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**TRANSITIVE ISOMETRY GROUPS WITH NONCOMPACT  
ISOTROPY**

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## TRANSITIVE ISOMETRY GROUPS WITH NON-COMPACT ISOTROPY

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**Let  $G$  be a connected Lie group acting effectively and transitively by isometries on a riemannian manifold  $M$ . Then  $G$  is a Lie subgroup of the full isometry group, which is not necessarily closed. In this paper we study the structure of the closure of  $G$  in  $I(M)$  and illustrate the results with examples, with non-compact isotropy, where the closure is described explicitly.**

**Introduction.** If  $M$  is a riemannian manifold the isotropy group in  $I(M)$  is compact; hence a homogeneous riemannian  $M$  can always be represented as a quotient  $G'/H'$  with  $H'$  compact.

Assume now that  $G$  is a connected Lie group acting effectively and transitively by isometries on  $M$ . Then  $G$  is a Lie subgroup of  $I(M)$  which will be closed in  $I(M)$  if and only if the isotropy subgroup  $H$  is compact.

In this paper we study in detail the closure of  $G$  in  $I(M)$ . Also if  $G$  is any connected Lie group and  $H$  a closed subgroup we compare three standard conditions on  $H$  which ensure that  $G/H$  admits a riemannian invariant structure. The rest of the paper is devoted to illustrate the fact that it is rather common to have transitive, effective, non-closed Lie subgroups of  $I(M)$ , hence the isotropy subgroup is non-compact. This situation arises quite frequently, even when  $M$  is compact (Lemma 1.4). Also, any connected semisimple Lie group with infinite center admits a closed non-compact subgroup  $H$  such that  $G$  acts effectively on  $G/H$  and  $G/H$  carries a  $G$ -invariant riemannian structure. In this case, that is, when  $G$  is semisimple, we give an upper bound for the dimension of  $\overline{G}_L$  and provide examples showing that these bounds are sharp (see (2.4), Proposition 2.3 and Remark 2.4).

In [DMW] the use of  $\overline{G}_L$  proved to be convenient in the study of bounded isometries on a riemannian manifold acted on transitively and effectively by a semisimple Lie group without local compact factors. Also, some examples where  $G_L$  is not closed in  $I(M)$  ( $G$  semisimple) are given in [DMW] Example (3.10). The authors would like to thank J. A. Wolf for very useful comments on a first version of this paper and, in particular, for suggesting a simpler proof of Proposition 2.2.

**1. Closure of  $G$ .** Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$  and let  $H$  be a closed subgroup of  $G$  with Lie algebra  $\mathfrak{h}$ . We will say that  $H$  satisfies condition (1) (respectively (2), (3)) if

(1)  $\overline{\text{Ad}_G(H)}$  is compact in  $\text{GL}(\mathfrak{g})$ , the closure with respect to the  $\text{GL}(\mathfrak{g})$  topology;

(2)  $\text{cl}(\text{Ad}_G(H))$  is compact in  $\text{Ad}(G)$ , where  $\text{cl}$  denotes closure with respect to the  $\text{Ad}(G)$  topology;

(3)  $\mathfrak{h}$  is a compactly embedded subalgebra of  $\mathfrak{g}$  (or equivalently see [H, p. 130])  $\text{Ad}_G(H_0)$  is compact in  $\text{Ad}(G)$ , where  $H_0$  denotes the identity component of  $H$ .

Along this section we will analyze the relationship among the different conditions (1), (2), (3).

*Condition (1).* It is well known (cf. [Ch E]. Ch. III) that if  $G$  acts effectively on  $G/H$  then  $H$  satisfies (1) if and only if  $G/H$  admits a  $G$ -invariant riemannian structure. Furthermore for each such structure,  $G$  is a Lie subgroup of the full isometry group  $I(M)$ ,  $M = G/H$ , via  $x \rightarrow L_x$ ,  $L_x$  the left translation on  $M = G/H$ . If  $S$  is a Lie subgroup of  $G$ , denote  $S_L = \{L_x | x \in S\}$ . The next theorem describes the closure of  $G_L$  in  $I(M)$ .

**1.1. THEOREM.** *Let  $G$  be a connected Lie group acting effectively on  $M = G/H$ ,  $H$  a subgroup of  $G$  satisfying (1). Then*

$$\overline{G}_L = \{L_x \circ \varphi : x \in G, \varphi \in \overline{H}_L\}.$$

*In particular,  $G_L$  is closed if and only if  $H$  is compact. Moreover*

(i)  $H_L$  is isomorphic to  $\text{Ad}(H)$  (as Lie groups).

(ii) There is an exact sequence

$$\{1\} \rightarrow \{(x, \text{Ad}(x^{-1})) : x \in H\} \rightarrow G \rtimes \overline{\text{Ad}(H)} \rightarrow \overline{G}_L \rightarrow \{1\}$$

where  $\rtimes$  denotes semidirect product.

*Proof.* If  $L_{x_n}$  converges in  $I(M)$  then  $x_n H \rightarrow x^* H$  in  $G/H$ , for some  $x^*$  in  $G$ . Hence there is a sequence  $h_n$  in  $H$  such that  $x_n h_n \rightarrow x^*$  in  $G$ . Now  $L_{h_n^{-1}} \in H_L$  and since  $H_L$  is contained in the isotropy subgroup in  $I(M)$ ,  $L_{h_n^{-1}} \rightarrow \phi$ ,  $\phi \in \overline{H}_L$ . As a consequence  $L_{x_n} = L_{x_n h_n} \circ L_{h_n^{-1}}$  converges to  $L_{x^*} \circ \phi$  and the first assertion follows.

