APPROXIMATION OF PRIME ELEMENTS IN DIVISION ALGEBRAS OVER LOCAL FIELDS AND UNITARY REPRESENTATIONS OF THE MULTIPLICATIVE GROUP

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Let $K$ be a locally compact, non-Archimedean field of residual characteristic $p$, and let $D$ be a central division algebra of dimension $n^2$ over $K$. In constructing the irreducible unitary representations of $D^\times$, a technical question repeatedly arises. Let $x \in D$, and let $x_1$ be "close" to $x$ (in the sense that, for the usual absolute value on $D$, $|x - x_1| < |x|$). Let $D_x$, $D_{x_1}$ be the subalgebras of elements commuting with $x$, $x_1$ respectively. Is it possible to pick a prime element $\eta_1 \in D_{x_1}$ and an element $\eta_0 \in D_x$ that are also close, and how close can $\eta, \eta_1$ be to one another? The first part of this paper analyzes this problem. It turns out that $\eta, \eta_1$ can be chosen close enough to one another so that Clifford-Mackey theory easily permits the construction of $(D^\times)^A$ only if $p^2 = n$ or $p^2 \neq n$. The construction has been given in earlier papers except for the case where $p|n, p \neq n,$ and $p^2|n$; the second part of the paper is a construction of $(D^\times)^A$ in this remaining case.

We recall some details of the construction of $(D^\times)^A$. Let $\pi$ be a prime of $K$, and let $K_n$ be an unramified extension of $K$ of degree $n$, embedded in $D$. One can choose a prime $\eta \in D$ such that $\eta^n = \pi$ and conjugation by $\eta$ is an automorphism of $K_n$ generating $Gal(K_n/K)$. We write $\eta a \eta^{-1} = a^\sigma$, $a \in K_n$; $\sigma \in Gal(K_n/K)$ is the Hasse invariant of $D$ up to isomorphism. (See [11] for these and other unreferenced facts about division algebras.) Let $R$ be the group of roots of unity in $K_n$ of order prime to $p$; let $O = O_D$ be the ring of integers in $D$, and let $P = \eta O$ be the prime ideal in $O$. Then one has

$$D^\times = U\langle \eta \rangle, \quad U = (1 + P)R,$$

where $U$ is the group of units of $O$, the products are semidirect products, and the first subgroup is normal. The main step in computing the irreducible representations of $D^\times$ is that of computing the irreducibles of $G = 1 + P$, and it is on this step that we concentrate. The general idea is this: a representation $\pi_0$ of $G$ will be trivial on some normal subgroup $(1 + P^{m+1})$ of $G$. Choose $m$ as small as possible; for convenience of exposition, assume $m$ odd, and let $2m' = m + 1$. Then $\pi_0$ is a representation of $G/(1 + P^{m+1})$, and $(1 + P^{m'}/(1 + P^{m+1})$ is an Abelian sub-
group. Furthermore, \((1 + P^m')/(1 + P^{m+1}) \cong P^m'/P^{m+1}\) via the map \(1 + y \mapsto y\). In this way, \(\pi_0(1 + P^m)\) can be regarded as a direct sum of \((1\text{-dimensional})\) characters of \(P^m\), all trivial on \(P^{m+1}\). To describe these characters, let \(\chi\) be an additive character of \(K\) that is trivial on \(P \cap K\) but nontrivial on \(O \cap K\); for \(x \in D\), let \(\chi_x(y) = \chi(\text{Tr}_{D/K}(xy))\). Then every character on \(D\) is of the form \(\chi_x\) for some \(x \in D\); \((P^{m+1})^\perp = \{ x \in P^{-m} \}\), and restricting to \(P^m\) means that \(x\) is determined only mod \(P^{-m+1}\). Given \(\pi_0\), we thus get \(x \pmod{P^{m+1}}\), and up to conjugation by \(G\) by the condition that \(\chi_x\) occurs in \(\pi_0|_{1 + P^m}\). Determining \(\pi_0\) then becomes a problem in Mackey (or Clifford) theory.

It is here that the relation between \(n\) and \(p\) becomes important. One needs to determine the \(w \in G\) for which \(\chi_x(wyw^{-1}) = \chi_x(y)\) for all \(y \in 1 + P^m\). It is easy to see that \(w\) satisfies this condition if \(w\) commutes with \(x\) mod some sufficiently high power of \(P\). In the tamely ramified case and in the case \(n = p\), one can arrange to have \(w\) and \(x\) commute; this simplifies matters. (See [1], [5], [4], [3], and [10] for details and further results.) For \(n = p^2\), however, matters are less simple. The problem is that one can pick elements \(x \in D\) such that \([K(x):K] = p^2\), but such that \(K(x)\) contains no extension of degree \(p\) over \(K\). However, it may be possible to choose \(x_1\) such that \([K(x_1):K] = p\), and \(x\) and \(x_1\) agree modulo some moderately high power of \(P\). Certain elements commuting with \(x_1\) commute with \(\chi_x\) and not with any element in \(K(x) \sim K\). In [2], this problem was handled by showing that in the division algebras \(D_{x_1}\), \(D_x\) of elements commuting with \(x_1\), \(x\) respectively, one could find prime elements \(\eta_1, \eta_0\) respectively that were congruent mod some moderately high power of \(P\); thereafter, the analysis could proceed roughly as before.

It is therefore useful to consider the following type of approximation question: suppose that \(x \in P^{j_0} \sim P^{j_0+1}\), and suppose that \(x_1 \equiv x \pmod{P^{j_1}}\), \(j_1 > j_0\). Let \(D_x, D_{x_1}\) be the division algebras of elements in \(D\) commuting with \(x, x_1\) respectively. How close to one another can one choose an element \(\eta_0 \in D_x\) and a prime element \(\eta_1 \in D_{x_1}\)? This is the topic of the first half of this paper. It turns out, unsurprisingly, that the difficulties arise primarily with wild ramification; much of the analysis, therefore, deals with the case of totally wildly ramified extensions, and the notation for this part (§§2, 3) is somewhat different from the notation in other sections.

The results are rather negative; they show that the methods of [2] apply directly only to the cases \(p^2|n\) and \(p^2 = n\). (One needs to replace \(P^s_{j+1} - s_j\) in Theorem 4.2 by \(P^s_{j+1} - s_j\) to use these methods, and in general that is impossible.) In the second part of this paper, we construct the
irreducible representations for the case \( n = pn_0 \), with \( n_0 \) prime to \( p \). The cases \((n, p) = 1\) and \( n = p^2 \) have previously been treated, as noted earlier.

**PART I. APPROXIMATION THEOREMS**

2. **Some results on finite fields.** Let \( k \) be a finite field of characteristic \( p \); for each integer \( r \geq 0 \), denote by \( k_r \) the extension field with \([k_r:k]=p^r\).

In particular, \( k_0 = k \). Fix \( n \geq 1 \); we shall be interested primarily in the \( k_r \) with \( r \leq n \). Let \( \sigma \) generate \( \text{Gal}(k_n/k) \). Write \( \text{Tr}_{i/j} \) for \( \text{Tr}_{k_i/k_j} \) (if \( i \geq j \)), and write \( \text{Tr}_i \) for \( \text{Tr}_{i/0} \); write \( N_{i/j} \), \( N_i \) for the corresponding norm maps.

**PROPOSITION 2.1.** (a) There exist elements \( a_1, \ldots, a_n \in k_n \) such that

\[
(2.1) \quad \alpha_r^\sigma - \alpha_r = \prod_{i=1}^{r-1} \alpha_i^{p-1}, \quad 1 \leq r \leq n \quad \text{(and \( \alpha_1^\sigma - \alpha_1 = 1 \))}
\]

moreover, \( k_r = k(\alpha_1, \ldots, \alpha_r) = k(\alpha_r) \).

(b) For \( 0 \leq j < p^n \), define \( \beta_j = \beta(j) \) by

\[
\beta_j = \prod_{i=1}^r \alpha_i^{m_i}, \quad \text{where } \sum_{i=1}^r m_i p^{i-1} = j; \quad \beta_0 = 1.
\]

Then the \( \beta_j \) with \( j < p^r \) give a vector space basis for \( k_r/k \).

(c) Let \( V_j \) be the vector space spanned by the \( \beta_i \) with \( i < j \); in particular, \( V_0 = \{0\} \). Then for all \( j \geq 1 \), there is a nonzero \( c \in \mathbb{F}_p \) (the prime field) such that

\[
(2.2) \quad \beta_j^\sigma - \beta_j - c\beta_{j-1} \in V_{j-1}.
\]

(d) \( \text{Tr}_m(\beta_{p^m-1}) = (-1)^m; \) for \( j < p^{m-1} \), \( \text{Tr}_m(\beta_j) = 0 \).

**Proof.** We proceed by induction on \( n \).

(1) Let \( n = 1 \). As \( \text{Tr}_1 1 = 0 \), there exists \( \alpha_1 \in k_1 \) with \( \alpha_1^\sigma - \alpha_1 = 1 \). Then \( \alpha_1 \not\in k_0 \), the fixed field of \( \sigma \), and we must have \( k_0(\alpha_1) = k_1 \). Now (a) and (b) follow. For (c) and (d), we use the formula

\[
(2.2) \quad (\alpha_i^j)^\sigma = (\alpha_i + 1)^j = \alpha_i^j + j\alpha_i^{j-1} + \sum_{i=2}^{j} \binom{j}{i} \alpha_i^{j-i}.
\]

Now (c) follows immediately; just let \( j = 1, 2, \ldots, p - 1 \). As

\[
j\alpha_i^{j-1} \equiv (\alpha_i^j)^\sigma - \alpha_i^j \mod V_{j-1}, \quad j \leq p - 1,
\]
the second part of (d) is also immediate. Finally, the case \( j = p \) gives
\[
(\alpha_1^p)^\sigma = (\alpha_1 + 1)^p = \alpha_1^p + 1.
\]
Hence \( \sigma \) fixes \( \alpha_1^p - \alpha_1 \), or \( \exists \delta \in k \) with \( \alpha_1^p - \alpha_1 - \delta = 0 \). The roots of \( X^p - X - \delta = 0 \) are therefore \( \alpha_1 \) and its conjugates. Write the coefficient of \( X \) as a symmetric function of the roots:
\[
-1 = (-1)^{p-1} \sum_{j=0}^{p-1} N(\alpha_1)/\alpha_1^{\sigma_j} = (-1)^{p-1} \text{Tr}_\nu(N(\alpha_1)/\alpha_1),
\]
where \( N = \text{norm in } k_1/k \).

But the conjugates of \( \alpha_1 \) are \( \alpha_1 + 1, \ldots, \alpha_1 + p - 1 \), and thus \( N(\alpha_1)/\alpha_1 \equiv \alpha_1^{p-1} \mod V_{p-1} \). Now the first part of (d) is obvious, since \( (-1)^{p-1} \equiv 1 \mod p \) for all \( p \).

(2) Now assume the result for \( n - 1 \). Then
\[
\text{Tr}_n \beta(p^{n-1} - 1) = p \text{Tr}_{n-1} \beta(p^{n-1} - 1) = 0,
\]
so that we can find \( \alpha_n \in k_n \) with \( \alpha_n^\sigma - \alpha_n = \beta(p^{n-1} - 1) \). As \( \text{Tr}_{n-1} \beta(p^{n-1} - 1) \neq 0 \), \( \alpha_n \notin k_{n-1} \). Just as in (1), we get all of (a) and (b) except for the claim that \( k_n = k(\alpha_n) \). We do have \( k_n = k_{n-1}(\alpha_n) \). Also, \( \beta(p^{n-1} - 1) \in k(\alpha_n) \), and (b) and (c), applied to \( k_{n-1} \), show that \( k(\beta(p^{n-1} - 1)) = k_{n-1} \). Hence
\[
k(\alpha_n) = k(\alpha_n, \beta(p^{n-1} - 1)) = k_{n-1}(\alpha_n) = k_n,
\]
as required.

