JONES POLYNOMIALS OF PERIODIC LINKS

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Let $L$ be a link in $S^3$ which has a prime period and $L_*$ be its factor link. Several relationships between the Jones polynomials of $L$ and $L_*$ are proved. As an application, it is shown that some knot cannot have a certain period.

1. Introduction. Let $L$ be an oriented link that has period $r > 1$. That is, there exists an orientation preserving auto-homeomorphism $\phi: S^3 \to S^3$ of order $r$ with a set of fixed points $F \cong S^1$ disjoint from $L$ and which maps $L$ onto itself. By the positive solution of Smith Conjecture, $F$ is unknotted. Let $\Sigma^3 = S^3/\phi$ be the quotient space under $\phi$. Since $F$ is unknotted, $\Sigma^3$ is again a 3-sphere, and $S^3$ is the $r$-fold cyclic covering space of $\Sigma^3$ branched along $F$.

Let $\psi: S^3 \to \Sigma^3$ be the covering projection. Denote $\psi(L) = L_*$, which is called the factor link, and let $V_L(t)$ and $V_{L_*}(t)$ denote, respectively, the Jones polynomials of $L$ and $L_*$. In this paper, we will prove some relationships between $V_L(t)$ and $V_{L_*}(t)$ which are analogous to those between their Alexander polynomials [M2]. In fact, we will prove

**Theorem 1.** Let $r$ be a prime and $L$ a link that has period $r^q$, $q \geq 1$. Then

\[
V_L(t) \equiv \left[ V_{L_*}(t) \right]^{r^q} \mod (r, \xi_r(t)),
\]

where $\xi_r(t) = \sum_{j=0}^{r-1} (-1)^j - r^{(r-1)/2}$.

If $L$ is not split, then we are able to prove a slightly more precise formula.

Let $\text{lk}(X, Y)$ denote the linking number between two simple closed curves $X$ and $Y$ in $S^3$. Then we have

**Theorem 2.** Let $r$ be a prime and $L$ a non-split link that has period $r^q$, $q \geq 1$. 

319
(1) If \( \text{lk}(L, F) \equiv 1 \pmod{2} \), then
\[
V_L(t) \equiv \left[ V_{L*}(t) \right]^{r^a} \pmod{r, \eta_r(t)},
\]
where \( \eta_r(t) = \left[ \sum_{j=0}^{r-2} (j + 1)(-t)^j \right](1 + t') - t'^{-1}. \)

(2) If \( \text{lk}(L, F) \equiv 0 \pmod{2} \), then
\[
V_L(t) \equiv \left[ V_{L*}(t) \right]^{r^a} \pmod{r, \xi_r(t)}.
\]

Note that \( \eta_r(t) \equiv 0 \pmod{r, \xi_r(t)} \). (See Lemma 6 in §3.) As a simple consequence, we obtain

**Corollary 3.** Let \( b \) be an \( n \)-braid and let \( V_b(t) \) be the Jones polynomial of the closure \( \hat{b} \) of \( b \). Let \( r \) be a prime and \( q \geq 1 \). Then
\[
V_{b^q}(t) \equiv \left[ V_b(t) \right]^{r^a} \pmod{r, \xi_r(t)}.
\]

Formulas (1.1), (1.2), and (1.3) involve slightly larger ideals than those in the corresponding formulas about the Alexander polynomials \([M2]\). However, they are the best possible. To see this, consider an \( n \)-component trivial link \( L \). \( L \) has any period \( r \) and a factor link \( L^* \) is also an \( n \)-component trivial link. Since \( V_L(t) = V_{L*}(t) = (-1)^{n-1}(\sqrt{t} + 1/\sqrt{t})^{n-1} \), the formula \( V_L(t) \equiv \left[ V_{L*}(t) \right]^{r} \pmod{I} \) holds only if the ideal \( I \) contains \( \xi_r(t) \). We should note that while the Alexander polynomial of a link may vanish, the Jones polynomial of a link never vanishes.

Corollary 3 is also verified for \( n = 3 \) by a direct computation using Theorem 21 \([J]\) and Theorem \([M2]\).