Clearly, if  $H$  is compact,  $\overline{H}_L = H_L \subset G_L$ . Conversely if  $G_L$  is closed in  $I(M)$  then  $H_L$  is compact in  $G_L$  since  $H_L$  is the intersection of  $G_L$  with the isotropy subgroup in  $I(M)$ .

Next we prove (i). Let  $\mathfrak{m}$  be an  $\overline{\text{Ad}(H)}$  invariant complement of  $\mathfrak{h}$  in  $\mathfrak{g}$  where  $\mathfrak{h}$  denotes the Lie algebra of  $H$ . It is not hard to show that the canonical map  $\overline{H}_L \rightarrow \overline{\text{Ad}(H)}|_{\mathfrak{m}}$  sending  $\varphi \rightarrow (d\varphi)_{eH}$  is an isomorphism of Lie groups. Hence (i) will follow once we prove that  $\overline{\text{Ad}(H)}$  and  $\overline{\text{Ad}(H)}|_{\mathfrak{m}}$  are isomorphic.

Let  $\pi: \overline{\text{Ad}(H)} \rightarrow \overline{\text{Ad}(H)}|_{\mathfrak{m}}$  denote the restriction homomorphism. By compactness of  $\overline{\text{Ad}(H)}$  it follows that  $\pi$  is onto. If  $T$  is in  $\overline{\text{Ad}(H)}$  and  $T|_{\mathfrak{m}}$  is the identity, then  $T|_{\mathfrak{m}_1}$  is the identity for any  $\mathfrak{m}_1$ , an  $\overline{\text{Ad}(H)}$  invariant complement of  $\mathfrak{h}$ . Hence we may take  $\mathfrak{m}_1$  to be the orthogonal complement of  $\mathfrak{h}$  with respect to an  $\overline{\text{Ad}(H)}$  invariant inner product and consequently  $\mathfrak{m}_1$  will satisfy  $[\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1$ . If  $X \in \mathfrak{h}, Y \in \mathfrak{m}_1$

$$[TX - X, Y] = [TX, Y] - [X, Y] = T[X, Y] - [X, Y] = 0.$$

Hence  $\text{Ad}(\exp(TX - X))$  is the identity in  $\mathfrak{m}_1$  or equivalently  $(dL_{\exp(TX - X)})_{eH}$  is the identity in  $G/H$ . Since we are assuming that the action of  $G$  on  $G/H$  is effective it follows that  $\exp(TX - X) = e$  for all  $X$  in  $\mathfrak{h}$ , hence  $T$  is the identity on  $\mathfrak{g}$ .

Now (ii) follows immediately from the description of  $\overline{G}_L$ , after observing that the kernel of the Lie group homomorphism  $G \rtimes \overline{H}_L \rightarrow \overline{G}_L, (x, \varphi) \rightarrow L_x \circ \varphi$  consists of pairs  $(x, L_{x^{-1}})$  where  $x$  is in  $H$ .  $\square$

Let  $G$  be a connected Lie group and  $H$  a subgroup of  $G$  satisfying (1). Set

$$\text{Aut}(G, H) = \{ \varphi \in \text{Aut}(G) : \varphi(H) = H, (d\varphi)_e \in \overline{\text{Ad}(H)} \}.$$

If  $\varphi \in \text{Aut}(G, H), \overline{\varphi}: G/H \rightarrow G/H$  will denote the induced diffeomorphism,  $\overline{\varphi}(xH) = \varphi(x)H$ . If  $G$  acts effectively on  $M = G/H$  then, for any  $G$ -invariant riemannian structure on  $M, \overline{\varphi}$  is an isometry of  $M$ . Furthermore, if  $(d\varphi)_e = \lim \text{Ad}(h_n)$ , the sequence  $L_{h_n} \rightarrow \psi, \psi$  in  $\overline{H}_L$  (by taking a subsequence if necessary), hence  $(dL_{h_n})_{eH} \rightarrow (d\psi)_{eH}$ , that is,  $\overline{\varphi} = \psi$  since they both fix  $eH$ .

1.2. PROPOSITION. *Assume the hypotheses of Theorem 1.1 hold. Then the correspondence  $\varphi \rightarrow \overline{\varphi}, \text{Aut}(G, H) \rightarrow \overline{H}_L$  is a group isomorphism.*

*Proof.* It is clearly a homomorphism. To check it is surjective assume  $\psi = \lim L_{h_n}, h_n \in H$ . By taking a subsequence, if necessary, we have that  $\text{Ad}(h_n) \rightarrow T, T$  an automorphism of  $\mathfrak{g}$ . If  $\phi$  denotes the lifting of  $T$  to  $\tilde{G}, \tilde{G}$  the simply connected covering group of  $G$ , then  $\phi$  is the identity restricted to the center of  $\tilde{G}$ . In fact, let  $\tilde{h}_n$  in  $\tilde{G}$  be such that  $d(I_{\tilde{h}_n})_e = \text{Ad}(h_n)$ . Then  $I_{\tilde{h}_n}(\exp X) = \exp \text{Ad}(h_n) X \rightarrow \exp TX = \phi(\exp X)$  and since

$\tilde{G}$  is connected  $I_{h_n}$  converges pointwise to  $\phi$ ; the assertion follows. As a consequence  $\phi$  induces an automorphism  $\varphi: G \rightarrow G$  lifting  $T$  and since  $I_{h_n}$  converges pointwise to  $\varphi$  then  $\varphi(H) \subset H$ . By repeating the above argument with  $T^{-1}$  we conclude that  $\varphi \in \text{Aut}(G, H)$ . Also from  $I_{h_n}(x) \rightarrow \varphi(x)$ ,  $x$  in  $G$ , it follows that  $L_{h_n}(xH) \rightarrow \bar{\varphi}(xH)$  and consequently  $\bar{\varphi} = \psi$ .

