POLYNOMIAL EQUATIONS OF IMMERSED SURFACES

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S. AKBULUT AND H. KING

If $V$ is a nonsingular real algebraic set we say $H_i(V; \mathbb{Z}_2)$ is algebraic if it is generated by nonsingular algebraic subsets of $V$.

Let $V^3$ be a 3-dimensional nonsingular real algebraic set. Then, we prove that any immersed surface in $V^3$ can be isotoped to an algebraic subset if and only if $H_i(V; \mathbb{Z}_2)$ $i = 1, 2$ are algebraic. This isotopy above carries the natural stratification of the immersed surface to the algebraic stratification of the algebraic set. Along the way we prove that if $V$ is any nonsingular algebraic set then any simple closed curve in $V$ is $\epsilon$-isotopic to a nonsingular algebraic curve if and only if $H_1(V; \mathbb{Z}_2)$ is algebraic.

Let $V^3$ be a 3-dimensional nonsingular real algebraic set. We call a homology group of $V$ algebraic if it is generated by nonsingular algebraic subsets. In this paper we prove:

**Theorem.** The following are equivalent:

(a) If $f: M^2 \hookrightarrow V^3$ is any immersion of a closed smooth surface in general position, then $f(M^2)$ is isotopic to an algebraic subset $Z$ of $V^3$ by an arbitrarily small isotopy. This isotopy carries the natural stratification of $f(M^2)$ to the algebraic stratification of $Z$.

(b) $H_1(V; \mathbb{Z}_2)$ and $H_2(V; \mathbb{Z}_2)$ are algebraic.

To be more precise for $i = 1, 2$ let $AH_i(V^3; \mathbb{Z}_2)$ be the subgroup of $H_i(V^3; \mathbb{Z}_2)$ generated by nonsingular algebraic subsets. Then $H_i(V; \mathbb{Z}_2)$ is algebraic if it is equal to $AH_i(V; \mathbb{Z}_2)$. In particular zero homology groups are algebraic. We will refer to elements of $AH_i(V^3; \mathbb{Z}_2)$ as algebraic homology classes. This definition is consistent with the conventions of [AK1].

In case $f$ is an imbedding this theorem reduces to a special case of Proposition 1 below, which is Theorem 4.1 and Remark 4.2 of [AK1]. Recall, if $W^n$ is a nonsingular algebraic set of dimension $n$, then $AH_{n-1}(W; \mathbb{Z}_2)$ is the subgroup of $H_{n-1}(W; \mathbb{Z}_2)$ generated by nonsingular algebraic subsets. Also if $M \subset W$ is a closed submanifold, denote the
\(Z_2\)-homology class in \(W\) induced by the fundamental class of \(M\) by \([M]_2\). Then

**Proposition 1.** A codimension one closed smooth submanifold \(M\) of \(W\) is \(\varepsilon\)-isotopic to a nonsingular real algebraic subset if and only if \([M]_2 \in AH_{n-1}(W; \mathbb{Z}_2)\). Furthermore, this isotopy can fix any smooth submanifold \(L\) of \(M\) which is already a nonsingular algebraic set.

**Remark.** Proposition 1 remains true if \(L\) is a union of nonsingular algebraic sets in \(M\) ([T]).

We first prove a codimension two version of this proposition for \(V^3\), which is an interesting result in itself.

**Proposition 2.** A simple closed curve \(C \subset V^3\) is \(\varepsilon\)-isotopic to a nonsingular algebraic curve if and only if \([C]_2 \in AH_1(V; \mathbb{Z}_2)\). Furthermore this isotopy can fix any collection of points in \(C\).

**Remark.** This proposition remains true if \(V^3\) is replaced by a nonsingular algebraic set of any dimension. The proof is essentially the same.

**Lemma 3.** Let \(C \subset V^3\) be a nonsingular algebraic curve and \(L \subset V^3\) be a smooth manifold. Then \(C\) can be moved by an \(\varepsilon\)-isotopy to a nonsingular algebraic curve \(C'\) which is transversal to \(L\).