We need to prove (c) only for those \( \beta(j) \) with \( j > p^{m-1} \). We first prove it for \( \alpha_n^i, 1 \leq i \leq p - 1 \). For \( i = 1 \), the result is immediate from the definition of \( \alpha_n \); in general, it follows from the formula
\[
(\alpha_n^i)^\sigma = (\alpha_n + \beta(p^{n-1} - 1))^i.
\]
Now we deal with the remaining case: \( \beta_j = \alpha_n^i \beta(l), 1 \leq l < p^{n-1} \) and \( i > 0 \). Then
\[
\beta_j^\sigma = (\alpha_n^i + \gamma)(\beta(l) + c\beta(l - 1) + \delta),
\]
where \( \gamma \in V_{i+1}, \delta \in V_{l-1}, \) and \( c \in F^p \),
\[
= \beta_j + c\alpha_n^i \beta(l - 1) + \alpha_n^i \delta + \epsilon, \quad \epsilon \in V_{i+1},
\]
which is (c) for \( \beta_j \). The same argument as in (1) now shows that \( \text{Tr}_n \beta_j = 0 \) if \( j < p^{m-1} \).
We need more calculation to get \( \text{Tr}_n \beta(p^n - 1) \). For \( \gamma \in k_n \), define
\[
\gamma(0), \gamma(1), \ldots \text{ inductively by}
\]
\[
\gamma(0) = \gamma, \quad \gamma(j + 1) = \gamma(j) \circ - \gamma(j).
\]
An easy induction gives
\[
(2.3) \quad \gamma^s = \sum_{j=0}^{s} \binom{s}{j} \gamma(j);
\]
in particular,
\[
(2.4) \quad \gamma^s = \gamma + \gamma(s) \quad \text{if } s \text{ is a power of } p.
\]
Now let \( \gamma = \beta_r \). Applying (c) \( p \) times and using (2.4), we get
\[
\beta_r^s - \beta_r \equiv c' \beta_{r-p} \mod V_{r-p} \quad \text{for some } c' \in F_p,
\]
and an easy induction shows that if \( s \) is a power of \( p \), then there is a \( c \in F_p \) such that
\[
\beta_r(s) = \beta_r^s - \beta_r \equiv c \beta_{r-s} \mod V_{r-s}.
\]
For \( r = p^n-1 \), we get
\[
\alpha_n^{p^n-1} = \alpha_n + c, \quad c \in F_p.
\]
But \( \sigma^{p^n-1} \) generates \( \text{Gal}(k_n/k_{n-1}) \), and \( k_n = k_{n-1}(\alpha_n) \). Thus, from (d) in the case \( n = 1 \),
\[
\text{Tr}_{n/n-1}(\alpha_n/c)^{p-1} = -1.
\]
As \( c^{p-1} = 1 \), we have
\[
\text{Tr}_{n/n-1} \alpha_n^{p-1} = -1,
\]
or
\[
\text{Tr}_{n/n-1} \beta(p^n - 1) = -\beta(p^{n-1} - 1).
\]
The inductive hypothesis on \( \beta(p^{n-1} - 1) \) now gives the first part of (d).

We note two corollaries. The first was essentially proved in the course of the above proof.

**Corollary 1.** If \( s = ap^r \) with \( p \nmid a \), then for each \( j \) there is a nonzero \( c \in F_p \) with
\[
\beta_j^s - \beta_j - c \beta_{j-p'} \in V_{j-p'}.
\]
(If \( j - p' < 0 \), then \( \beta_{j-p'} \) is taken to be 0.)
Proof. This is an easy calculation from (c) and (2.3).

**COROLLARY 2.** If \( r < n \), and if \( \gamma \in k_n \) satisfies \( \text{Tr}_{n/r}\gamma \in k \), then \( \gamma \in V_{p^n - p^r + 1} \).

**Proof.** This follows from (d) (applied to \( k_n/k_r \)) and the linearity of the trace.

We now prove a result about “partial traces”.

**LEMMA 2.2.** Let \( r < s \); suppose that \( p \nmid a \). Then \( \exists \) a nonzero \( c \in \mathbb{F}_p \) such that

\[
\left( \sum_{i=0}^{ap^{s-r}-1} \beta_{j}^{\alpha p^{r}} \right) \equiv c \beta \left( j - p^s + p^{s-r} \right) \mod V_{j - p^s + p^{s-r}}.
\]

**Proof.** For \( \gamma \in k_n \), define \( \gamma\{ j \} \) as in the proof of Lemma 2.1. Then (2.4) and induction give

\[
\gamma^{\alpha p^{r}} = \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) \gamma \{ j p^r \}.
\]

Hence

\[
\sum_{i=0}^{ap^{s-r}-1} \gamma^{\alpha p^{r}} = \sum_{i=0}^{ap^{s-r}-1} \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) \gamma \{ j p^r \}
\]

\[
= \sum_{j=0}^{ap^{s-r}-1} \sum_{i=j}^{ap^{s-r}-1} \left( \begin{array}{c} i \\ j \end{array} \right) \gamma \{ j p^r \} = \sum_{j=0}^{ap^{s-r}-1} \left( \begin{array}{c} ap^{s-r} \\ j + 1 \end{array} \right) \gamma \{ j p^r \}.
\]

As

\[
\left( \begin{array}{c} ap^{s-r} \\ j + 1 \end{array} \right) \equiv 0 \mod p \quad \text{if} \quad p^{s-r} + j - 1
\]

and

\[
\left( \begin{array}{c} ap^{s-r} \\ p^{s-r} \end{array} \right) \equiv a \mod p,
\]

we have

\[
\sum_{i=0}^{ap^{s-r}-1} \gamma^{\alpha p^{r}} = a \gamma \{ p^r(p^{s-r} - 1) \} + \gamma',
\]

where \( \gamma' \) is a \( k \)-linear combination of terms of the form \( \gamma \{ p^r(ip^{s-r} - 1) \} \),
Let $\gamma = \beta_j$. Since (c) of Proposition 2.1 and an easy induction shows that for each $i$ there is $c \in \mathbb{F}_p$ such that $\gamma^i - c\beta_{j-i} \in V_{j-i}$, the result follows.

Recall that $N_{i/j}$ is the norm map from $k_i$ to $k_j$ ($j < i$). We shall also need the following result about norms and traces in $k_m$.

**Proposition 2.3.** Let $s < m$. Given $\alpha \in k^\times$, there exists a nonzero $\lambda \in k_n$ with $N_{n/s}\lambda \in k$ and $\text{Tr}_n(\alpha\lambda) = 0$.

**Proof.** We use the following lemma:

**Lemma 2.4.** Let $q$ be a power of $p$; let $q' = (q^{p^s} - 1)/(q - 1)$. Then for all $n > s$, $q'$ and $(q^{p^n} - 1)/q'$ are relatively prime.

**Proof of the Lemma.** Since

$$q^{p^n} - 1 = (q^{p^s} - 1) \sum_{j=0}^{p^{n-s}-1} q^{jp^s}$$

and $q^{jp^s} \equiv 1 \mod q^{p^s} - 1$, we see that

$$(q^{p^n} - 1)/(q^{p^s} - 1) \equiv p^{n-s} \mod q^{p^s} - 1.$$  

Hence $q'(q - 1)$ and $(q^{p^n} - 1)/q'(q - 1)$ are relatively prime. Similarly, $q'$ and $q - 1$ are relatively prime, and this proves the lemma.

**Proof of Proposition 3.3.** Let $|k| = q$, $(q^{p^n} - 1)/(q^{p^s} - 1) = r$. As $N_{n/s}\lambda = \lambda'$, we need to pick $\lambda$ so that its order divides $(q - 1)r = (q^{p^s} - 1)/q'$. Now let $\beta$ generate $k_n^\times$ as a cyclic group, and let $\alpha = \beta^t$. From Lemma 3.4, we can find an integer $a$ such that

$$(q - 1)r|t + aq'.$$

Let $\lambda = \beta^{aq'}$. Then the order of $\lambda$ divides $(q - 1)r$, and $\alpha\lambda$ has order dividing $q'$. But then $\alpha\lambda \in k_s$, so that $\text{Tr}_n\alpha\lambda = 0$.

3. **Prime elements in sub-division algebras:** Totally wildly ramified case. We begin by describing the notational conventions in this section, since they are somewhat different from those for other sections. We assume that the index of $D$ over $K$ is $p^n$ (so that $\dim_K D = p^{2n}$); we choose $x \in D$, and assume that $K(x)$ is totally ramified over $K$. We assume further that $x$ is in general position, or that $|x| < |x + z|$ for all $z \in K$. Define $n$ by $|x| = p^n$. 


We recall some results from [7]. For each \( j \in \mathbb{Z} \), let \( x_{(j)} \) be an element such that \( x_{(j)} \equiv x \mod P^{-m+j+1} \) and \([K(x_{(j)}):K]\) is minimal (subject to the above condition). For \( j < 0 \), we have \( x_{(j)} \in K \), and we take \( x_{(j)} = 0 \) there; for sufficiently large \( j \), we may take \( x_{(j)} = x \). The fields \( K(x_{(j)}) \) are all totally ramified over \( K \). There are integers \( s_0 = 0, s_1, \ldots, s_{r-1} \) such that for each \( r \), \([K(x_{(s_r)}):K] > [K(x_{(s_{r-1})}):K]\). These integers are the jump points of \( x \). Set \( s = -\infty \) and define \( x_j = x_{(s_r-1)} \), with \( x = x \). We may (and henceforth do) assume that \( x_{(j)} = x_r \) if \( s_{r-1} < j < s_r \). We call the \( x_j \) the approximating elements for \( x \). Let \( D_r \) be algebra of elements commuting with \( x_r \). We shall be interested in how closely we can approximate a prime element in \( D_r \) by one in \( D_{r-1} \).

Write \( K_n \) for the unramified extension of degree \( p^n \) that is normalized by \( \eta \), and let \( K_b \) be the subfield of \( K_n \) of degree \( p^b \) over \( K \). (Note: \( K_n \) is what was called \( K_{p^n} \) in the introduction.) We write \( k_n \) for the residue class field of \( K_n \)(and of \( D \)); this corresponds to the notation in \( \S 2 \). The residue class map \( \theta \rightarrow \theta/P \equiv k_n \) is bijective on \( R \cup \{0\} \), and we generally identify \( R \cup \{0\} \) with \( k_n \), for notational ease. Thus we write a typical element \( y \in D \) as \( \sum_{j=0}^{n-1} \delta_j \eta^j, \delta_j \in k_n \). We use \( \alpha_j \) \((1 \leq j \leq n) \) and \( \beta_j = \beta(j) \) \((0 \leq j \leq p^n - 1) \) for the elements of \( k_n \) so denoted in \( \S 2 \), and write

\[
(3.1) \quad x = \sum_{j=-m}^{\infty} \gamma_j \eta^j.
\]

Let \([K(x_{(j)}):K] = p^{a_j} = e_j \), and let \( a_j + b_j = n \). Since \( x \) is in general position, \( m \) is divisible by \( p^{b_1} \) but not by \( p^{b_1+1} \). Furthermore, \((\gamma_m \eta^{-m}) p^{a_1} \in K \). It follows that we may assume \( \gamma_m \in K \), possibly by replacing \( \eta \) with some \( \epsilon \eta, \eta \in k_n \). (Every element of the form \( c \eta^{-mp^{a_1}} \), \( c \in k \), can be written as \((c' \eta^{-m}) p^{a_1} \) with \( c' \in k \), essentially because \( c' \mapsto (c')^p \) is an automorphism of \( k \). See p. 55 of [2].) It also follows from p. 55 of [2] that we may assume (after conjugating \( x \) by an element of \( G \)) that

\[
(3.2) \quad K_{b_j} \subseteq D_j.
\]

Note, incidentally, that \( a_j \) increases with \( j \), while \( b_j \) decreases.

Our first job is to find a "normal form" for the \( x_j \).

**Proposition 3.1.** By possibly conjugating \( x \) in \( D \), we may assume that

\[
x_i - x_{i-1} = \sum_{j=-m+s_{i-1}}^{\infty} \gamma_i(j) \eta^j,
\]

with \( \gamma_{i:-m+s_{i-1}} = c\beta(p^m - p^{b_{i-1}}) \) for some nonzero \( c \in k \).
Proof. Induction on $i$. For $i = 1$, this is just our assumption that \( \gamma_n \in k \); thus assume $i \geq 2$. Apply Satz 8 of [7] with $\nu = x_{i-1}$, $\omega$ a prime element in $D$, such that $\omega \equiv \eta \mod P^2$, and $\beta = \gamma_{i-n+s_{i-1}} \omega^{-n+s_{i}}$; if $\psi_i$ is the irreducible polynomial satisfied by $x_{i-1}$, then

$$
\psi_i(x_i) \equiv (\text{Tr}_{m/b_{i-1}} \gamma_{i-n+s_{i-1}}) \eta^b \mod P^{b+1},
$$

where $b$ is an integer divisible by $p^{b_{i-1}}$ but not by $p^{b_{i-1} - 1}$. Set

$$
\delta = \text{Tr}_{n/b_{i-1}} (\gamma_{i-n+s_{i-1}}).
$$

Then $\delta \in k_{b_{i-1}}$. Furthermore,

$$
(\delta \eta^b)^{p^\nu} = (N_{n/b} \delta) \eta^{bp^\nu} \in K;
$$

otherwise, $N_{n/b} \delta \in k_n \setminus k$ and Hensel's lemma implies that $K(x_i)$ is not totally ramified over $K$. But since $\delta \in k_{b_{i-1}}$,

$$
N_{n/b} \delta = (N_{b_{i-1}/b} \delta)^{p^{n-b_{i-1}}}.
$$

Hence $N_{i/i-1} \delta \in k$, and there exists $d \in k$ with $N_{i/i-1}(d/\delta) = 1$.