Assume  $\bar{\varphi}$  is the identity on  $G/H$ . Then  $y^{-1}\varphi(y) \in H$ , for all  $y$  in  $G$ . Hence

$$\left. \frac{d}{dt} \right|_{t=0} \exp(-tY)\varphi(\exp tY) = -Y + (d\varphi)_e Y \in \mathfrak{h}$$

for all  $Y$  in  $\mathfrak{g}$ . As a consequence  $(d\varphi)_e|_{\mathfrak{m}}$  is the identity,  $\mathfrak{m}$  an  $\text{Ad}(H)$  invariant complement of  $\mathfrak{h}$ . Hence  $(d\varphi)_e$  is the identity on  $\mathfrak{g}$  (see the proof of (i), Theorem 1.1) or equivalently,  $\varphi = \text{Id}$ .  $\square$

*Condition (2).* Let  $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$  be the adjoint representation,  $Z$  the center of  $G$ ,  $\text{ad}: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$  the derivative of  $\text{Ad}$ . Since  $\text{Ad}(G)$ , the connected Lie subgroup of  $\text{Aut}(\mathfrak{g})$  with Lie algebra  $\text{ad}(\mathfrak{g})$ , is isomorphic to  $G/Z$ , condition (2) is equivalent to

$$(2') \quad \pi(\overline{ZH}) \text{ compact in } G/Z,$$

where  $\pi: G \rightarrow G/Z$  is the standard projection. Furthermore, since  $\text{Aut}(\mathfrak{g})$  carries the relative topology of  $\text{Gl}(\mathfrak{g})$ , Condition (2) implies Condition (1) (the converse does not hold in general, see Example 1.4 below). For subgroups  $H$  of  $G$  satisfying (2), one can give a simple description of  $\overline{G}_L$ , in terms of  $G$  and the closure of  $ZH$  in  $G$  (see also [DMW], (3.5)).

**1.3. PROPOSITION.** *Let  $G$  be a connected Lie group, acting effectively on  $M = G/H$ ,  $H$  satisfying (2). Then  $\overline{G}_L = \{L_x \circ R_{y^{-1}}: x \in G, y \in \overline{ZH}\}$  and there is an exact sequence*

$$\{1\} \rightarrow S \rightarrow G \times \overline{ZH} \rightarrow \overline{G}_L \rightarrow \{1\}$$

where  $S = \{(z, zh) | z \in Z, h \in H\}$ . Furthermore, if  $H$  is not compact,  $\overline{ZH}/H$  is an abelian Lie subgroup of positive dimension of  $Z(\overline{G}_L)$ , the center of  $\overline{G}_L$ .

*Proof.* We observe first that for any  $G$ -invariant structure on  $M = G/H$ , if  $y \in \overline{ZH}$  the map  $R_y$  is in the closure of  $G$  in  $I(M)$ , where  $R_y(zH) = zyH$ . In fact, since  $\overline{ZH}$  is contained in  $N(H)$ , the normalizer of  $H$ ,  $R_y$  is a diffeomorphism. Furthermore the condition  $y = \lim z_n h_n$  says

that  $xyH = \lim L_{z_n}(xH)$ , hence  $R_y = \lim L_{z_n}$  in  $I(M)$  ([H], p. 203, Lemma 2.4).

The map  $\alpha: G \times \overline{ZH} \rightarrow \overline{G}_L$ ,  $\alpha(x, y) = L_x \circ R_{y^{-1}}$  is a Lie group homomorphism. To check that  $\alpha$  is surjective it is enough (by Theorem 1.1) to consider  $\varphi$  in  $\overline{H}_L$ ,  $\varphi = \lim L_{h_n}$ . Since  $H$  satisfies (2'), there exists a subsequence  $h_{n_j}$  such that  $\pi(h_{n_j}) \rightarrow \pi(u)$ ,  $u \in \overline{ZH}$ . Hence  $z_{n_j} h_{n_j} \rightarrow u$  for some  $z_{n_j}$  in  $Z$  and  $L_{h_n}(xH) = h_n z_n x z_n^{-1} h_n^{-1} H \rightarrow L_u \circ R_{u^{-1}}(xH)$ , that is,  $\varphi = \alpha(u, u)$ . Let  $(x, y)$  in  $\ker \alpha$ . Then  $z^{-1} x z y^{-1} \in H$  for all  $z$  in  $G$ . Since  $y x^{-1} \in H$  (taking  $z = e$ ) it follows that  $L_x \circ R_{x^{-1}} \in \text{Aut}(G, H)$  induces the identity on  $G/H$ . By Proposition 1.2,  $L_x \circ R_{x^{-1}}$  is the identity on  $G$  and the description of  $S = \ker \alpha$  follows.