These formulas may have more theoretical values than practical values. (See Proposition 7 in §4.) Nevertheless, we can prove that \( 10^{105} \) cannot have period 7 (Proposition 10). This solves one of several undecided cases for knots with 10 crossings.

**2. Proof of Theorem 1.** Since it suffices to prove Theorem 1 for \( q = 1 \), we assume that \( L \) has a prime period \( r \). In this section, we prove that Theorem 2 implies Theorem 1.

Suppose that \( L \) has period \( r \) and let \( \phi \) be an orientation preserving auto-homeomorphism of \( S^3 \) that maps \( L \) onto itself. Suppose that \( L \) splits into \( k \) components \( L_1, L_2, \ldots, L_k \). Then \( \phi \) must map a split component not having period \( r \) onto another split component not having period \( r \). Therefore, split components of \( L \) are divided into \( h + 1 \) sets
\[
A_1 = \{ L_1, \ldots, L_r \}, \quad A_2 = \{ L_{r+1}, \ldots, L_{2r} \}, \ldots, \quad A_h = \{ L_{(h-1)r+1}, \ldots, L_{hr} \}
\]
and
\[
B = \{ L_{hr+1}, \ldots, L_k \}
\]
such that any two links in \( A_i \) \( (i = 1, 2, \ldots, h) \) are ambient isotopic and a link in \( B \) has period \( r \). The factor link \( L_* \), then, has \( h + (k - hr) \) \( (= k - h(r - 1)) \) split components. Noting that the factor link of the \( r \)-split component link \( L_{sr+1} \cup \cdots \cup L_{(s+1)r} \) is
Now unshading the domain containing 0, we have the graph $\Gamma^*$ of $L^*$. We may take 0 as one vertex of $\Gamma^*$. Furthermore, we
can assign $+1$ or $-1$ to each edge of $\Gamma_*$ [M4]. Similarly, we have the graph $\Gamma$ of $\tilde{L}$ by unshading the domain containing 0. $\Gamma$ is also an oriented graph. Using $\Gamma$ and $\Gamma_*$, we can evaluate $V_L(t)$ and $V_{L*}(t)$ as follows. (See [M4].)

Let $p'$ and $n'$ be, respectively, the number of positive and negative edges in $\Gamma$. Let $S(a, b)$, $0 \leq a \leq p'$ and $0 \leq b \leq n'$, be the collection of subgraphs obtained from $\Gamma$ by removing exactly $a$ positive edges and $b$ negative edges. $S(a, b)$ contains $(p'_a)(n'_b)$ subgraphs.

For $\gamma \in S(a, b)$, let $\mu(\gamma) = b_0(\gamma) + b_1(\gamma)$, where $b_i(\gamma)$, $i = 0, 1$, denotes the $i$th Betti number of $\gamma$ as a 1-complex. Then the bracket polynomial $P_L(A)$ defined in [K] associated with the link diagram $\tilde{L}$ is given by the following formula:

$$P_L(A) = \sum_{0 \leq a \leq p'} A^{p'-2a-n'+2b} \sum_{\gamma \in S(a, b)} \left[ -(A^2 + A^{-2}) \right]^{\mu(\gamma) - 1}. $$

Let $p$ and $n$ be, respectively, the number of positive and negative edges in $\Gamma_*$. Then $\Gamma$ has exactly $rp$ positive and $rn$ negative edges, i.e. $p' = rp$ and $n' = rn$. Let $S_*(a, b)$ be the collection of subgraphs of $\Gamma_*$ which is defined in a similar way to $S(a, b)$. Then we have

$$P_{L*}(A) = \sum_{0 \leq a \leq p} A^{p-2a-n+2b} \sum_{\gamma \in S_*(a, b)} \left[ -(A^2 + A^{-2}) \right]^{\mu(\gamma_*) - 1}. $$

Since the rotation $\xi: \mathbb{R}^2 \to \mathbb{R}^2$ maps $\tilde{L}$ onto itself, we may assume that $\xi$ maps the graph $\Gamma$ onto itself, preserving the sign of each edge. In other words, $\xi$ defines an automorphism of the oriented graph $\Gamma$.