Suppose that $\delta \notin k$. Set $h = b_{i-1} - b_i$; let $-m + s_i = g$. Then $p^{b_i}|g$, but $p^{b_{i-1}} \nmid g$. Hence

$$
(\delta \eta^g)^{p^h} = (N_{i/i-1} \delta) \eta^{gp^h} = (d \eta^g)^{p^h},
$$

and so $\delta \eta^g$ and $d \eta^g$ satisfy the same equation over $K$. Therefore, they are conjugate in $D$ (and, in fact, by an element of $K_m$). That is, there is a prime $\eta'$ such that

$$
\delta \eta^g = d(\eta')^g.
$$

Hence $(\eta')^{gp^h} = d^{-p^h}(d \eta'^g)^{p^h} = d^{-p^h}(\delta \eta^g)^{p^h} = d^{-p^h}(d \eta^g)^{p^h} = \eta'^{gp^h}$, or $(\eta')^{p^h-1} = \eta'^{p^h-1}$. But $x_{i-1}$ commutes with $K_{b_{i-1}}$; hence in the expansion of $x_{i-1}$,

$$
x_{i-1} = \sum_{j=-m}^{\infty} \gamma_{j-i+1} \eta^j, \text{ say,}
$$

we must have $\gamma_{j-i+1}^{(i-1)} = 0$ unless $p^{b_{i-1}}|j$. It follows that the expansion of $x_{i-1}$ is the same if we use powers of $\eta'$ instead of powers of $\eta$. On the other hand, we have

$$
x_i - x_{i-1} = \sum_{j=m-s_{i-1}}^{\infty} \gamma_{j-i} \eta^j, \quad \gamma_{i-n+s_{i-1}} = \gamma_{i-n+s_{i-1}} d/\delta,
$$

so that

$$
\text{Tr}_{m/b_{i-1}} \gamma_{i-n+s_{i-1}} \in k.
$$
As \( \eta, \eta' \) are related by a conjugation in \( D \), this all means that we may assume that \( \delta \in k \). Now Corollary 2 of Proposition 2.1 says that

\[
\gamma_{i:n+s_i} \in V_{p^{n}-p_{b_{i-1}}}.
\]

To limit \( \gamma_{i:n+s_{i-1}} \) more, we conjugate. Write

\[
\gamma_{i:n+s_{i-1}} = a\beta\left(p^{m} - p^{b_{i-1}}\right) + \gamma, \quad \gamma \in V_{p^{n}-p_{b_{i-1}}} \quad \text{and} \quad a \in k.
\]

The inductive hypothesis gives \( \gamma_{i-1:m+s_{i-2}} = c\left(p^{n} - p^{b_{i-2}}\right), \quad c \neq 0 \). Write

\[
\beta = \beta\left(p^{m} - p^{a_{i-2}}\right); \quad \text{conjugate} \ x \ \text{with} \ 1 + \delta \eta^{s_{i-2}}, \quad \delta \in k_{a_{i-2}}. \quad \text{Then} \ 1 + \delta \eta^{s_{i-2}} \text{commutes with the} \ \gamma_{i}\eta' \ \text{such that} \ j < -m + s_{i-1}, \text{and the effect of the conjugation (mod} \ P^{n}-p_{b_{i-1}} \text{) is to change} \ \gamma \ \text{to}
\]

\[
\gamma - c\left(\beta \delta^{\sigma^{m+s_{i-1}} - \beta^{\sigma^{n-s_{i-1}}} \delta}\right).
\]

Let \( \sigma' = \sigma^{s_{i-2}}, \) so that \( \sigma' \) generates \( \text{Gal}(k_{n}/k_{b_{i}}) \); set \( \sigma^{-m+s_{i-1}} = \sigma'' \). Then \( \sigma'' = (\sigma')^{a\beta p^{b_{i-1}} - h_{i}}, \quad p \not\equiv a, \) and, in the notation of \( \S 2,\)

\[
\beta \delta^{\sigma''} - \delta \beta^{\sigma'} = \beta\left(\delta + c_{0}\delta\left(p^{h_{i}}\right) + \delta'\right) - \left(\beta + c_{0}\beta\left(p^{h_{i-1}}\right) + \beta'\right)\delta
\]

(where \( c_{0}, d_{0} \in k_{0}; \quad \delta' \in \text{span}\left(\delta\left(p^{h_{i}} + 1\right), \ldots, \delta\left(p^{n}\right)\right); \quad \text{and} \ \beta' \in \text{span}\left(\beta\left(p^{h_{i-1}} + 1\right), \ldots, \beta\left(p^{n}\right)\right) = c_{0}\beta\delta\left(p^{h_{i}}\right) - d_{0}\delta\beta\left(p^{h_{i-1}}\right) + \beta\delta' - \delta \beta' \)

But \( \delta \in V_{p^{h_{i-1}}}, \) and it is not hard to verify that \( \beta \delta\left(p^{h_{i}}\right) \) can be any linear combination of \( \beta\left(p^{m} - p^{h_{i-2}}\right), \ldots, \beta\left(p^{m} - p^{h_{i-1}-1}\right) \), while the remaining terms are in \( V_{p^{m+1}-p_{b_{i-2}}}. \) Hence we can choose \( \delta \) so that

\[
\gamma - c\left(\beta \delta^{\sigma''} - \beta^{\sigma''} \delta\right) \in V_{p^{n}-p_{b_{i-2}}},
\]

and we may therefore assume that \( \gamma \in V_{p^{n}-p_{b_{i-2}}}. \) Continue inductively; the next step is to conjugate with \( 1 + \delta \eta^{s_{i-2}} \), where \( \delta \in k_{b_{i-3}}, \) and thus to move \( \gamma \) into \( V_{p^{m+1}-p_{b_{i-3}}}. \) Since \( b_{0} = m \), we eventually move \( \gamma \) to 0. Hence we may assume that

\[
\gamma_{i:-m+s_{i}} = a\beta\left(p^{m} - p^{h_{i}}\right), \quad a \in k.
\]

If \( a = 0, \) then \( x(s_{i}) = x(s_{i-1}). \) This is impossible, since \( s_{i} \) is a jump point. Hence \( a \neq 0, \) and the proof is complete.

**Lemma 3.2.** (a) Let \( \eta_{1} = \eta + \delta_{2}\eta^{2} + \delta_{3}\eta^{3} + \cdots \) be a prime element of \( D; \) suppose that \( y \in P^{t} \) commutes with \( \eta_{1} \) mod \( P^{s+2}, \) with \( s \geq t \) (i.e., \( [y, \eta_{1}] \in P^{s+2} \)). Then there exists \( y_{1} \in K(\eta_{1}) \) such that \( y_{1} \equiv y \) mod \( P^{s+1}. \) (Note that \( K(\eta_{1}) \) is the algebra of elements commuting with \( \eta_{1}, \) since \( [K(\eta_{1}); K] = p^{n}. \))

(b) If \( \eta_{1} \) is such that \( \delta_{j} = 0 \) unless \( j \equiv 1 \) mod \( p^{r} \) (where \( r \leq m \)), and if \( y = \sum_{j=1}^{\infty} \epsilon_{j}\eta^{j}, \) with \( \epsilon_{j} = 0 \) unless \( p^{r} \mid j, \) then one can pick \( y_{1} \) as above so that \( K(y_{1}) \) is totally ramified of degree \( \leq p^{n-r}. \)
Proof. Let \( y = \sum_{r}^{\infty} \varepsilon_{r} \eta^{r} \); let \( \varepsilon, \eta^{i} \) be the first nonzero term. If \( i \geq s + 1 \), there is nothing to prove. If \( i \leq s \), then \( \varepsilon, \eta^{i} \) commutes with \( \eta \) (as one sees by computing \( \{ y, \eta_{1} \} \)); thus \( \varepsilon_{i} \in k \). Now \( y - \varepsilon_{i} \eta^{i} \) commutes with \( \eta_{i} \mod P_{i+2}^{2} \), and \( |y - \varepsilon_{i} \eta^{i}| < y \). Proceed inductively to produce \( y_{1} \).

(b) Continue with the notation of (a). We must have \( p^{r}|i \), and our construction gives \( y_{1} = \sum_{j=r}^{\infty} \varepsilon_{j} \eta^{j} \), where \( \varepsilon_{j} \in k \) and \( \varepsilon'_{j} = 0 \) unless \( p^{r}|j \).

Hence \( K(y_{1}) \subseteq K(\eta^{p_{r}}) \subseteq K(\eta_{i}) \), so that \( K(y_{1}) \) is totally ramified. As \( \eta_{r}^{p_{r}} \) commutes with \( k_{r} \), the division algebra \( D_{y_{1}} \) of elements commuting with \( y_{1} \) has index \( \geq p^{r} \) over \( K(y_{1}) \), and this implies that \( [K(y_{1}) : K] \leq p^{n-r} \).

We are now ready for the main result of this section.

**Theorem 3.3. (Approximation Theorem.)** Let notation be as mentioned previously.

(a) There exist \( \eta_{1}, \ldots, \eta_{t} \) for \( D_{1}, \ldots, D_{t} \) respectively, such that for \( 2 \leq i \leq t \),

\[
\eta_{i} = \sum_{j=1}^{\infty} \sum_{1 \leq l < i} \delta_{i,j}^{(l)} \eta^{j},
\]

where \( \delta_{i-1,1} = 1; \delta_{i,j} \in k_{b_{i-1}}; \delta_{i,j}^{(l)} = 0 \) for \( j \leq s_{i-1} - s_{i-1} \) and \( l < i - 1 \); and \( \delta_{i-1,j}^{(l)} \in k_{b_{i-1}} \) for \( j \leq s_{i-1} \).

(b) We have \( \eta_{i} = \sum_{j=1}^{\infty} \delta_{i,j} \eta^{j} \), with \( \delta_{i,j} \) is 0 unless \( p^{b_{i}}|j - 1 \).

**Remark.** From (a), \( \eta_{i} \) is congruent \( mod P_{s_{1-1} + s_{2} + 2}^{\infty} \) to the prime element \( \sum_{j=1}^{\infty} \delta_{i,j} \eta^{j} \) of \( D_{i-1} \). As the proof will show, we cannot generally do better.

**Proof.** We use induction on \( t \). For \( t = 1 \), the first statement is vacuous, while the second simply states that \( D_{1} \) has a prime \( \eta_{1} \) such that conjugation by \( \eta_{1} \) generates Gal\((K_{b_{1}}/K)\).

Assume the theorem for \( t - 1 \). From Proposition 3.1,

\[
x_{t} - x_{t-1} \equiv c \beta(p^{m} - p^{b_{t-1}})\eta^{-m+s_{t-1}} \mod P_{m+s_{t-1}+1}^{m}, \quad c \in k \sim \{0\}.
\]

We first find a prime \( \eta'_{i} \) in \( D \) such that \( \eta_{1}, \ldots, \eta_{t-1} \) (from the inductive hypothesis), and \( \eta'_{i} \) satisfy (a), (b) and

(c) \( \eta'_{i} \) commutes with \( x_{t} \mod P_{m+s_{t-1} + 2}^{m} \),

We do this by following the procedure in the proof of Proposition 3.1. Modulo \( P_{m+s_{t-1}+2} \), we have

\[
[\eta_{t-1}, x_{t}] = [\eta_{t-1}, c \beta p_{t-1} \eta^{-m+s_{t-1}}]
\]

\[
= c(\beta_{p_{t-1} - p^{b_{t-1}}} - \beta_{p_{t-1} - p^{b_{t-1}}}) \eta^{-m+s_{t-1}+1}
\]

\[
= \gamma \eta^{-m+s_{t-1}+1},
\]

We are now ready for the main result of this section.

**Theorem 3.3. (Approximation Theorem.)** Let notation be as mentioned previously.

(a) There exist \( \eta_{1}, \ldots, \eta_{t} \) for \( D_{1}, \ldots, D_{t} \) respectively, such that for \( 2 \leq i \leq t \),

\[
\eta_{i} = \sum_{j=1}^{\infty} \sum_{1 \leq l < i} \delta_{i,j}^{(l)} \eta^{j},
\]

where \( \delta_{i-1,1} = 1; \delta_{i,j} \in k_{b_{i-1}}; \delta_{i,j}^{(l)} = 0 \) for \( j \leq s_{i-1} - s_{i-1} \) and \( l < i - 1 \); and \( \delta_{i-1,j}^{(l)} \in k_{b_{i-1}} \) for \( j \leq s_{i-1} \).

(b) We have \( \eta_{i} = \sum_{j=1}^{\infty} \delta_{i,j} \eta^{j} \), with \( \delta_{i,j} \) is 0 unless \( p^{b_{i}}|j - 1 \).