**Proof.** Let \(F^2\) be the boundary of a small closed tubular neighborhood of \(C\) in \(V\). \(F\) is a circle bundle over \(C\) and hence has a section, so after a small isotopy of \(F\) we can assume that \(C \subset F\). Since \(F\) is null homologous, by Proposition 1, it is \(\varepsilon\)-isotopic to a nonsingular algebraic surface \(Z\) with \(C \subset Z\). By the terminology of \([AK]\) \(C\) is a stable algebraic set. Stable algebraic sets have the required property (Proposition 4.3 of \([AK]\)).

**Lemma 4.** If \(V^3\) is orientable and \(F^2 \subset V^3\) is a compact orientable surface with \(\partial F^2 = C \cup A\) where \(A\) is a nonsingular algebraic curve, then \(C\) is \(\varepsilon\)-isotopic to a nonsingular algebraic curve.

**Proof.** Since \(V\) is orientable \(F\) has a trivial normal bundle in \(V\). Let \(F' = \partial(F \times I) \subset V^3\) corners smoothed, and \(C \cup A = \partial(F \times 0) \subset F'\). \(C \cup A\) separates \(F'\). Since \([F']_2 = 0\) by Proposition 1 \(F'\) is \(\varepsilon\)-isotopic to a nonsingular algebraic surface \(Z\) with \(A \subset Z\). After a small isotopy of \(C\)
we can assume $C \subset Z$. Then $C \cup A$ separates $Z$; this means $[C]_2 = [A]_2 \in AH_1(Z; \mathbb{Z}_2)$. Hence by Proposition 1 $C$ is $\epsilon$-isotopic to a nonsingular algebraic curve $C^*$ in $Z$. $C^*$ is the required algebraic curve. \( \square \)

**Remark.** We can assume that the isotopy $C \sim C^*$ fixes any finite number of points of $C$. This is because by Proposition 1 we can arrange that $Z$ and $C^*$ fix these points.

**Lemma 5.** If $S \subset V^3$ is an orientable surface and

$$i_* : H_1(V - S; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$$

is the map induced by the inclusion, then $\ker(i_*) \subset AH_1(V - S; \mathbb{Z}_2)$.

**Proof.** From the homology exact sequence

$$H_2(V, V - S; \mathbb{Z}_2) \xrightarrow{\partial} H_1(V - S; \mathbb{Z}_2) \xrightarrow{i_*} H_1(V; \mathbb{Z}_2) \quad \text{im}(\partial) = \ker(i_*)$$

Also we have isomorphisms

$$H_2(V, V - S; \mathbb{Z}_2) \xrightarrow{\text{excision}} H_2(N, \partial N; \mathbb{Z}_2) \xrightarrow{\text{Thom}} H_1(S; \mathbb{Z}_2)$$

where $N$ is a small closed tubular neighborhood of $S$ in $V$. In particular $N$ is an $I$-bundle over $S$, and $\partial N$ is an $\hat{I}$-bundle over $S$ ($\hat{I} = S^0$). From the above isomorphism we see that elements of $\text{im}(\partial)$ are represented by the induced $\hat{I}$-bundles $\hat{\gamma}$ over the curves $\gamma$ of $S$.

Let $E$ be a small closed tubular neighborhood of $\gamma$ in $S$, since $S$ orientable $E \cong \gamma \times I$. Let $E'$ be the induced $I$-bundle over $E$. Let $F^2 = \partial E'$. Clearly $F^2$ is a null homologous surface in $V$ containing $\hat{\gamma}$. Furthermore $\hat{\gamma}$ separates $F^2$. By Proposition 1 $F^2$ can be $\epsilon$-isotoped to a
nonsingular algebraic surface $Z$. After a small isotopy of $\tilde{\gamma}$ we can assume that $\tilde{\gamma} \subset Z$. Since $\tilde{\gamma}$ separates $Z$, by Proposition 1 $\tilde{\gamma}$ is $\varepsilon$-isotopic to a nonsingular algebraic curve $\gamma^*$ in $Z$. By construction $\gamma^* \subset V - S$ and $[\gamma^*]_2 \in AH_1(V - S; \mathbb{Z}_2)$. □