If $\text{lk}(L, F) \equiv 1 \pmod{2}$, then the unbounded domain is shaded. Therefore, $\xi$ fixes only the origin 0. If $\text{lk}(L, F) \equiv 0 \pmod{2}$, however, the unbounded domain is unshaded, and hence, $\xi$ keeps exactly two vertices 0 and $\infty$ fixed, where $\infty$ is a point associated with the unbounded domain. Therefore, if $\text{lk}(L, F) \equiv 0 \pmod{2}$, $\xi$ may be considered as an automorphism of the graph $\Gamma$ in $S^2$ which keeps the north and south poles fixed.

**Case A.$,$ $\text{lk}(L, F) \equiv 1 \pmod{2}$.**

In this case, $\Gamma$ is the $r$-fold cyclic covering of $\Gamma$ branched at 0. Take $\gamma \in S(a, b)$.

**Case 1.$,$ $\gamma$ is not fixed under $\xi$, i.e. $\xi(\gamma) \neq \gamma$.**

This is, of course, the case when $a \neq 0 \pmod{r}$ or $b \neq 0 \pmod{r}$. In this case, $\gamma, \xi(\gamma), \xi^2(\gamma), \ldots, \xi^{r-1}(\gamma)$ are all distinct, but, since any two of
these are isomorphic, we have exactly $r$ identical terms in $P_L(A)$, and they vanish by reducing modulo $r$.

**Case 2.** $\gamma$ is fixed under $\xi$ setwise, i.e. $\xi(\gamma) = \gamma$.

In this case, $a \equiv b \equiv 0 \pmod{r}$. Write $a = ra'$ and $b = rb'$. Then $\gamma$ defines a unique quotient subgraph $\gamma^* = (\gamma / \xi) \in S_*(a', b')$.

Let $\alpha$ and $\alpha^*$, be, respectively, the terms in $P_L(A)$ and $P_{L^*}(A)$ which are associated with $\gamma$ and $\gamma^*$. Since $p' = rp$ and $n' = rn$, we have

\[(3.3) \quad (1) \quad \alpha = A^{r(p^{-2a'−n+2b'})}[−(A^2 + A^{-2})]^{\mu(\gamma)−1}, \quad \text{and} \]
\[(2) \quad \alpha^* = A^{p^{-2a'−n+2b'}}[−(A^2 + A^{-2})]^{\mu(\gamma^*)−1}.
\]

We will compare $\mu(\gamma)−1$ and $\mu(\gamma^*)−1$.

If we use the fact that $\gamma$ is the $r$-fold cyclic cover of $\gamma^*$, it is not difficult to find some relationship between $b_1(\gamma)$ and $b_1(\gamma^*)$.

Consider connected components of $\gamma$. Let $D_0, D_1, \ldots, D_k, \ D_{1,1}, \ldots, D_{1,r}, D_{2,1}, \ldots, D_{2,r}, \ldots, D_{m,1}, \ldots, D_{m,r}$ be connected components of $\gamma$ such that

\[(3.4) \quad (1) \quad D_0 \text{ contains the origin } \{0\}, \text{ and } \xi(D_0) = D_0, \]
\[(2) \quad D_i \ (i = 1, 2, \ldots, k) \text{ is a component } (\notin \{0\}) \text{ of } \gamma \text{ such that } \xi(D_i) = D_i, \]
\[(3) \quad \{D_{j,1}, \ldots, D_{j,r}\}, \ (j = 1, 2, \ldots, m) \text{ is a set of components of } \gamma \text{ which permutes by } \xi\]

Then connected components of $\gamma^*$ consist of the sets: $D'_i = D_i / \xi \ (i = 0, 1, 2, \ldots, k)$ and $D'_{j,1} = D_{j,1} \ (j = 1, 2, \ldots, m)$.

We compare $b_1(D_i)$ and $b_1(D_{j,\lambda})$ with $b_1(D'_i)$ and $b_1(D'_{j,\lambda})$.