**Remark.** From (a), \( \eta_{i} \) is congruent \( mod P_{s_{1-1} + s_{2} + 2}^{\infty} \) to the prime element \( \sum_{j=1}^{\infty} \delta_{i,j} \eta^{j} \) of \( D_{i-1} \). As the proof will show, we cannot generally do better.

**Proof.** We use induction on \( t \). For \( t = 1 \), the first statement is vacuous, while the second simply states that \( D_{1} \) has a prime \( \eta_{1} \) such that conjugation by \( \eta_{1} \) generates Gal\((K_{b_{1}}/K)\).

Assume the theorem for \( t - 1 \). From Proposition 3.1,
say, where \( \gamma \in V_{n-p^{b_r-1}}. \) Now consider
\[
\eta_{t-1} + \delta \eta_{t-1}^{-s_{t-2}+1}, \quad \delta \in k_{h_{t-2}}.
\]
Modulo \( P^{-m+s_{t-1}+2} \), we have
\[
\left[ \delta \eta_{t-1}^{-s_{t-2}+1}, x_t \right] = \left[ \delta \eta_{t-1}^{-s_{t-2}+1}, x_{t-1} \right]
\]
(since \( x_t - x_{t-1} \in P^{s_{t-1}+1} \))
\[
= (\delta x_{t-1} - x_{t-1} \delta) \eta_{t-1}^{-s_{t-2}+1}.
\]
But \( x_{t-1} = x_{t-2} + \beta_{p^{n-p^{b_{t-2}}}} + \text{higher order terms} \) (which disappear in the commutator once we work mod \( P^{-m+s_{t-1}+2} \)), and \( x_{t-1} \) commutes with \( \delta \). Thus (mod \( P^{-m+s_{t-1}+2} \))
\[
\left[ \delta \eta_{t-1}^{-s_{t-2}+1}, x_t \right] = \beta_{p^{n-p^{b_{t-2}}}} (\delta - \delta^{s_{t-1}-m}) \eta_{t-1}^{-s_{t-2}+1}.
\]
Also, \( \delta^{s_{t-1}-n} = \delta + c' \delta \{ p^{b_{t-1}} \}, \quad c' \in k \setminus \{ 0 \} \); as \( \delta \) runs through \( k_{h_{t-2}} \), \( \delta \{ p^{b_{t-1}} \} \) runs through the elements of \( V_{p^{b_{t-2}}-p^{h_{t-1}}} \). By choosing \( \delta \) appropriate, we may arrange to have
\[
\left[ \eta_{t-1} + \delta \eta_{t-1}^{-s_{t-2}}, x_t \right] = \gamma_{(t-2)} \eta^{-n+s_{t-1}+1} \mod P^{-n+s_{t-1}+1},
\]
\[
\gamma_{(t-2)} \in V_{p^{n-p^{b_{t-2}}}+1}.
\]
Continue inductively; the next step involves adding \( \delta' \eta_{t-2}^{-s_{t-3}} \) to \( \eta_{t-1} + \delta \eta_{t-1}^{-s_{t-2}} \) and showing in the same way that for an appropriate \( \delta' \in k_{h_{t-3}} \), we get the commutator to be
\[
\gamma_{(t-3)} \eta^{-m+s_{t-1}+1} \mod P^{-m+s_{t-1}+2}, \quad \gamma_{(t-2)} \in V_{p^{n-p^{b_{t-3}}}}.
\]
After \( (t-1) \) steps, we get \( \eta' \).

Now let \( x_t' \) be an element in \( K(\eta_t') \) with \( x_t' \equiv x_t \mod P^{-m+s_{t-1}+2} \) and
\( [K(x_t') : K] = [K(x_t) : K] \); this is possible because of Lemma 3.2 and the minimality of \( [K(x_t) : K] \). If \( x_t = x_t' \), then we are done. If not, then for some \( r \geq -m + s_{t-1} + 2 \) with \( p^h/r \) and some \( \gamma \in V_{p^m-p^{b_{t+1}}} \),
\[
x_t - x_t' \equiv \gamma \eta' \mod P^{r+1}.
\]
The argument to prove this is like the one at the start of the proof for Proposition 3.1. Let \( F \) be the minimal polynomials satisfied by \( x_t \), and suppose that \( x_t - x_t' \in P^{r} \). Then we have \( x_t - x_t' \equiv \gamma \eta' \mod P^{r+1} \), with \( \gamma \neq 0 \); moreover, \( F(x_t') = (\text{Tr}_{m/h} \gamma) \eta^h \mod p^{h+1} \), where \( p^h | h-r \).
But \( K(F(x_t)) \subseteq K(x_t) \), which shows that \( p^h | h \). Hence \( p^h | r \). Next, \( F(x_t') \cdot (\eta')^{-h} \) commutes with \( \eta' \); hence \( \text{Tr}_{m/h} \gamma \) commutes with \( \eta' \), and this shows that \( \text{Tr}_{m/h} \gamma = 1 \). Now apply Corollary 2 of Proposition 2.1.

Modulo \( P^{r+2} \) (as the next calculations are also to be understood), we have
\[
[\eta', x_t] = \gamma_1 \eta'^{r+1}, \quad \gamma_1 \in V_{p^m-p^{b_{t}}}.\]
We now argue as in the first part of the proof to remove $\gamma_1$. For $\delta \in \mathbb{k}_{h-1}$, we get

$$\left[ \delta \eta_{l-1}^{-s_{l-1}+1}, x_i \right] = c \left( \delta \beta \left( p^n - p^{h_{l-1}} \right)^{\sigma_{r-s_{l-1}}^r} - \beta \left( p^n - p^{h_{l-1}} \right) \delta^{\sigma_r^r} \right) \eta_{r+1}^r,$$

$$0 \neq c \in k.$$

Since

$$\delta \beta \left( p^n - p^{h_{l-1}} \right)^{\sigma_{r-s_{l-1}}^r} - \beta \left( p^n - p^{h_{l-1}} \right) \delta^{\sigma_r^r} \in - \delta \{ p^h \} \beta \left( p^n - p^{h_{l-1}} \right) + V_{p^n - p^{h_{l-1}}},$$

and since $\delta \{ p^h \}$ can be arbitrary in $V_{p^{h_{l-1}} - p^{h_{l+1}}}$, we can find $\delta$ so that

$$\left[ \eta' + \delta \eta_{l-1}^{-s_{l-1}+1}, x_i \right] = \gamma_{l-1}^{r+1}, \quad \gamma_{l-1}^r \in V_{p^n - p^{h_{l-1}}}.$$

Now iterate, next adding $\delta_{l-2} \eta_{l-2}^{-s_{l-2}+1}$ to put the commutator in $(V_{p^n - p^{h_{l-1}}} \eta_{l-2}^{-r+1})$. The same inductive argument as earlier shows that we can find $\eta'$ commuting with $x_i \mod P^{r+2}$ and satisfying (a) and (b). The theorem follows by using induction on $r$ and then taking limits.

**4. The general approximation theorem.** We now remove the restrictions on $D$ that were imposed for §3; thus $[D:K] = n^2$ and $n$ is arbitrary. Let $x \in D$ and write $x = \sum_{j=0}^\infty \gamma_j \eta_j'$; we assume for notational convenience that $x$ is in general position. Let $s_0, \ldots, s_{l-1}$ be the jump points of $x$, and let $x_1, \ldots, x_t = x$ be approximating elements for $x$; note that $s_0 = j_0$. Write $L_j = L(x_j)$, and let $e_j, f_j$ be the ramification index and residue class degree respectively of $K(x_j)/K$; let $e'_j = e_j/e_{j-1}$, $f'_j = f_j/f_{j-1}$. (We define $e_0 = f_0 = 1$.) Finally, set $D_j = \text{algebra of elements commuting with } x_j$.

**Lemma 4.1.** Let $E_j$ be the maximal tamely ramified extension in $L_j$. By conjugating $x$ in $G$, we may arrange to have $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_t$.

**Proof.** Let $x_0 = \alpha_{j_0} \pi_{j_0}$, and let $(n, j_0) = m_0, n = n_0 m_0$. Then $K(x_0^{n_0})$ is unramified over $K$. Let $n_0 = n_1 p^{n_2}$, where $(n_1, p) = 1$, then $E_1 = K(x_0^{n_2})$ is the maximal tamely ramified extension in $K(x_0)$. Let $y = x_0^{n_2}$, and let $D_y$ be the algebra of all elements commuting with $y$.

Let $l$ be the residue class field of $D_y$, $k_n$ the residue class field of $D$. Then $[k_n:l] = e(E_0/K)$ is prime to $p$. Define

$$S_j = \left\{ \delta \in k_n : \delta \in D_y \right\},$$

$$T_j = \left\{ \varepsilon \in k_n : \text{Tr}_{k_n/l} \delta^{-1} \varepsilon = 0 \text{ for all nonzero } \delta \in S_j \right\}.$$
It is easy to verify the following facts (proved as Lemma 2 of [3]):

(a) $S_0 = l_iS_j$ is a vector space over $l$ of dimension 1 if $f(E_0/K) | j$ and of dimension 0 otherwise.

(b) $T_j$ is also a vector space over $l$, and $S_j \oplus T_j = k_n$. (This uses the fact that $[k_n : l]$ is prime to $p$.)

(c) If $0 \neq \delta \in S_j$, then $S_{j+j'} = \delta S_{j}^{\sigma'} = S_j^{\delta \sigma'}$ and $T_{j+j'} = \delta T_{j}^{\sigma'} = T_j^{\delta \sigma'}$.

Furthermore, $\gamma_{j_0} \in S_{j_0}$.

We show first that we can conjugate $x$ by an element of $G$ so that $x \in D_y$. The proof is by induction (plus an easy convergence argument). Suppose that (by conjugating if necessary) every term in the expansion of $x$ through $\gamma_j \eta^j$ commutes with $y$ (i.e., $\gamma_i \in S$, if $i \leq j$). For $\varepsilon \in T_{j+1-j_0}$, consider

$$(1 + \varepsilon^{j+1-j_0}) x (1 + \varepsilon \eta^{j+1-j_0})^{-1} \equiv x + \left( \varepsilon \gamma_{j_0}^{\sigma_{j+1-j_0}} - \gamma_{j_0} \varepsilon^{\sigma_{j_0}} \right) \eta^{j+1} \mod P^{j+2}$$

$$\equiv x + \xi(\varepsilon) \eta^{j+1}, \text{ say.}$$

From (c), $\xi(\varepsilon) \in T_{j+1}$. On the other hand,

$$\xi(\varepsilon) = 0 \iff [\varepsilon \eta^{j+1-j_0}, \gamma \eta^{j_0}] = 0 \iff [\varepsilon \eta^{j+1-j_0}, y] = 0$$

$$\Rightarrow \varepsilon \in S_{j+1-j_0}.$$ 

Hence $\xi$ is injective from $T_{j+1-j_0}$ to $T_{j+1}$, and (a)–(c) imply now that $\xi$ is bijective. Hence we can choose $\varepsilon$ so that the coefficient of $\eta^{j+1}$ after conjugation lies in $S_{j+1}$, and this is the inductive step.

Thus we have $x \in D_y$. A simple application of Hensel’s lemma shows that every $K(x_\sigma)$ contains a tamely ramified extension conjugate to $E_1$. As this extension is in $D_y$ and the center of $y$ is $E_1$, the extension must be $E_1$.

Now the lemma follows by induction on $t$, since henceforth we can work inside $D_y$.

**Theorem 4.2.** With notation as above, $D_{j+1}$ has a prime congruent mod $P^{s_{j+1}-s_j+1}$ to an element of $D_j$.

**Proof.** We work by induction on $j$. In view of Lemma 4.1, $D_j$ and $D_{j+1}$ both contain the tamely ramified extension $E_j$; thus (by passing, if necessary, to the algebra of elements commuting with $E_j$) we may assume that $K(x_j)$ is totally wildly ramified over $K$. Similarly, we may assume that $K(x_{j+1})$ is totally ramified over $K$. 

Let $E_{j+1}$ be the tamely ramified piece of the extension for $K(x_{j+1})$, and let $L$ be a totally ramified extension of $L_j$ in $D$. We may assume further that $x_{j+1}, x_j$ commute with $L$. Let $D_L$ be the algebra of elements commuting with $L$, and let, e.g., $D_{L,x_j}$ be the subalgebra of elements in $D_L$ commuting with $x_j$. From Theorem 3.3, we can choose primes $\eta_j, \eta_{j+1}$ in $D_{L,x_j}$ and $D_{L,x_{j+1}}$ respectively with $\eta_j \equiv \eta_{j+1} \mod P^{s_{j+1}-s_j-1}$. Then $\eta_{j+1}$ is also a prime in $D_{x_{j+1}}$, because $L$ is totally ramified over $E_{j+1}$, while $\eta_j \in D_{x_j}$.