**Lemma 6.** Every element of $AH_1(V; \mathbb{Z}_2)$ can be represented by a connected nonsingular algebraic curve.

**Proof.** Let $\alpha \in AH_1(V; \mathbb{Z}_2)$ then $\alpha$ is represented by a union of nonsingular algebraic curves $C = C_1 \cup \cdots \cup C_k$. By Lemma 3 we can assume that they are disjoint. Let $S$ be the boundary of a closed tubular neighborhood of $C$. Since the normal bundle of $C$ has nowhere zero section, after an $\varepsilon$-isotopy of $S$ we can assume that $C \subset S$. Then by tubing the components of $S$ we get a connected surface $S'$ with $C \subset S'$. Let $C'_i$ be $\varepsilon$-isotopic copies of $C_i$ on $S'$ which are in general position with $C_i$. Connect $C'_i$, $i = 1, \ldots, k$, by tubes in $S'$ to get a connected curve $C' = C'_1 \# \cdots \# C'_k$ such that $C'$ is homologous to $C$ in $S'$.

By construction $[S']_2 = 0$ in $H_2(V; \mathbb{Z}_2)$, so by Proposition 1 we can $\varepsilon$-isotop $S'$ to a nonsingular algebraic surface $Z$ with $C \subset Z$. Continue to denote the isotopic copy of $C'$ in $Z$ by $C'$. Again since $[C']_2 = [C]_2 \in AH_1(Z; \mathbb{Z}_2)$ by Proposition 1, $C'$ is $\varepsilon$-isotopic to a nonsingular algebraic curve $C^*$ in $Z$. $C^*$ is connected and $\alpha = [C]_2 = [C^*]_2 \in AH_1(V; \mathbb{Z}_2)$. □

**Proof of Proposition 2.** We will prove this in three steps,

**Case 1.** $V^3$ is orientable.

Let $c = [C] \in H_1(V; \mathbb{Z})$. Since $[C]_2$ is algebraic there is a nonsingular algebraic curve $A \subset V$ such that $[C] = [A] + 2b$ for some $b \in H_1(V; \mathbb{Z})$. This means if $B \subset V$ is a simple closed curve with $b = [B]$, then $A \cup 2B$ bounds an orientable surface. Here $2B$ denotes the link $B \cup B'$ where $B'$ is a parallel copy of $B$, so $2B$ is a boundary of an orientable surface $B \times I$ in $V$. By Lemma 4 we can assume that $2B$ is a nonsingular algebraic curve. Again by Lemma 4 $C$ is $\varepsilon$-isotopic to a nonsingular
algebraic curve. By the Remark following Lemma 4 we can assume that this isotopy fixes any finite number of points of \( C \).

**Case 2.** \([C]_2 = 0\) in \( H_1(V; \mathbb{Z}_2)\)

Let \( S \subset V \) be a surface representing the dual of the first Steifel-Whitney class \( w_1(V) \) of \( V \). We can assume that \( C \cap S = \emptyset \). This is because by homological reasons \( C \cap S \) must be an even number of points, and we can modify \( S \) as in the picture below without affecting its homology class.

\[
\begin{array}{c}
\text{old } S \\
\uparrow \\
C \\
\downarrow \\
\text{new } S
\end{array}
\]

Hence \( C \subset V - S \), and by assumption \([C]_2 \in \ker(i_*)\) where

\[
i_*: H_2(V - S; \mathbb{Z}_2) \to H_2(V; \mathbb{Z}_2)
\]

is the induced map by inclusion. Since \([S]_2 = w_1(V)\), \( S \) is orientable (exercise), so by Lemma 5 \([C]_2 \in AH_1(V - S; \mathbb{Z}_2)\). Since \( V - S \) is orientable, by Case 1 \( C \) is \( \varepsilon \)-isotopic to a nonsingular algebraic curve in \( V - S \), fixing any finite number of points of \( C \).