**Lemma 5.**

\[(3.5) \quad (1) \quad b_1(D_0) = rb_1(D'_0), \]
\[(2) \quad b_1(D_i) - 1 = r\{b_1(D'_i) - 1\} \text{ for } 1 \leq i \leq k, \]
\[(3) \quad b_1(D_{j,\lambda}) = b_1(D_{j,\lambda}) = b_1(D'_{j,\lambda}) \text{ for } 1 \leq j \leq m \text{ and } 1 \leq \lambda \leq r.\]

**Proof.** (1) Let $d'_0$ and $e'_0$, denote, respectively, the number of vertices and edges of $D'_0$. Then, since $D_0$ is the $r$-fold cyclic covering of $D'_0$ branched at 0, the number of vertices and edges of $D_0$ are given by
\(r(d'_0 - 1) + 1\) and \(re'_0\) respectively. Therefore

\[
1 - b_1(D_0) = r(d'_0 - 1) + 1 - re'_0 = r(d'_0 - e'_0) - r + 1
\]

\[
= r(1 - b_1(D'_0)) - r + 1 = 1 - rb_1(D'_0),
\]

and hence, \(b_1(D_0) = rb_1(D'_0)\).

(2) Since \(D_i\) is the \(r\)-fold (unbranched) cyclic covering of \(D'_0\), it follows that \(\chi(D_i) = r\chi(D'_i)\), where \(\chi\) denotes the Euler characteristic. Since \(\chi(D_i) = 1 - b_1(D_i)\), we have

\[
1 - b_1(D_i) = r\chi(D'_i) = r\{1 - b_1(D'_i)\}
\]

and hence, \(b_1(D_i) - 1 = r\{b_1(D'_i) - 1\}\).

(3) is obvious.

Now we compare \(\mu(\gamma) - 1\) and \(\mu(\gamma*) - 1\). Using Lemma 5, we obtain

\[
\mu(\gamma) - 1 = b_1(\gamma) + b_0(\gamma) - 1
\]

\[
= b_1(D_0) + \sum_{i=1}^{k} b_1(D_i) + \sum_{j=1}^{m} \sum_{\lambda=1}^{r} b_1(D_{j,\lambda}) + k + 1 + rm - 1
\]

\[
= rb_1(D'_0) + \sum_{i=1}^{k} \{rb_1(D'_i) - r + 1\} + \sum_{j=1}^{m} rb_1(D'_{j,1}) + k + rm
\]

\[
= r\left[b_1(D'_0) + \sum_{i=1}^{k} b_1(D'_i) + \sum_{i=1}^{m} b_1(D'_{j,1}) + k + 1 + m - 1\right]
\]

\[
- rk - rm - rk + k + k + rm
\]

\[
= r[b_1(\gamma*) + b_0(\gamma*) - 1] - 2k(r - 1)
\]

\[
= r[\mu(\gamma*) - 1] - 2k(r - 1).
\]

Using this equality, we have

(3.6) \(\alpha \equiv \alpha^*_r \mod \left\{ (A^2 + A^{-2}) \right\}^{2(r-1)} - 1 \).

In fact, a simple computation shows that

\[
\alpha = A^{r(p - 2a' - n + 2b')}\left[-(A^2 + A^{-2})\right]^{\mu(\gamma) - 1}
\]

\[
= A^{r(p - 2a' - n + 2b')}\left[-(A^2 + A^{-2})\right]^{(\mu(\gamma*) - 1) - 2k(r-1)}
\]

\[
= \left\{ A^{p - 2a' - n + 2b'}\left[-(A^2 + A^{-2})\right]^{\mu(\gamma*) - 1}\right\}^{\mu(\gamma*) - 1} \left[-(A^2 + A^{-2})\right]^{-2k(r-1)}
\]

\[
= \alpha^*_r \left[-(A^2 + A^{-2})\right]^{-2k(r-1)}
\]

\[
\equiv \alpha^*_r \mod \left\{ (A^2 + A^{-2}) \right\}^{2(r-1)} - 1 \).
\]

Case B. \(\text{lk}(L, F) \equiv 0 \mod 2\).
We consider connected components of $\gamma \in S(a, b)$. Let $D_0, D_1, \ldots, D_k, D_\infty, D_{1,1}, \ldots, D_{1,r}, D_{2,1}, \ldots, D_{2,r}, \ldots, D_{m,1}, \ldots, D_{m,r}$ be connected components of $\gamma$ which satisfy (3.4) (2) and (3). Furthermore, $D_0$ and $D_\infty$ are such that

\[(3.7) \quad D_0 \text{ contains } \{0\} \text{ and } D_\infty \text{ contains } \{\infty\}, \text{ and } \xi(D_0) = D_0 \text{ and } \xi(D_\infty) = D_\infty.\]

It may occur that $D_0 = D_\infty$. We should note that $\gamma$ is the $r$-fold cyclic covering of $\gamma_*$ branched at 0 and $\infty$.