REMARK 1. It is natural to ask whether the result of the theorem is best possible. If $(n, p) = 1$, the answer is certainly “no”; in fact, Lemma 4.1 says that in that case, we can find primes $\eta_j$ for $D_{x_j}$ such that $\eta_j \in D_{x_i}$ for $i < j$. If $n$ is a power of $p$, the answer (for totally ramified extensions) is “yes” in general; it is easy to construct examples by paralleling the constructions in the proof of Theorem 3.3. In the general case it appears that one cannot do better than Theorem 4.2, but I have not checked an example in detail.

REMARK 2. The proof of Theorem 4.2 actually proves a bit more than what is stated. Since the stronger result will be useful in what follows, we state it here as a corollary.

COROLLARY 4.3. In the situation of Theorem 4.2,

(a) If $j_0$ is the largest index $< j_1$ such that $s_{j_0}$ is a jump point with wild ramification, then there is a prime in $D_{j_1}$ congruent mod $P^{s_{j_1}-s_{j_0}+1}$ to an element of $D_{j_0}$;

(b) If there is no index $< j_1$ where wild ramification occurs, then for every $j < j_1$, $D_{j_1} \subset D_j$.

5. Commutators in division algebras. We shall later need a result about commutators, which we prove now. Let $G_j = 1 + P^j, G = G_1$.

PROPOSITION 5.1. Let $y \in [G, G] \cap G_h, h \geq 2$; let $r$ be any integer $> h$. Then we can write $y \mod G_r$ as a product of commutators,

$$y = (u_1, v_1)(u_1', v_1') \cdots (u_{r-h}, v_{r-h})(u_{r-h}', v_{r-h}')$$

where each $u_i, u_i'$ is of the form $1 + \delta \eta$ $(\delta \in k_\eta)$ and the $v_j, v_j'$ are of the form $1 + \epsilon \eta^h + \eta^{j-2}$.

Proof. By an obvious induction argument, it suffices to consider the case $r = h + 1$. In what follows, all calculations on $G$ are performed modulo $G_r$. Write $y = 1 + \sum_{j=h}^{\infty} \gamma_j \eta^j$. If $\chi$ is any character of $D^\times$ that
factors through the norm map, then \( \chi(y) = 1 \). This implies in particular that if \( w \in K \cap (1 + P^{1-2h}) \), then \( \text{Tr}_{D/K}w(y - 1) = 0 \); see, e.g. Theorem 1 of [2]. Thus \( \text{Tr}_{k_n/k}^{\gamma_h} = 0 \) if \( n \mid h \).

In general, we have

\[
(u_1, v_1) = 1 + \left( \delta_1 \epsilon_1^g - \epsilon_1 \delta_1^{\sigma_h-1} \right) \eta^h.
\]

If \( n \mid h \), let \( \delta_1 = 1 \). Then

\[
(u_1, v_1) = 1 + (\epsilon_1^g - \epsilon_1) \eta^h;
\]
as \( \gamma_h \) has trace 0, we can choose \( \epsilon_1 \) so that \( (u_1, v_1) \equiv y \mod G_r \). In this case, we can let \( u_1' = v_1' = 1 \).

If \( n \nmid h \), write

\[
\lambda_0 = \delta_1^{\sigma_h} \delta_1^{\sigma_h+1} \cdots \delta_1^{\sigma_h+(t-1)};
\]

where \( t \) is the smallest integer such that \( n = h + t \); let \( \lambda = \delta_1^{\sigma_h+1} \lambda_0 \). Then

\[
\lambda_0 \left( \delta_1 \epsilon_1^g - \epsilon_1 \delta_1^{\sigma_h-1} \right) = (\lambda \epsilon_1)^g - \lambda \epsilon_1.
\]

So for fixed \( \delta_1 \).

\[
\gamma_h = \delta_1 \epsilon_1^g - \epsilon_1 \delta_1^{\sigma_h-1}
\]

for some \( \epsilon_1 \in k_n \Rightarrow \text{Tr}_{k_n/k}(\lambda_0 \gamma_h) = 0 \).

Let \( \delta = \delta_1^{\sigma_h} \), so that \( \lambda_0 = \delta^g \delta \delta^{g-1} \cdots \delta^{g-1} \). From Hilfsatz 4 of [7], we can (by choosing \( \delta_1 \) appropriately) make \( \lambda_0 \) any element such that

\[
N_{k_n/k} \lambda_0 \in k, \quad s = (h, n).
\]

Write \( n = p^{n_0} n_1 \), where \( (p, n_1) = 1 \). Suppose first that \( p^{n_0} \nmid h \); let \( s = p^{n_0} s_1 \), and set \( s' = p^{n_0} n_1 \). From Proposition 2.3, we know that there exists \( \lambda_0' \) such that \( N_{k_n/k}^{\lambda_0'} \lambda_0' \in k^{s'}, \text{Tr}_{k_n/k}^{\lambda_0'} \gamma_h = 0 \). Since \( n/s' \) is a power of \( p \), \( N_{k_n/k}^{\lambda_0'} \) is an automorphism on \( k^{s'} \). Thus we can multiply \( \lambda_0' \) by an element of \( k^{s'} \) to get \( \lambda_0 \) with \( N_{k_n/k} \lambda_0 \in k, \text{Tr}_{k_n/k} \lambda_0 \gamma_h = 0 \). Thus we can find \( \delta_1, \epsilon_1 \) to prove the lemma. Here, too, we have \( u_1' = v_1' = 1 \).

Therefore we may suppose that \( p^{n_0} \mid h \). Restrict attention to elements \( \delta_1 \in k_n^* \); it is not hard to see that it suffices to consider the case \( n_0 = 0 \).

We are now in the tamely ramified situation. Note that

\[
(u_1, v_1)(u_1', v_1') = 1 + \left( \delta_1 \epsilon_1^g - \epsilon_1 \delta_1^{\sigma_h-1} \right) + \left( \delta_1' \epsilon_1^g - \epsilon_1' \delta_1^{\sigma_h-1} \right) \eta^h.
\]

We need to show that the sum in brackets can be made equal to any element of \( k_n^* \). It suffices, since \( \text{Tr}_{k_n/k} \) is faithful on \( k \), to show that then

(a) there exists \( \epsilon_1, \delta_1 \) such that \( \text{Tr}_{k_n/k}(\delta_1 \epsilon_1^g - \epsilon_1 \delta_1^{\sigma_h-1}) = 1 \); and

(b) if \( \text{Tr}_{k_n/k} \kappa = 0 \), then \( \exists \delta_1', \epsilon_1' \) with \( \kappa = \delta_1' \epsilon_1' - \epsilon_1' \delta_1^{\sigma_h-1} \).
Part (b) is easy; in fact, we can take $\delta_1 = 1$. As for (a), fix $\delta_1$ and suppose that $\text{Tr}_{k_n/k}(\delta_1 \epsilon_1^a - \epsilon_1 \delta_1^{a-1}) = 0$ for all $\epsilon_1$. Then $\text{Tr}_{k_n/k} \epsilon_1^a(\delta_1^{a} - \delta_1) = 0$ for all $\epsilon_1$; hence $\delta_1^{a} - \delta_1 = 0$, or $\sigma^h$ fixes $\delta_1$. We need only choose $\delta_1$ to be outside the fixed field of $\sigma^h$ to complete the proof.

Part II. Representations of Division Algebras of Index $pn_0$, $p \nmid n_0$.

6. Some simpler cases. Let $D$ be a division algebra of index $pn_0$ over its center $K$, where $K$ has residual characteristic $p$ and $(p, n_0) = 1$. We use the notation of §§1 and 4.

We wish to determine the irreducible unitary representations of $D^\times$. In general, we work by determining those of $G$. Any such representation has a kernel containing some $(1 + P^{m+1}) = G_{m+1}$ for an $m \geq 0$; choose $m$ to be as small as possible. In this and the next few sections, we assume that $m$ is odd; we remove this assumption in §9. Let $m = 2m' - 1$, and let $\chi$ be a character on the Abelian group $G_{m'}/G_{m+1}$ that is nontrivial on $G_m$. As noted in §1, one can write $\chi = \chi_x$ for some $x \in P^{-m} \setminus P^{-m+1}$. Let $s_0, \ldots, s_{t-1}$ be the jump points for $x$. We shall assume until §9 that the $s_j$ are all odd.

The construction of the representations of $G$ is done by (mathematical) induction. We assume that the representations of the corresponding group $G'$, and of $D'^\times$, are known if $D'$ is a division algebra whose index over its center $K'$ is a proper divisor of $n$ (of course, $K'$ also has residual characteristic $p$). We also assume that all irreducibles of $G$ containing $\chi_{x'}$ are known when $x' \in P^{-m+1}$. In this section we deal with some relatively easy cases, leaving the hard work for §§7 and 8.

Case I. $x$ is not in general position. Then there is a central element $x_0 \in P^{-m}$ such that $x - x_0 \in P^{-m+1}$. We may let $x_0 = \gamma_{-m} \eta^{-m}$, in fact. If $\chi$ is any character of $K^\times \cap G$ satisfying $\chi(1 + \delta \eta^m) = \psi(\gamma_{-m} \delta)$, then $\chi \circ N_{D/K}$ agrees with $\chi_{x_0}$ on $P^m$. Therefore $\chi_{x'} = \chi_x(\chi \circ N_{D/K})^{-1}$ is a character on $P^{m-1}$. Moreover, $\pi_0$ contains $\chi_{x'} \Leftrightarrow \pi_0 \otimes (\chi \circ N_{D/K})$ contains $\chi_x$, and the representations containing $\chi_{x'}$ are assumed known.

Henceforth we assume that $x$ is in general position. Write

$$x = \sum_{j=-m}^{\infty} \gamma_j \eta^j, \quad x_0 = \gamma_{-m} \eta^{-m} \not\in K.$$

Case II. $t = 1$. Then $K(x_0)$ and $K(x)$ have the same ramification index and residue class degree. The following argument is like that in [6]; indeed, it applies whenever $t = 1$ (regardless of $n$). It is also not strictly
necessary for the construction in our case, but I think that it may be useful to have the following result stated explicitly.

**Theorem 6.1.** Suppose that \(x\) is in general position and that \(t = 1\). Then \(y \in D^x\) commutes with \(\chi^x\) on \(G_{m'} \leftrightarrow y \in G_{m'} \cdot D^x\). Moreover, \(\chi^x\) extends to a character on \(G_{m'} \cdot D^x\). Let \(\chi\) be any such extension. For each representation \(\pi_0\) of \(D^x\) trivial on \(G_{m'} \cap D^x\), extend \(\pi_0\) to \(G_{m'} \cdot D^x\) by making it trivial on \(G_{m'}\). Then \(\chi \otimes \pi_0\) induces to an irreducible representation of \(D^x\); moreover, every representation of \(D^x\) containing \(\chi^x\) is obtained in this way.

**Proof.** Satz 2 of [8] gives the result about elements commuting with \(\chi^x\). If \(w \in G_{m'}\) and \(y \in G_{m'} \cdot D^x\), then \(\chi^x((w, y)) = 1\) because \(y\) commutes with \(\chi^x\), while \(\chi^x\) is 1 on \((D^x, D^x \cap G_{m'})\) because \(\chi^x\) factors through the norm map on \(D^x\). Hence \(\chi^x\) extends to \(G_{m'} \cdot D^x\). \(\chi^x \otimes \pi_0\) is a multiple of \(\chi^x\) on \(G_{m'}\), and Theorem 6 of [9] implies that it induces to an irreducible representation of \(D^x\). Finally, we show that we obtain all representations of \(D^x\) containing \(\chi^x\) which are trivial on \(\eta^n\). We may assume that \(\chi(\eta^n) = 1\). The set \(S = \{\pi_0 \in (D^x)^{\wedge} : \pi_0\) is trivial on \(G_{m'} \cap D^x\) and on \(\eta^n\}\) satisfies

\[
\sum_{\pi_0 \in S} (\dim \pi_0)^2 = [G \cap D^x : G_{m'} \cap D^x] \cdot e_1(q^{f_1} - 1),
\]

since the left-hand side is \([D^x : \langle \eta^n \rangle \cdot G_{m'}]\). On the other hand,

\[
[D^x : D^x \cdot G_{m'}] = [G : G_{m'}][(G \cap D^x : G_{m'} \cap D^x)]^{-1}(n/e_1) \left(\frac{q^n - 1}{q^{f_1} - 1}\right).
\]

Since \(\chi^x\) appears in \(\pi = \text{Ind}_{G_{m'}D^x}^{G_{m'}}(\chi^x \otimes \pi_0)\) exactly \((\dim \pi_0)\) times, we see that the \(\pi\)-primary subspace in \(\text{Ind}_{G_{m'}\langle \eta^n \rangle}^{G_{m'}} D^x \chi^x\) has dimension \((\dim \pi_0)^2[D^x : D^x \cdot G_{m'}]\). Hence, by Frobenius reciprocity, these subspaces account for a subspace of dimension

\[
\sum_{\pi_0 \in S} (\dim \pi_0^2)[D^x : D^x \cdot G_{m'}] = [G : G_{m'}] n(q^n - 1) = [D^x : G_{m'} \langle \eta^n \rangle],
\]

or for all of \(\text{Ind}_{G_{m'}\langle \eta^n \rangle}^{G_{m'}} D^x \chi^x\). This proves the theorem.