It is not hard to check that  $\overline{ZH}/H$  is an abelian Lie group which is a Lie subgroup of  $Z(\overline{G}_L)$ . Assume  $\dim \overline{ZH}/H = 0$ . The map

$$\beta: Z \times H \rightarrow \overline{ZH}, \quad (z, h) \rightarrow zh$$

is a one-to-one Lie group homomorphism. By regularity  $\beta(Z \times H) = ZH$  is open in  $\overline{ZH}$ , hence closed. Now if  $ZH$  is closed and  $H$  satisfies (2') it follows that  $H$  is compact. This completes the proof.  $\square$

We will next show that, even the case  $M$  compact riemannian, it is quite common to have a representation with non-compact isotropy subgroup. This question was posed to us by Cristian Sanchez.

1.4. LEMMA. *Let  $G$  be a compact, connected semisimple Lie group,  $H$  a closed subgroup of  $G$  such that  $G$  acts effectively on  $G/H$ . Assume that  $M = G/H$  carries a  $G$ -invariant riemannian structure such that  $\text{rank}(U) \geq \text{rank}(H) + 2$ , where  $U = \{x \in N(H): R_x \text{ is an isometry}\}$ . Then  $M$  can be represented as a quotient  $G'/H'$ , where  $G'$  acts effectively by isometries on  $G'/H'$  and  $H'$  is a closed noncompact subgroup. Furthermore, if  $\phi: G' \rightarrow I(M)$  is the inclusion as a Lie subgroup, then  $\phi(G') = G_L \cdot S_R$  where  $S$  is a torus in  $U$ .*

*Proof.* Let  $\mathfrak{u}$  denote the Lie algebra of  $U$  and let  $T$  be a torus in  $U$  such that  $\dim(T) \geq 2$  and  $\mathfrak{t} \cap \mathfrak{h} = 0$  (here  $\mathfrak{t}$  is the Lie algebra of  $T$ ). Fix  $X \in \mathfrak{t}$  so that  $t \rightarrow \exp(tX)$  is one-to-one.

Set  $G' = G \times \mathbf{R}$ ; let  $\phi: G' \rightarrow I(M)$  be given by  $\phi(g, t) = L_g \circ R_{\exp(-tX)}$ . We assert that  $\phi$  is a monomorphism. Indeed,  $(g, t) \in \ker(\phi)$  if and only if  $x^{-1} g x \exp(-tX) \in H$ , for any  $x$  in  $G$ . Thus  $g = h \exp(tX) \in U$ , for some  $h$  in  $H$  and  $\text{Ad}(g)(Z) - Z \in \mathfrak{h}$ , for any  $Z \in \mathfrak{g}$ . Arguing as in the proof of Proposition 1.2 one concludes that  $\text{Ad}(g) = \text{Id}$ , or  $g \in Z(G)$ . On the other hand, the one-parameter group generated by  $X$

intersects  $H$  in the identity element, since  $T \cap H$  is a finite group. This easily implies that  $h = \exp(tX) = e$ , or  $t = 0$ . Hence  $M$  is isometric to  $G'/H'$  where  $H' = \{(h \exp(tX), t): h \in H, t \in \mathbf{R}\}$ .

Finally, by Proposition 1.3 it is easy to check that if  $S$  denotes the closure of  $\{\exp(tX): t \in \mathbf{R}\}$ , then

$$\overline{G'_L} = G_L \cdot (S \times \{0\})_R.$$

From this it easily follows that  $\overline{\phi(G')}$  is as asserted in the lemma. This completes the proof.  $\square$

1.5. REMARK. (i) As a particular case in the lemma we may take  $H = \{e\}$ . That is,  $M = G$  is any compact, semisimple, connected Lie group endowed with a left invariant metric such that  $\text{rank}(U) \geq 2$ . For instance, if  $\text{rank}(G) \geq 2$  and we use a bi-invariant metric, then  $G$  is a riemannian symmetric space admitting a representation  $G'/H'$ , with  $H'$  non-compact.

(ii) The proof above only uses that  $Z(G)$  is finite and that  $H$  is compact. Hence the lemma remains valid under these assumptions on  $G$  and  $H$ , only.

We recall that in a variety of cases Conditions (1) and (2) are equivalent. For instance when  $\text{Ad}(G)$  is closed in  $\text{GL}(\mathfrak{g})$  since both topologies coincide. This happens for  $G$  semisimple since any derivation of  $\mathfrak{g}$  is inner. Also  $\text{Ad}(G)$  is closed if  $G$  is reductive. It may fail to be so when  $G$  is solvable as the next example shows.

1.6. EXAMPLE. We provide here an example of a subgroup  $H$  of a connected Lie group  $G$ , satisfying (1) and not satisfying (2). Moreover since  $G$  acts effectively on  $G/H$ , we apply Theorem 1.1 to compute the closure of  $G$  in  $I(M)$ ,  $M = G/H$ .