Now (3.5) (2) and (3) are still valid under the present case. Only (3.5) (1) should be changed to the following.

\[(3.8) \quad \begin{align*}
(i) & \quad \text{If } D_0 \neq D_\infty, \text{ then } b_1(D_0) = rb_1(D_0') \text{ and } b_1(D_\infty) = rb_1(D_\infty'). \\
(ii) & \quad \text{If } D_0 = D_\infty, \text{ then } b_1(D_0) + 1 = r\{b_1(D_0') + 1\}.
\end{align*}\]

**Proof.** (i) follows from the fact that $D_0$ and $D_\infty$ are, respectively, the $r$-fold cyclic coverings of $D_0'$ and $D_\infty'$ branched at 0 and $\infty$.

(ii) $D_0(= D_\infty)$ is the $r$-fold cyclic covering of $D_0'$ branched at 0 and $\infty$. Let $d'$ and $e'$ denote the number of vertices and edges of $D_0'$. Then

\[
1 - b_1(D_0) = 2 + r(d' - 2) - re' = r(d' - e') - 2r + 2
\]

which yields $b_1(D_0) + 1 = r\{b_1(D_0') + 1\}$.

Using (3.8) (i) and (3.5) (1), (2), we obtain the following formulas.

(i) When $D_0 \neq D_\infty$,

\[
\mu(\gamma) - 1 = b_1(\gamma) + b_0(\gamma) - 1
\]

\[
= b_1(D_0) + \sum_{i=1}^{k} b_1(D_i) + b_1(D_\infty) + \sum_{j=1}^{m} \sum_{\lambda=1}^{r} b_1(D_{j,\lambda}) + k + 2 + rm - 1
\]

\[
= rb_1(D_0') + \sum_{i=1}^{k} \{rb_1(D_i') - r + 1\} + rb_1(D_\infty') + \sum_{j=1}^{m} rb_1(D_{j,\lambda}') + k + 1 + rm
\]

(continues)
Therefore, we have

\[ \mu(\gamma) - 1 = b_1(D_0) + \sum_{i=1}^{k} b_i(D_i) + \sum_{j=1}^{m} b_1(D_{j,1}) + k + 1 + rm - 1 \]

\[ = rb_1(D_0') + r - 1 + \sum_{i=1}^{k} rb_1(D_i') + \sum_{j=1}^{m} rb_1(D_{j,1}') + k + rm \]

\[ = r\left( b_1(D_0) + \sum_{i=1}^{k} b_i(D_i) + \sum_{j=1}^{m} b_1(D_{j,1}) + k + 1 + m - 1 \right) \]

\[ - rk - rm + r - 1 + k + rm \]

\[ = r[\mu(\gamma*) - 1] - (k - 1)(r - 1). \]

Therefore, we have

\[(3.9) \quad \alpha = \alpha_* \mod([-(A^2 + A^{-2})]^{r-1} - 1). \]

Now it only remains to show the following simple lemma.

**Lemma 6.** For any prime \( r \),

(1) \((t + 1)^{2(r-1)} - t^{r-1} \equiv \eta_r(t) \mod r.\)

(2) \((t + 1)^{r-1} - t^{r-1/2} \equiv \xi_r(t) \mod r.\)

**Proof.** If \( r = 2 \), the lemma is obvious. Therefore, we assume that \( r \) is an odd prime. Then it suffices to prove the following.