**Remarks.** 1. We have dealt with \(D^x\) rather than \(G\) in this theorem; obviously, there is a similar theorem for \(G\). When we come to deal with the case \(m\) even, the representation \(\chi^x\) will not extend to \(G_{m'} D^x\), and we need to use a Weil representation; see, e.g., [4].
2. In general, \( K(x) \) is not determined up to conjugacy by \( x \mod p^{-m'} \); this is most evident in the case where \( K(x) \) is inseparable. This fact makes it more difficult to arrange for a good parametrization of \( (D^x)^\wedge \).

**Case III.** The first jump point involves only tame ramification. From Lemma 4.1, we may assume that the \( \gamma_j \eta^j \) all commute. Now the construction of [5] (or [1]) yields all irreducible representations of \( D^x \) that agree with \( \chi_x \) on \( G_m \), and hence all such representations agreeing with \( \chi_x \) on \( G_{m'} \). The proofs are exactly as in [1]; we omit details.

**Case IV.** The first jump point is not totally wildly ramified. Let \( K_0 \) be the largest tamely ramified field in \( K(x_0) \); we may assume again (Lemma 4.1) that every \( \gamma_j \eta^j \) commutes with \( K_0 \). Let \( D_0 \) be the algebra of elements commuting with \( K_0 \), and construct all representations \( \sigma \) of \( D_0^x \cap G \) containing \( \chi_x \mid D_0^x \cap G_{m'} \). The same construction as in [1] shows that there is a subgroup \( N_0 \) of \( G_{m'} \) on which \( \chi_x \) is trivial, which is normalized by \( D_0^x \cap G \), and which satisfies

\[
N_0(D_0^x \cap G) = G_{m'}(D_0^x \cap G), \quad N_0 \cap (D_0^x \cap G) \subseteq G_{m+1}.
\]

Then we can extend \( \sigma \) to \( G_{m'}(D_0^x \cap G) \) by making it trivial on \( N_0 \). Induce \( \sigma \) to \( G \) to get an irreducible \( \pi \) containing \( \chi_x \). That \( \pi \) is irreducible and that every \( \pi \) containing \( \chi_x \) is obtained in this way can be proved essentially as in Case III, by following the corresponding proofs in [1].

**7. Extending \( \chi_x \).** We henceforth assume that

(a) the element \( x \) is in general position;

(b) the first jump point of \( x \), \( s_0 = -m \), is totally wildly ramified.

Let \( -s_j = 2s'_j - 1 \) (recall that we are assuming that the \( s_j \) are all odd), and define \( H = H_x \) to be the group

\[
G_{s_0}(G_{s_1} \cap D_{s_1}) \cdots (G_{s_{r-1}} \cap D_{s_{r-1}})(G \cap D_{s_r}).
\]

We wish to show that \( \chi_x \) extends to a character of \( H \). This is equivalent to:

**Theorem 7.1.** If \( y \in [H, H] \cap G_{s_0} \), then \( \chi_x(y) = 1 \).

**Proof.** This follows the lines of the proof of Lemma 8 of [2]. We write \( y \) as a product of commutators. We note that

\[
(7.1) \quad (v_1v_2, w) = (v_1v_2v_1^{-1}, v_1wv_1^{-1})(v_1, w)
\]
and
\[(7.2) \quad (v, w_1 w_2) = (v, w_1) (w_1 v w_1^{-1}, w_1 w_2 w_1^{-1}).\]

In this way, we can let \( y \) be a product of commutators of the form \((v, w)\), where \( v = 1 + \gamma \eta_0 \), \( w = 1 + \gamma \eta \), and \( \eta_0, \eta \) are specified primes, while \( \gamma, \delta \in k_n \). Similarly, we can commute commutators by using
\[
u_2 u_1 = u_1 (u_1^{-1} u_2 u_1),
\]
where, if \( u_2 = (v, w) \), then \( u_1^{-1} u_2 u_1 = (u_1^{-1} v u_1, u_1^{-1} w u_1) \).

We proceed by a lengthy sequence of steps.

(a) If \( v, w \in H \) and \( v w v^{-1} \in G_{s_0} = G_{m'} \), then \( w \in G_{m'} \) (since \( G_{m'} \) is normal in \( G \)), and \( \chi_x(v w v^{-1}) = \chi_x(w) \), since \( \chi_x(v w v^{-1}) = \chi_{v^{-1}x_0}(w) \) and elements of \( H \) preserve \( \chi_x \). In particular, \( \chi_x((v, w)) = 1 \).

(b) The following computation will arise repeatedly in the proof: if \( x \) and \( v \) commute, then
\[\text{Tr}_{D/K} x(v w v^{-1} - uv') = \text{Tr}_{D/K} (v x w v^{-1} - x u v') = 0,\]
since \( \text{Tr}_{D/K} (ab) = \text{Tr}_{D/K} (ba) \).

(c) Let \( H' = H_x = G_{s_0} (G_{s_j} \cap D_{x_j}) \cdots (G_{s_{j-1}} \cap D_{x_{j-1}}) \). If \( u, v \in H' \), then \((u, v) \in G_{s_0} \) and \( \chi_x((u, v)) = 1 \). To prove this, it suffices to consider the case where \( u \in G_{s_j} \cap D_{x_j} \) and \( v \in G_{s_j} \cap D_{x_j} \), as repeated use of (7.1) and (7.2) shows. Assume \( i < j \), for definiteness; write \( u = 1 + u_0, v = 1 + v_0 \). Then modulo \( P^{m+1} \),
\[
(u, v) = 1 + (u_0 v_0 - v_0 u_0) + (v_0 u_0^2 - u_0 v_0 u_0) + (v_0 u_0 v_0 - u_0 v_0^2)
= (1 + u_0 v_0 - v_0 u_0)(1 + v_0 u_0^2 - u_0 v_0 u_0)(1 + v_0 u_0 v_0 - u_0 v_0^2).
\]
Since \( u_0 v_0 - v_0 u_0 \in P^{m+s_j+1} \), we have
\[
\chi_x(1 + u_0 v_0 - v_0 u_0) = \chi_{x_j}(1 + u_0 v_0 - v_0 u_0) = 1,
\]
from (b) (note that \( x_j \) and \( v_0 \) commute). The other terms are taken care of similarly.

(d) Now (a) and (c) reduce us to considering
\[(7.3) \quad w = (u_1, v_1) \cdots (u_r, v_r),\]
where one of each \( u_j, v_j \) is in \( G \cap D_x \). We shall assume for notational convenience that \( u_j \in G \cap D_x \) for all \( j \); this will not affect the proof. We may also assume that \( u_j = 1 + u_{j,0}, v_j = 1 + v_{j,0} \), where the \( u_{j,0}, v_{j,0} \) are "monomials":
\[(7.4) \quad u_{j,0} = \delta \eta_i^a, \quad v_{j,0} = \varepsilon \eta_i^b, \quad a = a_j, b = b_j\]
where $\delta, \epsilon \in k_n$, $v_{j,0} \in G_{s_{i-1}} \cap D_{x_i}$, and $\eta_i, \eta_t$ are primes for $D_{x_i}, D_{x_i}$ respectively. We fix these primes so that $\eta_i$ is congruent mod $P^{s_{i-1}/s_i}$ to an element $\eta_i'$ of $D_{x_i}$.

We may assume also that for the first $r_0$ commutators, and only for these, we have $v_j \in D_{x_i}$. Then the product of these commutators is in $G_{s_{i-1}+1}$ (since every other commutator is in $G_{s_{i-1}+1}$), and we may, therefore, assume from Proposition 5.1 that $a_j + b_j \geq s_{i-1}$ for all $j$. (In fact, we can have $a_j = \text{the order of a prime in } D_{x_i}$ for $j \leq r_0$).

Write $u_{j,0}' = 1 + u_{j,0}', v_j' = 1 + v_{j,0}'$, where

\begin{equation}
\label{7.5}
u_{j,0}' = \delta(\eta_t')^a, \quad v_{j,0}' = \epsilon\eta_t^b \quad \eta_i, \eta_t \text{ primes in } D_{x_i} \text{ related to } \eta_i, \eta_t
\end{equation}

as in Theorem 4.2 and Corollary 4.3); the $a, b, \delta, \epsilon$ in (7.5) agree with those in (7.4). Let

$y' = (u_1', v_1') \cdots (u_r', v_r')$.

Then $y'$ is a commutator in $D_{x_i}$, so that

$\chi_{x_i}(y') = 1$.

The proof consists of showing that $\chi_x(y) = \chi_{x_i}(y)$.

(e) Write $(u_j, v_j) = 1 + w_{j,0} + w_{j,1}$, where

$$w_{j,0} = \sum_{i=1}^{\infty} (-1)^i (v_j u_j' - u_j v_j u_j'^{-1});$$

write $y = 1 + w_0 + w_1$, where $w_0 = \sum_{j=1}^{\infty} w_{j,0}$. If one multiplies out all the commutators, $w_1$ consists of all terms of degree $\geq 2$ in the $v_j$. For instance, we have (mod $G_{m+1}$)

$$w_{j,1} = \sum_{i=1}^{\infty} (-1)^i (u_{j,0} v_{j,0} u_{j,0}^{-1} v_{j,0} - v_{j,0} u_{j,0} u_{j,0}^{-1} v_{j,0} + u_{j,0} v_{j,0} v_{j,0}^2 - u_{j,0} v_{j,0} u_{j,0}^{-1} v_{j,0} v_{j,0})$$

and $w_1$ is a sum of the $w_{j,1}$, plus products of the $w_{j,0}$ and $w_{j,1}$. Thus $w_1 \in P^{m'}$; as $w \in G_{m'}$, we have $w_0 \in P^{m'}$, and

$$\chi_x(1 + w_0 + w_1) = \chi_x(1 + w_0) \chi_x(1 + w_1).$$

Similarly, we write $(u_j', v_j') = 1 + w_{j,0}' + w_{j,1}'$ and

$y' = (u_1', v_1') \cdots (u_r', v_r') = 1 + w_0' + w_1'$;

the same argument as above shows that

$$\chi_{x_i}(y') = \chi_{x_i}(1 + w_0') \chi_{x_i}(1 + w_1'), \quad w_0' \text{ and } w_1' \in P^{m'}.$$
(f) We have
\[ \chi_x(1 + w_0) = 1 \]
from (b), since each \( u_j \) commutes with \( x \). Similarly, \( \chi_x(1 + w_0') = 1 \).

Moreover, \( w_1 \) and \( w_1' \) are congruent mod \( P^{-s_1} + 1 \). The reason is that each term of \( w_1 \) is at least quadratic in the \( u_{j,0} \); moreover, if \( u_{j,0} \) appears, so does \( v_{j,0} \). By Corollary 4.3 (applied to \( j = 0 \)), \( u_{j,0}v_{j,0} = u_{j,0}v_{j,0}' \) mod \( P^{s_1-s_0+c-1} \), where \( v_j \in D_x \) and \( c \) is the order of \( u_{j,0}v_{j,0}' \). This order is at least \( P^{s_1} + 1 \). Thus \( u_{j,0}v_{j,0} = u_{j,0}v_{j,0}' \) mod \( P^{m-s_1} \). Now suppose that the term of \( w_1 \) contains a product \( u_{j,0}v_{j,0}u_{i,0}v_{i,0} \), where \( v_{i,0} \) is a prime of \( D_y \) that is close to an element of \( D_{j,0} \), and \( r' \geq r \). Then \( u_{j,0}v_{j,0} \) and \( u_{j,0}v_{j,0}' \) are in \( P^{s_1} + 1 \) (and congruent mod \( P^{m-s_1} \)); hence the products are congruent mod \( P^{m-s_1+s_1'+1} \), and therefore congruent mod \( P^{m+1} \). The remaining terms in \( w_1 \) are of the form \( u_{j,0}v_{j,0} \) or \( v_{j,0}u_{j,0}v_{j,0}' \). But, e.g., \( u_{j,0}v_{j,0} = u_{j,0}v_{j,0}' \) mod \( P^{m-s_1+s_1'+1} = P^{m} \), and \( \chi_x, \chi_{x_1} \) agree on \( G_m \).