If  $t_i \in S^1$ ,  $z_i \in \mathbf{C}$  for  $1 \leq i \leq n + 1$ , denote

$$s(t_1, \dots, t_{n+1}) = \begin{vmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_{n+1} & & \\ & & & & 1 \end{vmatrix},$$

$$n(z_1, \dots, z_{n+1}) = \begin{vmatrix} 1 & & & & z_1 \\ & \ddots & & & \vdots \\ & & 1 & & z_{n+1} \\ & & & & 1 \end{vmatrix}$$

and set

$$S = \{s(t_1, \dots, t_{n+1}): t_i \in S^1\}, \quad N = \{n(z_1, \dots, z_{n+1}): z_i \in \mathbf{C}^{n+1}\},$$

$G_1 = NS$ . Since

$$s(t_1, \dots, t_{n+1})n(z_1, \dots, z_{n+1})s(t_1, \dots, t_{n+1})^{-1} = n(t_1z_1, \dots, t_{n+1}z_{n+1})$$

then it is clear that  $G_1$  is a closed subgroup of  $GL(n + 2, \mathbf{C})$  isomorphic to the semidirect product  $N \rtimes S$  where  $S$  acts on  $N$  by

$$s(t_1, \dots, t_{n+1}) \cdot (n(z_1, \dots, z_{n+1})) = n(t_1z_1, \dots, t_{n+1}z_{n+1}).$$

We note that if  $g = ns$ ,  $g$  commutes with  $N$  if and only if  $s$  commutes with  $N$ , that is,  $s = 1$ . Hence, if  $G_2$  is a Lie subgroup with  $N \subset G_2$  and if, for each  $j$ , there is  $s(t_1, \dots, t_{n+1})$  in  $S \cap G_2$  with  $t_j \neq 1$ , then  $Z(G_2) = 1$ .

Now let

$$\phi: \mathbf{R}^{n+1} \rightarrow G_1, \phi(x_1, \dots, x_{n+1}) = s(e^{2\pi i x_1}, \dots, e^{2\pi i x_{n+1}}).$$

Let  $\mathbf{V}$  be an  $n$ -dimensional subspace of  $\mathbf{R}^{n+1}$  and let  $v \in \mathbf{V}$  be such that  $\phi$  is injective when restricted to  $\mathbf{V}$  and  $\mathbf{R}v + \sum_1^{n+1} \mathbf{Z}e_j$  is dense in  $\mathbf{R}^{n+1}$ ,  $\{e_1, \dots, e_{n+1}\}$  the standard basis of  $\mathbf{R}^{n+1}$ ; (see appendix, Lemma A.4).

Set  $G = N\phi(\mathbf{V})$ ,  $H = \phi(\mathbf{R}v)$ . Then  $G$  is a normal dense Lie subgroup of  $G_1$  and  $H$  is dense in  $S$ . On the other hand,  $H$  is a closed Lie subgroup of  $G$ . Now, by the choice of  $H$ , it is not hard to see that  $\overline{\text{Ad}(H)} = \text{Ad}_G(S)$  and this is, in turn, isomorphic to  $T^{n+1}$ . Hence (1) holds. However, given that  $Z = \{e\}$ , and  $H$  is not compact, (2) is not satisfied.

One checks that  $\overline{\text{Ad}(H)} = \{I_s: s \in S\}$ . From this and the fact that  $G$  acts effectively on  $G/H$ , it follows (see 1.1 and 1.2) that

$$\overline{G}_L = \{L_x \circ \bar{I}_s: x \in G, s \in S\}.$$

By a calculation one checks that  $Z(\overline{G}_L) = \{e\}$  in contrast to the case when (2) is satisfied (see Proposition 1.3).

**1.7. REMARK.** Clearly a connected subgroup  $H$  of  $G$  satisfying condition (3) will satisfy (2) (then (1)). Now if  $G$  acts effectively on  $G/H$ , the conditions  $\text{Ad}_G(H)$  compact and  $H$  compact are equivalent. Hence if  $H$  satisfies (3),  $G_L$  will be closed in  $I(M)$ . As 1.4, 1.5, and 1.6 show, it is not true in general that if  $G/H$  is riemannian homogeneous then  $H$  satisfies (3). On the other hand, we see later (see Remark 2.3) that if  $G$  is any simply connected semisimple Lie group with infinite center there exists  $H$  a non-compact, Lie subgroup satisfying (2) with  $G$  acting effectively on  $G/H$ . Such an  $H$  will not satisfy (3).

As a final observation we describe a way to construct new examples of pairs  $(G, H)$ ,  $H$  satisfying (1),  $H$  non-compact.

1.8. LEMMA. *Let  $(G, H)$  be a pair of a connected Lie group  $G$  and a closed subgroup  $H$  so that (1) holds. Let  $(\sigma, \mathbf{V})$  be a finite dimensional representation of  $G$  such that  $H$  acts unitarily. Set  $G_\sigma = \mathbf{V} \rtimes_\sigma G$ . Assume that  $\ker(\sigma) \cap \ker(\text{action of } G \text{ on } G/H) = \{e\}$ . Then  $G_\sigma$  acts effectively on  $G_\sigma/H$  and  $H$  satisfies (1). Furthermore, if  $\ker(\sigma) \cap Z(G) = \{e\}$ ,  $G$  acts without non-zero fixed vectors and  $H$  is not compact, then (2) will not be satisfied.*

*Proof.* Clearly  $H$  is closed in  $G_\sigma$  and  $\text{Ad}_{G_\sigma}(H) \subset O(\mathbf{V}) \times \text{Ad}_G(H)$ , hence (1) holds. On the other hand

$$(1.9) \quad \begin{aligned} (0, g_1) \cdot (v, g) \cdot (0, g_1^{-1}) &= (\sigma(g_1)v, g_1 g g_1^{-1}), \\ (v_1, 1) \cdot (v, g) \cdot (-v_1, 1) &= (v + v_1 - \sigma(g)v_1, g). \end{aligned}$$

