) is an epimorphism of frames.

The equivalence of the first two properties in the following proposition is the analogue for frames of Hoffmann's result for spaces. This characterization is the solution to a problem of Macnab [9]. We thank the referee for drawing our attention to this paper. In the following a frame is called \textit{noetherian} whenever each of its elements is compact. Using the Axiom of Choice, this is easily seen to be equivalent to the Ascending Chain Condition which says that every sequence \( a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \) in \( L \) is eventually constant.

\textbf{Proposition 1.} The following are equivalent:

(1) \( CL \) is compact,

(2) \( L \) is Noetherian,

(3) \( CL = \text{Cong } L \) (the congruence lattice of \( L \) as a lattice),

(4) the complemented elements of \( CL \) are precisely the compact ones,

(5) \( CL \) is coherent.

\textbf{Proof.} If \( CL \) is compact then every complemented element of \( CL \), and hence in particular each \( \nabla_a \), \( a \in L \), is compact. Since \( a \mapsto \nabla_a \) is a frame embedding, this makes \( a \in L \) compact. This establishes the implications \((1) \Rightarrow (4) \Rightarrow (2)\).

If \( L \) is noetherian then so is \( L \times L \). This implies that arbitrary joins in \( L \times L \) are actually finite joins, and hence any sublattice of \( L \times L \) (including top and bottom) is already a subframe. In particular, any lattice congruence on \( L \) is actually a frame congruence, and \( CL \) is just the congruence lattice of \( L \) as a lattice. It follows that \( CL \) is closed under up-directed unions, and since \( \nabla \) is generated by \((0, e)\) this makes it compact. It follows that \((2) \Rightarrow (3) \Rightarrow (1)\). That \((5)\) is an equivalent: any \( CL \) is generated by its complemented elements. If these are compact then it follows that \( CL \) is coherent.

\textbf{Remark.} Evidently, a frame \( L \) is noetherian iff the natural homomorphism \( \sigma_L : \mathcal{F}L \to L \) from its ideal lattice by taking joins is an isomor-
phism, that is, iff every ideal of \( L \) is principal. Hence the above proposition may be paraphrased thus: \( CL \) is compact iff \( \sigma_L: \mathcal{J}L \to L \) is an isomorphism. This is the frame counterpart of the early result by Brümmer [3] that \( SkX \) is compact Hausdorff iff the natural embedding of \( X \) into the prime spectrum of \( \Omega X \) (given by all lattice homomorphisms \( \Omega X \to 2 \)) is a homeomorphism. For a further development of related ideas see also Künzi-Brümmer [8].

In order to relate Proposition 1 to topological spaces, we have to consider the spectrum functor \( \Sigma \) from the category Frm to the category TOP of topological spaces and continuous maps. For any frame \( L, \Sigma L \) is the space whose elements, called the points of \( L \), are the frame homomorphisms \( \xi: L \to 2 \), and whose topology \( \Omega \Sigma L \) consists of the sets \( \Sigma_a = \{ \xi | \xi(a) = 1 \} \). Recall that \( L \) is called spatial whenever its points separate its elements, which is equivalent to the requirement that the frame homomorphism \( L \to \Omega \Sigma L \) given by \( a \mapsto \Sigma_a \) be an isomorphism. Proving the spatiality of certain types of frames usually requires some choice principle such as the Ultrafilter Theorem for Boolean algebras which we shall assume whenever needed. A particular class of frames to which this applies are the coherent frames (Banaschewski [1]).

The following description of the spectrum of the congruence lattice of a frame appears in [10] and is used by Simmons in [11] to study the properties of \( N\Omega(X) \).

**Proposition 2.** There exists a homeomorphism \( \gamma_L: \Sigma CL \to Sk \Sigma L \), natural in \( L \).

**Remark 1.** The homeomorphism \( \gamma_L \) is determined by the continuous one-one map \( \Sigma v_L: \Sigma CL \to \Sigma L \) (remember that \( \Sigma L \) and \( Sk \Sigma L \) have the same underlying set). Moreover \( \gamma_L \) determines a frame isomorphism \( \Omega Sk \Sigma L \to \Omega \Sigma CL \), and since the latter is the spatial reflection of \( CL \) this says: the Skula topology of the spectrum \( \Sigma L \) is the spatial reflection of the congruence lattice \( CL \).

**Remark 2.** Frith [5] shows that \( v_L: L \to CL \) is the universal frame homomorphism from \( L \) with the property that each element in the image is complemented. Using this, one has an alternative proof that every \( L \to 2 \) factors through \( v_L \).
We are now in the position to give the simple proof of the sufficiency of the noetherian condition in Hoffmann’s result for sober spaces quoted earlier [6]. For this, recall that a space \( X \) is called noetherian whenever its frame \( \Omega X \) of open sets is noetherian, and sober whenever the usual continuous map \( \varepsilon_X: X \to \Sigma \Omega X \), taking \( x \in X \) to the point \( \hat{x} \) of \( \Omega X \) given by \( \hat{x}(U) = \text{card}(U \cap \{x\}) \), is one-one and onto. Note that the latter implies \( X \) is \( T_0 \).

**Proposition 3.** For any topological space \( X \), \( SkX \) is compact Hausdorff iff \( X \) is sober and noetherian.

**Proof.** (\( \Rightarrow \)) This part of the argument is entirely topological, and the reader is referred to the straightforward proof given in [6].

(\( \Leftarrow \)) Since \( X \) is sober and hence \( T_0 \), \( SkX \) is Hausdorff by its definition and we need only check compactness. For noetherian \( X \), \( C \Omega X \) is coherent (Proposition 1) and consequently spatial (Johnstone [7]) and hence \( \Sigma C \Omega X \) is compact. On the other hand, if \( X \) is also sober one has \( SkX \cong Sk\Sigma \Omega X \), and then the homeomorphism \( Sk\Sigma \Omega X \cong \Sigma C \Omega X \) of Proposition 2 for \( L = \Omega X \) shows \( SkX \) is compact.

**References**


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Bernhard Banaschewski, J. L. Frith and C. R. A. Gilmour, On the congruence lattice of a frame ...................................................... 209
Paul S. Bourdon, Density of the polynomials in Bergman spaces .......... 215
Lawrence Jay Corwin, Approximation of prime elements in division algebras over local fields and unitary representations of the multiplicative group .......................................................... 223
Stephen R. Doty and John Brendan Sullivan, On the geometry of extensions of irreducible modules for simple algebraic groups .......... 253
Karl Heinz Doermann and Reinhard Schultz, Surgery of involutions with middle-dimensional fixed point set ........................................ 275
Ian Graham, Intrinsic measures and holomorphic retracts ................. 299
John Robert Greene, Lagrange inversion over finite fields ............... 313
Kristina Dale Hansen, Restriction to GL_2(\mathbb{C}) of supercuspidal representations of GL_2(F) ...................................................... 327
Kei Ji Izuchi, Unitary equivalence of invariant subspaces in the polydisk .... 351
A. Papadopoulos and R. C. Penner, A characterization of pseudo-Anosov foliations ................................................................. 359
Erik A. van Doorn, The indeterminate rate problem for birth-death processes ................................................................. 379
Ralph Jay De Laubenfels, Correction to: “Well-behaved derivations on C[0, 1]” ................................................................. 395
Robert P. Kaufman, Correction to: “Plane curves and removable sets” ...... 396
Richard Scott Pierce and Charles Irvin Vinsonhaler, Correction to: “Realizing central division algebras” ....................................... 397