(3.10) For \( j = 0, 1, \ldots, r - 1, \)

1. \( \begin{pmatrix} 2r - 2 \\ j \end{pmatrix} \equiv (-1)^j (j + 1) \mod r, \)

2. \( \begin{pmatrix} 2r - 2 \\ r + j \end{pmatrix} \equiv (-1)^j (j + 1) \mod r, \)

3. \( \begin{pmatrix} r - 1 \\ j \end{pmatrix} \equiv (-1)^j \mod r. \)
Proof. Firstly, (3) is obviously true for \( j = 0 \) and 1. Since \( \left( \frac{r-j}{r} \right) = \left( \frac{r-j}{r} \right) - 1 + \left( \frac{r-j}{r} \right) - 1 \), it follows by the induction hypothesis that \( 0 \equiv \left( \frac{r-j}{r} \right) + (-1)^{j-1} \) \((\text{mod } r)\) which yields \( \left( \frac{r-j}{r} \right) \equiv (-1)^j \) \((\text{mod } r)\). This proves (3). Secondly, (1) is trivially true for \( j = 0 \) and 1. Now for \( 1 \leq j \leq r - 1 \),

\[
\binom{2r-2}{j} = \binom{2r-2}{j-1} \frac{2r-j-1}{j}.
\]

Using the induction hypothesis, we can write

\[
\binom{2r-2}{j-1} = (-1)^{j-1} j + rk
\]

for some integer \( k \). Then

\[
\binom{2r-2}{j} = (-1)^j (j + 1) + (-1)^{j-1} r + \frac{rk}{j} (2r - j - 1).
\]

Since \( \binom{2r-2}{j} \) is an integer and \( r \) is a prime, \( j \mid k(2r - j - 1) \) and hence

\[
\binom{2r-2}{j} \equiv (-1)^j (j + 1) \pmod{r}.
\]

This proves (1). Finally, since \( r \) is odd and \( r - j - 2 \leq r - 1 \) for \( 0 \leq j \leq r - 1 \), (3.10) (1) implies that

\[
\binom{2r-2}{r+j} = \binom{2r-2}{r-j-2} \equiv (-1)^{r-j+2} (r - j - 2 + 1)
\]

\[
\equiv (-1)^{r-j+1} (j + 1) \equiv (-1)^j (j + 1) \pmod{r}.
\]

This proves (2).

Let \( I \) be the ideal in \( \mathbb{Z}[A, A^{-1}] \) generated by \( r \) and

\[
\left( - (A^2 + A^{-2}) \right)^{(r-1)} - 1 \pmod{r}
\]

when \( \text{lk}(L, F) \equiv 1 \pmod{2} \) (or \( \text{lk}(L, F) \equiv 0 \pmod{2} \)). The Lemma 6 yields that \( P_L(A) \equiv [P_{L*}(A)]^r \pmod{I} \). Let \( w(\tilde{L}) \) be the twisting number (or the writhe) of \( \tilde{L} \). Then, since \( w(\tilde{L}) = rw(\tilde{L}*) \), it follows that

\[
f_L(A) = (-A)^{-3w(\tilde{L})} P_L(A) = (-A)^{-3w(\tilde{L}*)} P_{L*}(A) \equiv \left( (-A)^{-3w(\tilde{L}*)} P_{L*}(A) \right)^r
\]

\[
= \left[ f_{L*}(A) \right]^r \pmod{I}.
\]

Here \( f_L(t^{-1/4}) = V_L(t) \) \([K]\) and Theorem 1 follows from Lemma 6. A proof of Theorem 2 is now complete.
4. Applications and remarks. Formula (1.1) may not be used to determine whether a knot (but not a link) $K$ has small prime period $r \leq 5$. In fact, we have the following

**Proposition 7.** Let $K$ be a knot. Then for $r = 2, 3$ or 5,

\[(4.1) \quad V_K(t) \equiv 1 \mod(r, \xi_r(t)).\]

**Proof.** First, note that $\xi_2(t) = 1 - t - \sqrt{t}$, $\xi_3(t) = 1 - 2t + t^2$ and $\xi_5(t) = 1 - t - t^2 + t^4$. Now, as is well known (Definition 17 [J]), $1 - V_K(t) \equiv 0 \mod \xi_5(t)$, and hence $V_K(t) \equiv 1 \mod \xi_5(t)$. Furthermore, congruences $1 - t + t^2 \equiv (1 - t - \sqrt{t})(1 - t + \sqrt{t}) \mod 2$, $(1 - t)(1 - t^3) \equiv (1 - t + t^2)(1 + t^2) \mod 2$ and $(1 - t)(1 - t^3) \equiv (1 - t^2 + t^3)$. $(1 + t + t^2) \mod 3$ prove Proposition 7.