(g) It follows that \( \chi_{x_1}(1 + w_1') = \chi_x(1 + w_1) \); (e) and (f) give
\[ \chi_x(w) = \chi_{x_1}(1 + w_0') \]
since \( (1 + w_0')(1 + w_1') \) is congruent mod \( P^{m+1} \) to a commutant in \( D_{x_1} \). This proves the theorem.

8. Construction of the representations. In this section, we construct “enough” representations of \( H = H_x \) so that inducing to \( G \) produces all the desired irreducibles. This is not too difficult, but takes some time.

Recall that \( K(x_1) \) is totally wildly ramified. Hence we may (and do) assume that \( \gamma_{-m} \in k \). This implies that the residue class fields of \( D_1, \ldots, D_t \) are all extensions of degree prime to \( p \). The residue class field of \( D_{x_1} \) is \( k_{n_0} \); let \( l_j \) be the residue class field of \( D_j \), and let \( k(x_j) \) have ramification index and residue class degree \( e_j, f_j \) respectively.

We know that \( H_x \) is generated by \( G_m \) and elements \( 1 + \delta \eta_j' \), where \( \eta_j \) is a prime of \( D_j \) that is close to an element of \( D_1, \delta \in l_j \), and \( r \geq s_1' \). Moreover, \( \eta_j = \delta' \eta_j' \) mod \( P^{f_j+1} \) for some \( \delta' \in k_{n_0} \). For each integer \( r \), let
\[ S_r(j) = \{ \delta \in k_{n_0} : \text{there is an element of } D_j \text{ congruent mod } P^{r+1} \text{ to } \delta \eta_j' \} \]
\[ T_r(j) = \{ e \in l_j : \text{Tr}_{k_{n_0}/l_j}(e\delta^{-1}) = 0 \text{ for all nonzero } \delta \in S_r(j) \} \].

We sometimes write \( S_r, T_r \) for \( S_r(t), T_r(t) \).

**Lemma 8.1.** (a) \( S_0(j) = l_j \).
(b) If \( \delta \in S_r(j) \) is nonzero, then
\[ S_{r+r'}(j) = \delta^{\sigma}\delta^{-1} \]
\[ S_{r+r'}(j) = \delta^{\sigma'}S_r(j) = \delta S_r(j)^{\sigma'} \]
and

\[ T_{r+r'}(j) = \delta^{a'} T_{r'}(j) = \delta T_{r'}(j)^{a'}. \]

(c) Both \( S_r(j) \) and \( T_r(j) \) are vector spaces over \( l_j \), and \( S_r(j) \oplus T_r(j) = k_{n_0} \).

(d) \( \text{Dim}_{l_j} S_r(j) = 1 \) if \( F_j \mid r \) and \( 0 \) otherwise.

Proof. This is essentially done as Lemma 2 of [3].

Now let \( N \) be the subgroup of \( H \) generated by \( G_{m'} \) and the elements \( 1 + \epsilon \eta_j \in D_j \cap H \) with \( \epsilon \in T_r \).

Lemma 8.2. (a) \( N \) is normal in \( H \),
(b) \( H/N \equiv (G \cap D_x)/(G_{m'} \cap D_x) \).

Proof. (a) Since

\[ (1 + \epsilon_1 \eta^i_j)(1 + \epsilon_2 \eta^i_j) \equiv 1 + \epsilon_1 \eta^i_j + \epsilon_2 \eta^i_j \mod P^{m'} \]

if the \( 1 + \epsilon_i \eta^i_j \) are generators of \( N \), and since \( \eta^i_j \) is congruent to an element of \( D_x \) modulo \( P^{m'} \), it is not hard to see that \( N \) is composed entirely of elements of the form

\[ w = 1 + \sum_{j=0}^{\infty} \epsilon_j \eta^i_j, \quad 2 j_0 \geq m' \text{ and } \epsilon_j \in T_j \text{ if } j < m', \]

while \( H \) is composed of elements of the form

\[ y = 1 + \sum_{j=1}^{\infty} \delta_j \eta^i_j, \quad \delta_j \in S_j \text{ if } 2 j < m'. \]

To prove (a), it suffices to show that every element of the form (8.2) normalizes the elements of the form (8.1).

Write

\[ w = w_{j_0} w_{j_0+1} \cdots w_{m'}, \quad w_j = 1 + \epsilon_j \eta^i_j \text{ for } j < m'; \]

then \( w_{m'} \in 1 + P^{m'} \). Similarly, one can write

\[ y = y_1 \cdots y_s, \quad y_j = 1 + \delta_j \eta^i_j \text{ with } \delta_j \in S_j \text{ for } 2 j < m'; \]

\[ y_s \in 1 + P^s \text{ and } 2s \geq m'. \]

Then \( y_s \) commutes with \( w \mod 1 + P^{m'} \) and \( w_{m'} \) commutes with each \( y_j \mod 1 + P^{m'} \). It thus suffices to show that \( y_i w_j y_i^{-1} \in N \) for \( i < s \) and
$j < m'$. This is a straightforward calculation:

$$y_{w_{j}y_{i}^{-1}} = 1 + \epsilon_{w_{j}y_{i}^{-1}} + \left( \delta_{i} \epsilon_{w_{j}y_{i}^{-1}} - \epsilon_{w_{j}y_{i}^{-1}} \delta_{i} \right) \eta_{i}^{+j}$$

$$+ \left( \delta_{i} \epsilon_{w_{j}y_{i}^{-1}} - \epsilon_{w_{j}y_{i}^{-1}} \delta_{i} \right) \eta_{i}^{2i+j} + \cdots,$$

and repeated application of Lemma 8.1 shows that $y_{w_{j}y_{i}^{-1}} \in N$.

As for (b), $G \cap D_{x}$ injects into $H$ and hence maps into $H/N$; from the form of elements in $N$ and $H$ given in (8.1) and (8.2), it is easy to verify that the map is surjective and has $G_{m'} \cap D_{x}$ as kernel.

Any representation of $G \cap D_{x}$ that is trivial on $G_{m'} \cap D_{x}$ can thus be a representation of $H$ trivial on $N$. Take an extension of $\chi_{x}$ to $G$ (guaranteed by Theorem 7.1); call the extension $\chi_{x}$ as well. Then $\chi_{x} \otimes \sigma$ is also a representation of $H$, and is a multiple of $\chi_{x}$ on $N$.

Let $H' = G_{m'}(G_{-s_{1}} \cap G_{s_{1}}) \cdots (G_{-s_{r-1}} \cap G_{s_{r-1}})(G \cap D_{x})$. The key results we need about $H'$ and $H$ are contained in the following proposition:

**Proposition 8.3.** (a) $[H' : H] = [H : G_{m'}(G \cap D_{x})]$.

(b) If $y \in G$ is such that $\chi_{x}(ywy^{-1}) = \chi_{x}(w)$ for all $w \in G_{m'}$, then $y \in H'$; conversely, $y \in H' \Rightarrow \chi_{x}(ywy^{-1}) = \chi_{x}(w)$ for all $w \in G_{m'}$.

(c) If $y \in H'$ is such that $\chi_{x}(ywy^{-1}) = \chi_{x}(w)$ for all $w \in H$ such that $ywy^{-1} \in H$, then $y \in H$; conversely, $\chi_{x}(ywy^{-1}) = \chi_{x}(w)$ if $w, y \in H$.

Before proving Proposition 8.3, we show how it solves the problem of constructing representations of $G$ containing $\chi_{x}$. For each $\sigma \in (D_{x} \cap G/D_{x} \cap G_{m'}) \wedge$, let $\pi_{\sigma} = \text{Ind}_{H}^{G}(\sigma \otimes \chi_{x})$.

**Theorem 8.4.** The $\pi_{\sigma}$ are distinct irreducibles of $G$, and $\text{Ind}_{G_{m}}^{G} \chi_{x} \cong \bigoplus \sigma[H' : H](\text{dim} \sigma) \pi_{\sigma}$.

**Proof.** The irreducibility follows from Proposition 8.3 (c) and Theorem 6 of [9]. Proposition 8.3 (c) and Theorem 7 of [9] also imply that the $\pi_{\sigma}$ are distinct.

Frobenius reciprocity says that $\pi_{\sigma}$ appears in $\text{Ind}_{G_{m}}^{G} \chi_{x}$ with a multiplicity equal to the multiplicity of $\chi_{x}$ in $\pi_{\sigma} \mid_{G_{m}}$. But

$$\pi_{\sigma} \mid_{G_{m}} \cong \bigoplus_{y \in G/H} \sigma \otimes \chi_{yxy^{-1}} \mid_{G_{m}} \cong (\text{dim} \sigma) \bigoplus_{y \in G/H} \chi_{yxy^{-1}},$$

and $\chi_{yxy^{-1}} = \chi_{x} \Leftrightarrow y \in H'/H$ from Proposition 8.3(b). Thus the multiplicity of $\pi_{\sigma}$ is $[H' : H] \text{dim} \sigma$. Finally,

$$[H' : H](\text{dim} \sigma) \text{dim} \pi_{\sigma} = [H' : H][G : H] \text{dim}^{2} \sigma$$

$$= [G : G_{m'}(D_{x} \cap G)] \cdot \text{dim}^{2} \sigma$$
by Proposition 8.3(a). Summing over $\sigma$ shows that we have accounted for
a subrepresentation of $\text{Ind}^{G_m}_G \chi_x$ of dimension
\[
\sum_{\sigma} [G : G_{m'}(d_x \cap G)] \dim^2 \sigma = [G : G_{m'}(D_x \cap G)] [D_x \cap G : D_x \cap G_m']
\]
and hence for all of $\text{Ind}^{G_m}_G \chi_x$.

We still need to prove Proposition 8.3. For part (b), write $w = 1 + w_0$, $w_0 \in P^{m'}$. Then
\[
\chi_x(ywy^{-1}) = \psi \circ \text{Tr}_{D/K}(xyw_0y^{-1}) = \psi \circ \text{Tr}_{D/K}(y^{-1}xyw_0).
\]
This is equal to $\psi \circ \text{Tr}_{D/K}(xw_0)$ for all $w_0 \in P^{m'}$ iff
\[
x - y^{-1}xy \in P^{-m'+1},
\]
and this congruence holds iff $y \in H'$ by Satz 2 of [8]. Half of part (c) is
easy; $y, w \in H \Rightarrow \chi_x(ywy^{-1}) = \chi_x(w)$ from Theorem 7.1. For the rest of
(c) and for (a), we need to do some more work.

The field $k_n$ is the compositum of $k_p$ and $k_{n_0}$. Define $\alpha, \beta, \gamma \in k_p$ as
in §2.

**Lemma 8.5.** Let $x - x_j = \xi_j \eta_j \bmod P^{-j+1}$, with $j \geq 1$. Then by con-
jugating, we may assume that $\xi_j = \beta_i \xi'_j$, $\xi'_j \in S_{s_j}(j)$ and $\xi'_j \neq 0$.

**Proof.** We have seen that $x - x_0 = \gamma - m \eta^{-m} \bmod P^{-m+1}$, with $\gamma - m \in k$
and $\gamma - m \neq 0$. Since $x_0$ is totally wildly ramified, we have $(m, p) = m_0$.

Assume the lemma for $j - 1$. We certainly have
\[
\xi_j = \sum_{i=0}^{p-1} \alpha_i \xi_{i,j}
\]
for appropriate $\xi_{i,j} \in k_{n_0}$.

We describe anything of the form
\[
\sum_{i=0}^{p-2} \alpha_i \gamma_j, \quad \gamma_j \in k_{n_0},
\]
as “small”. Note that if we conjugate $x$ with $1 + \gamma \eta^{s_j+m}$, we get $x +
\gamma_0(\gamma^{s_j+m} - \gamma) \eta_{s_j}(\bmod P^{s_j+1})$, and $\gamma_0(\gamma^{s_j+m} - \gamma)$
can be any element $\alpha$ of $k_n$ such that $\text{Tr}_{k_n/k_{n_0}} \alpha = 0$; that is, we can always
get rid of any small element by conjugating.