Thus

$$\bigcap_{x \in G_\sigma} xHx^{-1} = \left( \bigcap_{x \in G} xHx^{-1} \right) \cap \ker \sigma = \{e\},$$

by the assumption. Hence the action is effective. On the other hand, by (2.5)

$$Z(G_\sigma) = \{(g_0, v_0); g_0 \in Z(G) \cap \ker(\sigma), \sigma(g)v_0 = v_0 \text{ for any } g \in G\}$$

and under the assumptions  $Z(G_\sigma) = \{e\}$ , hence the second assertion is clear.  $\square$

1.9. REMARKS. (a) We note that Example 1.6 is a particular instance of the construction in the lemma with  $(G, H) = (\mathbf{R}^n, \mathbf{R})$ . In this case, the representation of  $\mathbf{R}^n$  in  $\mathbf{C}^{n+1}$  is faithful and unitary and the action of  $G$  on  $G/H$  is far from being effective but the action of  $G_\sigma$  on  $G_\sigma/H$  is so. This procedure could naturally be iterated by taking a finite dimensional representation  $(\tau, \mathbf{W})$  of  $G_\sigma$ , unitary on  $H$ , thus getting a new pair  $((G_\sigma)_\tau, H)$ .

(b) As another illustration of the lemma, let  $G$  be semisimple and  $H$  a closed non-compact subgroup such that (2) holds. Let  $(\sigma, \mathbf{V})$  be a non-trivial, finite dimensional, irreducible representation of  $G$ . We have that  $Z(G_\sigma) = \{(0, z) | z \in Z \cap \ker \sigma\}$ . Hence

$$p_\sigma(\overline{Z(G_\sigma)H}) = \{0\} \times p(\overline{(Z \cap \ker \sigma)H})$$

is compact, where  $p_\sigma: G_\sigma \rightarrow G_\sigma/Z(G_\sigma)$  is the canonical projection. Hence (2) holds and if, for instance,  $G$  acts effectively or  $\sigma$  is faithful the action of  $G_\sigma$  is effective.

**2. Non compact isotropy.** In this section we consider the case when  $G$  is semisimple. As observed in §1, for  $H$  a closed subgroup of  $G$  Conditions (1) and (2) are equivalent. The first result we prove is a characterization of connected non-compact subgroups  $H$  satisfying (2). We also give a way to compute  $\overline{G}_L$  explicitly, in the light of Proposition 1.2 and a classical result (see Theorem A.1 in the appendix). We also construct families of examples where  $\dim \overline{G}_L - \dim G_L$  is the maximum possible.

Let then  $G$  be a connected semisimple Lie group. If  $G = K \cdot \exp(\mathfrak{p})$  is a Cartan decomposition,  $K$  is compact if and only if  $Z$ , the center of  $G$ , is finite. Denote by  $\mathfrak{k}$  the Lie algebra of  $K$  and let  $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ . Then  $\mathfrak{k} = z_\mathfrak{k} \oplus \mathfrak{k}'$ ,  $z_\mathfrak{k}$  the center of  $\mathfrak{k}$ . Denote by  $Z_K$  and  $K'$ , respectively, the connected Lie subgroups of  $K$  corresponding to  $z_\mathfrak{k}$  and  $\mathfrak{k}'$ . Then  $K'$  is compact and if furthermore,  $G$  is simply connected,  $K$  is isomorphic to the direct product of  $Z_K$  and  $K'$ . In this case  $Z_K$  is a vector group.

2.1. PROPOSITION. *Let  $G$  be a simply connected semisimple Lie group and  $H$  a closed connected subgroup such that (2) is satisfied. Then there is a Cartan decomposition  $G = K \exp(\mathfrak{p})$  such that*

$$(2.1) \quad H = \exp(\mathbf{V}) \cdot (H \cap K')$$

where  $V$  is a vector subspace of the center of  $\mathfrak{h}$ ,  $\exp: \mathbf{V} \rightarrow \exp \mathbf{V}$  is an isomorphism onto a closed subgroup and  $\exp(\mathbf{V}) \cap (H \cap K') = 1$ . Furthermore  $\overline{ZH}$  can be explicitly computed.

*Proof.* Let  $p: G \rightarrow G/Z$  be the canonical projection. Since  $p(\overline{ZH}) = \overline{p(H)}$  is compact, by a theorem of Iwasawa, there is a maximal compact subgroup  $K_0$  of  $G/Z$  such that  $p(\overline{ZH}) \subset K_0$ . Thus  $\overline{ZH} \subset K = p^{-1}(K_0)$ . Let  $K = Z_K \cdot K'$  as above. If  $Z_K = \{e\}$ , then  $K$  is compact, hence  $H$ , and (2.1) is clear. We thus assume from now on that  $z_\mathfrak{k} \neq 0$ . Now,  $\mathfrak{h}$  admits an  $\text{Ad}(H)$ -invariant inner product, hence  $\mathfrak{h} = z_0 \oplus \mathfrak{h}'$  where  $z_0$  is the center of  $\mathfrak{h}$  and  $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ . Let  $H'$  (resp.  $Z_H$ ) be the connected Lie subgroup of  $H$  corresponding to  $\mathfrak{h}'$  (resp.  $z_0$ ). Then  $H'$  is compact and  $H' \subset K'$ . Write  $z_0 = \mathbf{V} \oplus z'$  with  $z' = z_0 \cap \mathfrak{k}'$  and  $\mathbf{V}$  a vector subspace of  $z_0$ . Fix  $\{(U_i, Y_i): U_i \in z_\mathfrak{k}, Y_i \in \mathfrak{k}', 1 \leq i \leq m\}$  a basis of  $\mathbf{V}$ . Then the set  $\{U_i: 1 \leq i \leq m\}$  is linearly independent since  $\mathbf{V} \cap \mathfrak{k}' = \{0\}$ . Also  $[Y_i, Y_j] = 0$ , for all  $i, j$ . Assume now that

$$\exp\left(\sum_1^m t_i(U_i, Y_i)\right) = \exp\left(\sum_1^m t_i(U_i, 0)\right)\exp\left(\sum t_i(0, Y_i)\right) \in K'.$$