It is also easy to show that for any prime $r \geq 5$, $\xi_5(t) | \xi_r(t)$.

**Proposition 8.** Let $r$ be an odd prime $\geq 5$. Let $\omega$ and $\tau$ denote, respectively, a primitive $(r - 1)/2$th-root and $(r + 1)/2$th-root of unity. If a link $L$ has period $r$, then

\[(4.2) \quad (1) \quad V_L(\omega) \equiv V_L(\omega) \mod r \]
\[ (2) \quad V_L(\tau) \equiv V_L(\tau^{-1}) \mod r.\]

**Proof.** From Theorem 1, we see that $V_L(t) \equiv V_L(t^r) \equiv V_L(t^r) \mod(r, \xi_r(t))$. Note that

\[ \xi_r(t) = \frac{1 + t^r}{1 + t} - t^{(r-1)/2} = \frac{1}{1 + t} (1 - t^{(r-1)/2})(1 - t^{(r+1)/2}).\]

Since

$\omega^r = (\omega^{(r-1)/2})^2 \omega = \omega$ and $\tau^r = (\tau^{(r+1)/2})^2 \tau^{-1} = \tau^{-1}$,

a substitution $\omega$ or $\tau$ for $t$ in $V_L(t)$ and $V_L(t^r)$ proves (4.2).

**Corollary 9.** Under the conditions of Proposition 8, if $L_*$ is unknotted, then

\[(4.3) \quad V_L(\omega) \equiv V_L(\tau) \equiv 1 \mod r.\]

Using Corollary 9, we can prove the following

**Proposition 10.** The knot $10_{105}$ in $[\mathcal{R}]$ has no period.
Proof. According to [B-Z, p. 312], 7 is the only possible period of $10_{105}$. Suppose that $K$ has period 7. Since $K$ is alternating and fibred [M1], the factor knot $K_\ast$ is either unknotted or fibred [M3]. Since $\Delta_K(t) = 1 - 8t + 22t^2 - 29t^3 + 22t^4 - 8t^5 + t^6 \equiv (1 + t)^6 \pmod{7}$, it follows from [M2] that $K_\ast$ must be unknotted. Therefore, by Corollary 9, $V_K(\omega) \equiv 1$ and $V_K(\tau) \equiv 1 \pmod{7}$, where $\omega = e^{2\pi i/3}$ and $\tau = e^{2\pi i/4} = \sqrt{-1}$. Since $V_K(t) = t^{-7} - 4t^{-6} + 8t^{-5} - 12t^{-4} + 15t^{-3} - 15t^{-2} + 14t^{-1} - 11 + 7t - 3t^2 + t^3$, we have $V_K(\sqrt{-1}) \equiv -1 \pmod{7}$. Therefore, $K$ cannot have period 7.

Remark. A similar argument reveals that if $K = 10_{101}$ in [R] has period 7, then the factor knot cannot be unknotted.

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Selman Akbulut and Henry Churchill King, Polynomial equations of immersed surfaces ................................................................. 209
Wayne C. Bell and John William Hagood, Separation properties and exact Radon-Nikodým derivatives for bounded finitely additive measures ..... 237
Dennis J. Garity, James P. Henderson and David G. Wright, Menger spaces and inverse limits ........................................................... 249
B. Brent Gordon, Algebraically defined subspaces in the cohomology of a Kuga fiber variety .............................................................. 261
Jeffrey A. Hogan, Weighted norm inequalities for the Fourier transform on connected locally compact groups ........................................... 277
Guojun Liao, A study of regularity problem of harmonic maps ............... 291
Chin-pi Lu, Modules satisfying ACC on a certain type of colons ................ 303
Kunio Murasugi, Jones polynomials of periodic links ............................. 319
Hans Schoutens, Approximation properties for some non-Noetherian local rings ........................................................................ 331
Peter Sjögren, Convergence for the square root of the Poisson kernel ...... 361
Alexandru Ion Suciu, The oriented homotopy type of spun 3-manifolds .... 393