Thus we have $\xi_j = \alpha^{p-1} \xi'_j$, $\xi'_j \in k_{n_0} = S_{s_j}(1)$. Now conjugate $x$ with
$1 + \gamma \eta^{s_j-s_i}$, where $\gamma \in k_{n_0} = l_i$; we get (mod $P^{s_j+1}$, as all future compu-
tations are made)
\[
x + (\delta \xi'_{s_j} - \xi_{s_j}) \eta_{s_j}, \quad \delta \text{ any element of } k_{n_0}.
\]
Now $\xi_1 = \alpha^{p-1} \xi'_1$ with $\xi'_1 \in k_{n_0}$. Moreover, Satz 8 of [7] shows that the largest tamely ramified extension of $K(\xi_1 \eta^{s_i})$ is (conjugate to) the largest tamely ramified extension of $x_1$; this implies that $\delta \in S_r(2) \iff \delta \eta^r$ and $\xi'_1 \eta^{s_1}$ commute. So set

$$F_1(\delta) = \delta \xi_1^{\sigma y^{-s_1}} - \delta \sigma y^r \xi'_1, \quad \delta \in k_{n_0} = S_{s_j-s_1}(1).$$

This is a $k$-linear map, and it is $0 \iff \delta \in S_{s_j-s_1}(2)$. For $\delta \in T_{s_j-s_1}(2)$, Lemma 8.1 implies that $F_1(\delta) \in T_{s_j}(2)$. Hence counting implies that $F_1$ maps $k_{n_0}$ into $T_{s_j}(2)$. Moreover,

$$\delta \xi_1^{\sigma y^{-s_1}} - \delta \sigma y^r \xi'_1 = \alpha^{p-1} F_1(\delta) + \text{a small term.}$$

Thus we can change $\gamma_{s_j}$ (the coefficient of $\eta^{s_j}$ in the expansion of $x$) by any element of the form $\alpha^{p-1} e$, $e \in T^{s_j}(2)$. It follows that we can arrange to have $\xi_j = \alpha^{p-1} \xi_{j,2}$, with $\xi_{j,2} \in S_j(2)$.

We continue inductively. Conjugate $x$ with $1 + \gamma \eta_2^{s_{j+1}}$, $\gamma \in l_2$; we get $x + (\delta \xi_2^{\sigma y^{-s_2}} - \delta \sigma y_2^{s_2}) \eta_2^{s_j}$, where $\delta$ is any element of $S_{s_j-s_1}(2)$. The same argument as before shows that $\delta \in S_{s_j}(2) \iff \delta \eta^r$ commutes with $\xi'_1 \eta^{s_1}$ and $\xi'_2 \eta^{s_2}$. So define $F_2(\delta) = \delta \xi_2^{\sigma y^{-s_2}} - \delta \sigma y_2^{s_2} \xi'_2$, $\delta \in S_{s_j-s_2}(2)$; by the same argument as above, $F_2$ maps $S^{s_j-s_2}(2)$ onto $T^{s_j}(3)$. It follows that we can have $\xi_j = \alpha^{p-1} \xi_{j,3}$, $\xi_{j,3} \in T_j(3)$, and the same inductive procedure gives the result.

**Corollary.** In the above setup, let $L_j$ be the maximal tamely ramified extension in $K(x_j)$. Choose $\xi_{j,0}$ such that $\xi_{j,0} \eta^{s_j}/l_j \equiv \xi'_j \eta^{s_j} \mod \eta_0^{s_j+1}$. Then $L_{j+1}$ is conjugate (mod an element of $G$) to the maximal tamely ramified extension in $L_j(\xi_{j,0} \eta^{s_j}/l_j)$. (This follows from Satz 8 of [7], as we observed in the course of the proof of Lemma 8.5.)

**Remark 1.** The corollary shows the following useful fact: suppose that $\delta \in S_{s_j}(j)$. Then $\delta \in S_{s_j}(j+1)$ if $\delta \eta^r$ and $\xi'_j \eta^{s_j}$ commute, or, equivalently, if $\delta \eta^r$ and $\xi_{s_j}^{s_j}/l_j$ commute mod $P^{s_j+r+1}$. The reason is that $\delta \eta^r$ then commutes with the largest tamely ramified extension in $K(\xi'_j \eta^{s_j})$. Another way of stating the condition is that $\delta \eta^r$ is congruent mod $P^{s_j+r+1}$ to an element of $K(x_{j+1})$ if $\delta' \in S_0(j)$ and $[\delta \eta^r, \xi_{s_j}^{s_j}/l_j] = 0$.

**Remark 2.** While the process of Lemma 8.5 may mean that the tamely ramified extensions in $K(x_j)$ are no longer contained in the algebra generated by $k_{n_0}$ and $\eta^p$, conjugating by elements of $G$ does not change the membership of the $S_{s_j}(j)$ or $T_r(j)$.
We now return to the proof of Proposition 8.3. Part (a) is a matter of counting. For each \( r \leq m' \), we compute \( \text{card}(H \cap G_r)/(H \cap G_{r+1}) \). If \( s'_j \leq r < s'_{j-1} \), this number is 1 if \( f_j \mid r \) and \( \text{card} l_j = q^{n/e_j} \) if \( f_j \nmid r \). Since \( f_j | (2s'_j - 1, 2s'_{j-1} - 1) \), we see that for \( j \leq t \),

\[
\prod_{s'_j \leq r < s'_{j-1}} [H \cap G_r : H \cap G_{r+1}] = q^{l(n/e_j)(s'_{j-1} - s'_j)}.
\]

A similar calculation for \( r < s'_t \) gives

\[
\prod_{1 \leq r < s'_t} [H \cap G_r : H \cap G_{r+1}] = q^{(n/e_j)(s'_{t-1} - s'_t)}, \quad 2s'_t - 1 = -s_t = f_t.
\]

So

\[
\log_q[H : H \cap G_{m'}] = \sum_{j=1}^t \frac{n}{n_j} (s'_{j-1} - s'_j), \quad n_j = [K_j : K].
\]

Similarly,

\[
\log_q[G \cap D_x : G_{m'} \cap D_x] = \frac{n}{n_t} (s'_0 - s'_t)
\]

and

\[
\log_q[H' : H' \cap G_{m'}] = \frac{n}{n_t} (s'_0 - (s_{t-1} - s_0) - s_t) + \sum_{j=1}^{t-1} \frac{n}{n_j} (-s_j + s_{j-1}).
\]

It is now easy to verify that

\[
2 \log_q[H : H \cap G_{m'}] = \log_q[G \cap D_x : G_{m'} \cap D_x] + \log_q[H' : H' \cap G_{m'}],
\]

which proves (a).

As for the second half of (c), assume that \( \chi_x(ywy^{-1}) = \chi_x(w) \) for all \( w \in H \) such that \( ywy^{-1} \in H \), but that \( y \notin H \). Since \( y \in H' \) (from (b)), we may assume (after multiplying by an element of \( H \)) that

\[ y = 1 + \varepsilon \eta_f \] + higher order terms,

where \( 1 + \varepsilon \eta_f \in G_{s_j} \cap D_j \) and \( 1 + \varepsilon \eta_f \notin G_{s'_j} \cap D_j \). We have \( \varepsilon \eta_f \in P^{r_j} - P^{r_j - 1} \), and we may assume that \( \varepsilon \eta_f \) is not congruent mod \( P^{r_j + 1} \) to an element of \( D_j \).

Let \( w_\delta = 1 + \delta_0 \delta \eta_f \), where \( (r + r')f_j = -s_j \), \( \delta \in l_j \) is fixed, and \( \delta \) runs over \( l_j \). The \( w_\delta \) are all in \( D_j \), and (with \( w_\delta = 1 + w_{\delta,0} \) and \( y = 1 + y_0 \))

\[
yw_\delta y^{-1}w_\delta^{-1} \equiv 1 + \sum_{i=1}^{\infty} (-1)^{i-1}(y_0w_{\delta,0}y_0^{i-1} - w_{\delta,0}y_0)
\]

\[ + \sum_{i=2}^{\infty} (-1)^{i-1}(w_{\delta,0}y_0w_{\delta,0}^{i-1} - y_0w_{\delta,0}^i) \mod P^{m+1}.
\]
Each term in the sums is in $P^{m'}$, and each term with an index $i \geq 2$ is in $P^{-s_{j+1}}$. We have
\[
\chi_x \left( yw_\delta y^{-1}w_\delta^{-1} \right) = \chi_x \left( 1 + y_0 w_{\delta,0} - w_{\delta,0}y_0 \right) \\
\cdot \prod_{i=2}^{\infty} \chi_x \left( 1 + (-1)^{i-1} \left( y_0 w_{\delta,0} y_0^{-1} - w_{\delta,0} y_0 \right) \right) \\
\cdot \prod_{i=2}^{\infty} \chi_x \left( 1 + (-1)^{i-1} \left( w_{\delta,0} y_0 y_0^{-1} - y_0 w_{\delta,0} \right) \right),
\]
and (b) of the proof of Theorem 7.1 shows that every term in the two infinite products is 1, since we can replace $\chi_x$ with $\chi_{x_j}$. Similarly, $\chi_{x_j} \left( 1 + y_0 w_{\delta,0} - w_{\delta,0} y_0 \right) = 1$, since $x_j$ commutes with $y$ and $w_\delta$. This implies that
\[
\xi_0 e^{\sigma_j} - \xi e^{\sigma_j} \equiv 0,
\]
since $\xi_0$, $e \in l_j$ and $[k_{n_0}:l_j]$ is prime to $p$. But then $\xi_0 \eta_j^{-r} = \eta_j^r$ and $\eta_j^r$ commute. As noted in the Remark after Lemma 8.5, this means that $\eta_j^r$ is congruent (mod $P^{rF+1}$) to an element of $D_{j+1}$, which contradicts our assumption on $y$. This finishes the proof of Proposition 8.3 and of Theorem 8.4.

9. Extending to $D_x^\times$; removing hypotheses. In this section, we deal with two issues: extending the representations of $G$ to representations of $D_x^\times$, and removing the assumption that the $s_j$ are odd. The procedures are essentially those of [1], [2] and [6], and our discussion will be brief.

The elements of $k_\times$ that commute with $\pi_0$ (obtained from $\chi_x$) are those in $D_x$. Indeed, if $\delta \in k_\times \cap D_x$, then $\delta$ commutes with $\chi_x$, and we could simply extend $\chi_x$ to $H(k_\times \cap D_x)$. If $\delta \in k_\times$ but $\delta \notin D_x$, then the argument of Theorem 8.4 is easily adapted to show that $\pi$ and the representation $\pi^\delta$ defined by $\pi^\delta(y) = \pi(\delta y \delta^{-1})$ are disjoint. Hence we can extend $\pi_0$ to $G(k_n \cap D_x)$ and induce to get the irreducibles of $G \cdot k_\times$.

The situation for extending to $D_x^\times$ is similar; arguing as in §5 of [2], one can show that the elements of $D_x^\times$ commuting with a representation $\pi_\sigma$ of $G \cdot k_\times$ that comes from $\chi_x$ are precisely the elements of $D_x^\times(G \cap k_\times)$. Thus we need to extend $\pi_1$ to $G \cdot k_\times \cdot \langle \eta_t \rangle$ (since $\eta_t$ is a prime in $D_x$). Since $\langle \eta_t \rangle \cong Z$, there is no Mackey obstruction; however, we have no good way of describing $\pi_1(\eta_t)$. Inducing to $D_x^\times$ then gives an irreducible $\pi$.

To see how to deal with even $s_j$, consider the case where $m$ is even. Define $m'$ by $2m' = -m$. Now $\chi_x$ is initially defined on $G_{m'+1}$. We want to define an extension of $\chi_x$ to
\[
H : G_m \left( D_{s_1} \cap G_{s_1'} \right) \cdots \left( D_{s_{r-1}} \cap G_{s_{r-1}'} \right) (D_{s_r} \cap G).
\]
The argument of §7 lets us extend $\chi_x$ to $G_{m'+1}(D_{x_i} \cap G_{s'_i}) \cdots (D_{x_i} \cap G) = H_0$. To go to $H$ from $H_0$, note that a set of coset representatives for $G_{m'}/G_{m'+1}$ consists of the elements $y_{\delta} = 1 + \delta \eta^{m'}$, $\delta \in k_n$. The map $\mu: (\delta_1, \delta_2) \mapsto \chi_x(y_{\delta_1}y_{\delta_2}y_{\delta_1}^{-1}y_{\delta_2}^{-1})$ is an antisymmetric bilinear form on $k_n$, and $\delta$ is in the radial of $\mu \Leftrightarrow y_\delta \in H$. There is a unique irreducible projective representation of $H/H_0$ with the above form $\mu$ as multiplier; when $p$ is odd, it corresponds to a Heisenberg-type representation on the $k$-vector space $k_n/\text{Rad } \mu$. Tensor this representation with $\chi_x$ to get a representation which might be called $\chi'_x$; on $H_0$, $\chi'_x$ is a multiple of $\chi_x$. The reasoning in §8 applies to show that $\text{Ind}^G_{H} \chi'_x \otimes \sigma$ is irreducible and that these irreducibles exhaust $\text{Ind}^G_{G_m} \chi_x$; even the changes in the counting arguments are not difficult. If some other $s_j$ is even, we again get a Heisenberg-type representation on $G_{s'_j}$, where $2s'_j = -s_j$; the details are similar to those sketched above. Extending to those elements of $k_n^\times$ that commute with $x$ now involves the Weil representation; see, e.g., [4].

REFERENCES


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