**Case 3. The general case.**

We choose a connected nonsingular algebraic curve \( D \) disjoint from \( C \) so that \([C]_2 = [D]_2\). Let \( S \) be the boundary of a closed tubular neighborhood of \( C \cup D \). As in the proof of Lemma 6 after a small isotopy of \( S \) we can assume that \( C \cup D \subset S \), and let \( S' \) be the connected surface obtained by tubing the two components of \( S \). By construction \( C \cup D \subset S' \). Let \( C' \) and \( D' \) be \( \varepsilon \)-isotopic transverse copies of \( C \) and \( D \) in \( S' \). Then by tubing \( C' \) and \( D' \) in \( S' \) we get a curve \( E = C' \# D' \) as in the picture.
By construction we have
\[ (a) \ [S']^2 = 0 \text{ in } H_2(V; \mathbb{Z}_2) \]
\[ (b) \ [E]^2 = [C \cup D]^2 \text{ in } H_1(S'; \mathbb{Z}_2) \]
\[ (c) \ [E]^2 = 0 \text{ in } H_1(V; \mathbb{Z}_2) \]

By Case 2 $E$ is $\varepsilon$-isotopic to a nonsingular algebraic curve $E^*$ in $V$ fixing the points $E \cap (C \cup D)$. After an $\varepsilon$-isotopy of $S'$ we may assume $C \cup D \cup E^* \subset S'$. By Proposition 1 (and by the remark following it) we can $\varepsilon$-isotope $S'$ to a nonsingular algebraic surface $Z$ with $D \cup E^* \subset Z$. Let $C'$ be the corresponding $\varepsilon$-isotopic copy of $C$ in $Z$. Since $[C']_2 = [D \cup E^*]^2 \in AH_1(Z; \mathbb{Z}_2)$ by Proposition 1. $C'$ is $\varepsilon$-isotopic to a nonsingular algebraic curve $C^*$ in $Z$. Furthermore given any finite number of points on $C_1$ by Proposition 1 we can require that all these isotopies fix these points.

Proof of the Theorem. First we show $(b) \Rightarrow (a)$. For every $y \in f(M^2)$ consider $n(y) = \max \{ n \mid \text{there are } n \text{ distinct points } x_1, \ldots, x_n \in M \text{ with } f(x_i) = y \text{ for } i = 1, 2, \ldots, n \}$ = the cardinality of $f^{-1}(y)$. $f(M)$ is a stratified set with strata $\{ L_i \}_{i=1}^3$ where $L_i$ are the $i$-fold point sets, $L_i = \{ y \in f(M) \mid n(y) = i \}$. Call $d(f) = \max \{ i \mid L_i \neq \emptyset \}$, then $d(f) \leq 3$ and if $d(f) = 3$, $L_3$ is a collection of points (the triple points). Let $M_3 = f^{-1}(L_3)$. By ([AKi], Lemma 2.3) there is a unique immersion $f'$ with $d(f') = 2$ making the following commute

\[ M' = B(M, M_3) \overset{f'}{\twoheadrightarrow} B(V, L_3) = V' \]
\[ \downarrow p' \quad \downarrow \pi' \]
\[ M \overset{f}{\twoheadrightarrow} V \]

where the vertical maps are the blowing up maps along the centers $M_3, L_3$. Since the points are algebraic, we can assume that $V' \xrightarrow{\pi'} V$ is the algebraic blow up of $V$ along $L_3$.

Since $d(f') = 2$ the 2-fold point set $L_2 \subset V'$ of the map $f'$ is a smooth manifold (i.e., collection of smooth circles). Let $M_2 = (f')^{-1}L_2$. Once again by [AKi] there is a unique immersion $f''$ with $d(f'') = 1$ (i.e., it is an imbedding) making the following commute

\[ M'' = B(M', M_2) \overset{f''}{\leftrightarrow} B(V', L_2) = V'' \]
\[ \downarrow p'' \quad \downarrow \pi'' \]
\[ M' \overset{f'}{\leftrightarrow} V' \]
where the vertical maps are the blowing up maps. In particular $M'' = M'$ and $p'' = \text{identity}$, since $M_2 \subset M'$ is codimension one.