Since  $\exp(\sum_1^m t_i U_i) = 1$ , then,  $t_i = 0$  for  $1 \leq i \leq m$ . Hence,  $\exp|_{\mathbf{V}}$  is one-to-one and  $\exp(\mathbf{V}) \cap K' = \{1\}$ . This implies that  $Z_H \cap K' = \exp(z')$ , that is,  $Z_H = \exp \mathbf{V} \cdot (Z_H \cap K')$ .

Let now  $v_n \in \mathbf{V}$  be so that  $\exp(v_n) \rightarrow u \in G$ . Then  $u \in Z_H$ , hence  $u = \exp(v)u'$ ,  $u' \in Z_H \cap K'$ . Write

$$v_n v^{-1} = \sum_1^m t_{i,n}(U_i, Y_i).$$

Since  $\exp(v_n v^{-1}) \rightarrow u' \in K'$  it follows that  $\exp(\sum_1^m t_{i,n} U_i) \rightarrow 1$  in  $Z_K$ . Hence  $t_{i,n} \rightarrow 0$ , for  $1 \leq i \leq m$  (again we use that  $U_1, \dots, U_m$  are linearly independent). Therefore  $u' = 1$  and  $\exp(\mathbf{V})$  is closed in  $G$ . This completes the proof of the first assertion. We now describe how to compute  $\overline{ZH}$ .

Fix  $Z' \subset Z \cap Z_K$ , a subgroup such that  $|Z/Z'| < \infty$  and let  $Z''$  be a system of representatives of  $Z/Z'$ . Then

$$(2.2) \quad \overline{ZH} = \overline{Z'H} \cdot Z'' = \overline{Z' \cdot \exp \mathbf{V} \cdot H \cap K' \cdot Z''}.$$

Let  $\{(U_i, Y_i) | 1 \leq i \leq m\}$  be as above. Also let  $\mathfrak{s} = \sum_1^m \mathbf{R}Y_i$ ,  $T' = \exp(\mathfrak{s})$  and let  $\mathfrak{t}'$  be the Lie algebra of  $T'$ . Choose  $\{X_j; 1 \leq j \leq d\}$  a basis of  $\mathfrak{t}'$  so that  $\sum \mathbf{Z}X_j = \ker(\exp|_{\mathfrak{t}'})$  and  $\{V_i; 1 \leq i \leq c\}$  a basis of  $\mathfrak{z}_{\mathfrak{t}'}$  such that  $Z' = \exp(\sum \mathbf{Z}V_i)$ .

Thus

$$(2.3) \quad \overline{Z' \exp(\mathbf{V})} = \overline{\exp\left(\sum_1^c \mathbf{Z}V_i + \sum_1^d \mathbf{Z}X_j + \mathbf{V}\right)}$$

and it then follows from (2.2) and (2.3) that  $\overline{ZH}$  can be computed (at least in principle) by means of Theorem A.1 in the appendix.  $\square$

2.2. REMARK. Assume that  $G$  is semisimple and the hypotheses of Proposition 1.3 hold. Then, given any  $G$ -invariant riemannian metric on  $G/H$ .

$$(2.4) \quad \dim \overline{G}_L \leq \dim G + \text{rank}(K) - \text{rank}(H)$$

where  $G = K \cdot \exp(\mathfrak{p})$  is a Cartan decomposition of  $G$ . In fact, since  $H$  satisfies (2),  $\overline{G}_L$  is given by Proposition 1.3, hence

$$\dim \overline{G}_L = \dim G + \dim \overline{ZH}/H.$$

Now  $\overline{ZH} \subset K = p^{-1}(K_0)$ ,  $K_0$  a maximal compact subgroup of  $G/Z$ ,  $p(\overline{ZH}) \subset K_0$ . Let  $\mathfrak{h}$  denote the Lie algebra of  $\overline{ZH}$ . It is clear that  $\mathfrak{h}$  is an ideal of  $\mathfrak{k}$  and there is an abelian ideal  $\mathfrak{a}$  of  $\mathfrak{k}$  with  $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{h} \subset \mathfrak{k}$ . Hence  $\dim \overline{G}_L = \dim G + \dim \mathfrak{a} \leq \dim G + \text{rank}(K) - \text{rank}(H)$  as asserted.

We next provide a family of examples where  $\dim \overline{G}_L - \dim G_L$  is the maximum possible. This will show that the bound in (2.4) is sharp.