$V' = V \# \mathbb{R}P^3$ so $H_i(V') = H_i(V) \oplus H_i(\# \mathbb{R}P^3)$ for $i = 1, 2$; in particular $H_1(V'; \mathbb{Z}_2)$ and $H_2(V'; \mathbb{Z}_2)$ are algebraic. By Proposition 2 the curve $L_2$ is $\epsilon$-isotopic to a nonsingular algebraic set. We can change $f(M)$ by a small isotopy in $V$ keeping $L_3$ fixed so that the corresponding double point set $L_2$ in $V'$ is this nonsingular algebraic set. Therefore we can take $\pi''$ to be the algebraic blow up along $L_2$, in particular $V''$ is a nonsingular algebraic set.

We claim that $H_2(V''; \mathbb{Z}_2)$ is algebraic. This can be seen by the homology exact sequences

$$
\cdots \rightarrow H_2(C') \rightarrow H_2(V') \rightarrow H_2(V'', C'') \rightarrow \cdots
$$

where all the homology groups have coefficient $\mathbb{Z}_2$, and $C', C''$ are closed tubular neighborhoods of $L_2$, $(\pi'')^{-1}(L_2)$ respectively. Since $\pi''$ is degree 1 $\pi''$ is onto, and by the above diagram $\ker \pi'' = \text{im}(i_*)$ where $i$ is the inclusion $C'' \hookrightarrow V''$. So $H_2(V''; \mathbb{Z}_2)$ is generated by the nonsingular algebraic sets $(\pi'')^{-1}(L_2)$, and $(\pi'')^{-1}(S_i)$ where $S_i$ are surfaces in $V'$. By Proposition 1 we can assume $S_i$ are nonsingular algebraic surfaces. By ([AK_1] Proposition 4.3) we can assume $S_i$ are transverse to $L_2$. Hence $H_2(V''; \mathbb{Z}_2)$ is generated by nonsingular algebraic sets.

By Proposition 1 we can $\epsilon$-isotop the smooth submanifold $f''(M'')$ to a nonsingular algebraic subset $Q$ of $V''$ by a smooth isotopy. By ([AK_1] Lemma 2.5) $\pi' \circ \pi''(Q)$ is an algebraic set. $\pi' \circ \pi''(Q)$ is isotopic to $f(M)$ by a small isotopy. More precisely, the last remark can be seen by applying ([AK_2] Proposition 5.5). Namely [AK_2] gives an isotopy $h_t: V'' \rightarrow V''$ such that

1. $h_0 = \text{Id},$
2. $h_1(f''(M'')) = Q,$
3. $h^{-1}_t(\pi^{-1}(x)) = \pi^{'}^{-1}(x)$ for all $x \in L \subset V$, where $\pi = \pi' \circ \pi''$, $L = L_3 \cup \pi'(L_2)$.

Then we can define an isotopy

$$
g_t: V \rightarrow V \text{ by } g_t(x) = \pi h_t(y) \text{ for } \begin{cases} y = \pi^{-1}(x), & \text{if } x \not\in L, \\ y \in \pi^{-1}(x), & \text{if } x \in L. \end{cases}$$
(Notice $\pi$ is a diffeomorphism over the complement of $L$.) $g_t$ gives an isotopy of $f(M)$ to $\pi(Q)$ fixing $L$ pointwise. Also $g_t$ is smooth in the complement of $L$.

It remains to show (a) $\Rightarrow$ (b). Clearly (a) implies $H_2(V; \mathbb{Z}_2)$ algebraic. To see $H_1(V; \mathbb{Z}_2)$ algebraic we write every simple closed curve $C \subset V^3$ as the double point of an immersion. $C$ has a normal bundle $C \times D^2 \subset V$. Then $C \times X \subset V$ where $X$ is the figure eight, so $C \times X = f(S^1 \times S^1)$ where $f: S^1 \times S^1 \to V$ is the obvious immersion. Hence by (a) $f(S^1 \times S^1)$ can be made algebraic and $C$ is the singular set of this algebraic set. 

**Note added in proof.** After writing this paper we have been informed by W. Kucharz that he had proved a special case of Proposition 2 when $V$ is orientable in “Topology of Real Algebraic Threefolds” Duke Math. Journal, vol. 53, No. 4, Dec. 1986.

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