**2.2. PROPOSITION.** *Let  $G$  be a simply connected semisimple Lie group such that the center of each simple factor is infinite. Then there exists a discrete subgroup  $H \simeq \mathbf{Z}$  such that  $\overline{ZH} = T$ , a compact Cartan subgroup, and  $G$  acts effectively on  $G/H$ . Furthermore if  $G$  is not locally isomorphic to  $\mathrm{SL}(2, \mathbf{R})$ , there exists a closed subgroup  $H' \simeq \mathbf{R}$  with  $\overline{ZH'} = T$  and  $G$  acting effectively on  $G/H'$ . Then for any  $G$ -invariant riemannian metric on  $G/H$  (resp.  $G/H'$ )*

$$\dim \overline{G}_L = \dim G + \mathrm{rank}(K) \quad (\text{resp. } \dim \overline{G}_L = \dim G + \mathrm{rank}(K) - 1).$$

Furthermore  $\overline{G}_L = I(G/H)_0$  (resp.  $I(G/H')_0$ ).

*Proof.* Let  $T$  be a Cartan subgroup,  $T \subset K$  and let  $\mathfrak{t}$  denote the Lie algebra of  $T$ . Let  $p: G \rightarrow G/Z$  be the canonical projection. Then  $T/Z$  is a torus. Fix  $X$  in  $\mathfrak{t}$  so that, if  $t = \exp(X)$ ,  $tZ$  is a generator of  $T/Z$  (i.e. the powers of  $tZ$  form a dense subgroup of  $T/Z$ ). Let  $H$  be the group generated by  $t$  and let  $H'$  be the one-parameter group generated by  $X$ . Now  $X = X_1 + X_2$ ,  $X_1 \in \mathfrak{z}_{\mathfrak{t}}$ ,  $X_2 \in \mathfrak{k}'$  and  $X_1 \neq 0$ , since  $tZ$  is a generator. This implies that both  $H$  and  $H'$  are closed subgroups.

On the other hand  $G$  acts effectively on  $G/H$  since  $\bigcap_{x \in G} x^{-1}Tx = Z$  and by the choice of  $t$  it is clear that  $H \cap Z = \{e\}$ . Similarly, if  $H' \cap Z = \{e\}$ , the action on  $G/H'$  is effective. Otherwise  $p(H')$  is a circle and then  $p(H') = T$ . Therefore  $G$  is isomorphic to the universal covering group of  $\mathrm{SL}(2, \mathbf{R})$ , against our assumption. Thus, by (2.4)  $\dim \overline{G}_L$  is as asserted, since, if  $S = H$  or  $H'$ , by Proposition 1.3  $\overline{G}_L = G_L \cdot (\overline{ZS})_R = G_L \cdot T_R$ . Finally, by a result of C. Gordon ([G], Theorem 4.1)

$$I(M)_0 = G_L \cdot (U_0)_R$$

where, as usual,  $U = \{x \in N(S): R_x \text{ is an isometry}\}$ . Now

$$T \subseteq U_0 \subset N(S)_0 \subset N(T)_0 = T.$$

Thus the last assertion is clear. This concludes the proof. □

**2.3. REMARK.** It is easy to modify the example in (3.10) [DMW] to show that any connected semisimple Lie group with infinite center  $G$  admits a closed, non-compact, discrete subgroup  $H$  satisfying (2) and  $G$  acting effectively on  $G/H$ .

APPENDIX. In this appendix we list some classical facts on the subgroups of  $\mathbf{R}^n$ . We refer to [B], Ch. VII for a detailed treatment. If  $x, y \in \mathbf{R}^n$  denote by  $x \cdot y$  the canonical inner product. Given  $G \subset \mathbf{R}^n$ , a subgroup, set  $G^* = \{g' \in \mathbf{R}^n \mid g' \cdot g \in \mathbf{Z}, \text{ for all } g \in G\}$ .

A.1. THEOREM (see [B], p. 74). *Let  $G \subset \mathbf{R}^n$  be a subgroup. Then  $\overline{G} = G^{**}$ . In particular  $G$  is dense if and only if  $G^* = \{0\}$ .*

A.2. COROLLARY. *Fix  $v = (a_1, \dots, a_n) \in \mathbf{R}^n$  and let  $\{e_j\}_1^n$  be the canonical basis. Let  $G_1 = \mathbf{Z}v + \sum_1^n \mathbf{Z}e_j$  (resp.  $G_2 = \mathbf{R}v + \sum_1^n \mathbf{Z}e_j$ ). Then  $G_1$  (resp.  $G_2$ ) is dense in  $\mathbf{R}^n$  if and only if  $1, a_1, \dots, a_n$  are  $\mathbf{Q}$ -linearly independent (resp.  $a_1, \dots, a_n$  are  $\mathbf{Q}$ -linearly independent).*

A.3. LEMMA. *If  $1 \leq k \leq n$ , let  $\phi: \mathbf{R}^n \rightarrow \mathbf{R}^k \times T^{n-k}$  be the canonical map,  $\phi(t_1, \dots, t_k, t_{k+1}, \dots, t_n) = (t_1, \dots, t_k, e^{2\pi i t_{k+1}}, \dots, e^{2\pi i t_n})$ . If  $G \subset \mathbf{R}^n$  is a subgroup, then  $\phi(G) = \phi(G + \sum_{k+1}^n \mathbf{Z}e_j)$ .*

A.4. LEMMA. *There exists an  $n$ -dimensional subspace  $\mathbf{V}$  of  $\mathbf{R}^{n+1}$  and  $v \in \mathbf{V}$  such that  $\mathbf{V} \cap \sum_1^{n+1} \mathbf{Z}e_j = \{0\}$  and  $\mathbf{Z}v + \sum_1^{n+1} \mathbf{Z}e_j$  is dense in  $\mathbf{R}^{n+1}$ .*

*Proof.* The lemma follows in a standard fashion from Theorem A.1. The proof will thus be omitted.

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