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The functor Tor is related to some classes of C_λ groups, notably the IT groups, and when $\lambda = \Omega$ the C_Ω groups of balanced projective dimensions 1. Separate necessary and sufficient conditions are given for $\mathrm{Tor}(G, H)$ to be a d.s.c. group when G and H are C_λ groups. Some generalizations of the fact that balanced subgroups of G and H determine balanced subgroups of $\mathrm{Tor}(G, H)$ are presented.

Introduction. In this paper, by the term "group" we will mean an abelian p-group for some fixed prime p.

Ever since the class of totally projective groups was introduced in **[11]**, they have had an intimate relationship with the Tor functor. In **[12]** it was asked when $\mathrm{Tor}(G, H)$ is a d.s.c. group. The problem was solved in this work when G and H have different lengths. The situation has proven to be considerably more complicated when G and H have equal length. In **[6]** it was shown that the only totally projective groups of the form $\mathrm{Tor}(G, H)$ for G and H reduced are, in fact, d.s.c. groups. Furthermore, certain sufficient conditions were given for $\mathrm{Tor}(G, H)$ to be a d.s.c. group. In **[5]** separate necessary and sufficient conditions were given for $\mathrm{Tor}(G, H)$ to be a direct sum of cyclic groups.

It is our purpose to generalize these results in two ways. First, we relate the Tor functor with some more general classes of groups, such as the C_λ groups of **[9]** and the IT groups of [7]. Using these more general classes we get somewhat cleaner results. Secondly, we apply the techniques of **[5]** to the situation where the groups involved have lengths which are limit ordinals $\lambda \leq \Omega$.

Briefly summarizing the content of this work, the first section is primarily concerned with constructing some exact sequences involving the Tor functor. These generalize the well-known fact that balanced subgroups of G and H generate balanced subgroups of $\mathrm{Tor}(G, H)$. We also provide a formula for generating λ-high subgroups of $\mathrm{Tor}(G, H)$ in terms of λ-high subgroups of G and H.

In the second section we concentrate mainly on C_λ groups. Our main result shows that this class is preserved by the Tor functor. We also consider the class of C_Ω groups which are unions of Ω-high towers (essentially introduced in [6]) and prove that the isotype subgroups of d.s.c. groups are summands of groups which belong to this class.

In the third section we use some standard "back-and-forth" techniques to give some necessary conditions and other sufficient conditions for $\mathrm{Tor}(G, H)$ to be a d.s.c. group. In particular, it is shown that if G and H are reduced C_Ω groups of length Ω and balanced projective dimension 1, then $\mathrm{Tor}(G, H)$ is a d.s.c. group.

1. We begin this section by reviewing some standard facts on the functor Tor. If K and L are subgroups of X and Y respectively then there is a natural embedding of $\mathrm{Tor}(K, L)$ in $\mathrm{Tor}(X, Y)$. We will therefore view $\mathrm{Tor}(K, L)$ as a subgroup of $\mathrm{Tor}(X, Y)$. In general, any map which is not specifically defined is induced by some obvious inclusion or projection.

The following are results of Nunke [12].

LEMMA 1. *Let X and Y be groups.*

(a) *If K_1, K_2 are subgroups of X and L_1, L_2 are subgroups of Y, then*

$$\mathrm{Tor}(K_1, L_1) \cap \mathrm{Tor}(K_2, L_2) = \mathrm{Tor}(K_1 \cap K_2, L_1 \cap L_2).$$

(b) *If α is an ordinal then* $\mathrm{Tor}(X, Y)(\alpha) = \mathrm{Tor}(X(\alpha), Y(\alpha))$.

(c) *If $0 \to K \to X \to X/K \to 0$ is a pure short exact sequence then the sequence*

$$0 \to \mathrm{Tor}(K, Y) \to \mathrm{Tor}(X, Y) \to \mathrm{Tor}(X/K, Y) \to 0$$

is also pure exact.

The following also appears in [13].

COROLLARY 1. *If K and L are isotype subgroups of X and Y respectively then* $\mathrm{Tor}(K, L)$ *is an isotype subgroup of* $\mathrm{Tor}(X, Y)$.

Proof. Follows from (a) and (b) above.

If G is a group we denote the length of G by $l(G)$, and if α is an ordinal, we let $G(\alpha) = \{g \in G : ht(g) \geq \alpha\}$.

If $0 \to A \xrightarrow{\phi} B \to C \to 0$ is a short exact sequence and the image of ϕ has some property as a subgroup of B we shall also say the short exact sequence has this property.

If λ is an ordinal a subgroup A of B is called *λ-isotype* if for every $\alpha \leq \lambda$ we have $A \cap B(\alpha) = A(\alpha)$. Observe that if λ is a limit ordinal we get this condition for free at $\alpha = \lambda$. We say A is *λ-nice* if for every $\alpha < \lambda$ we have $(B/A)(\alpha) = \{B(\alpha) + A\}/A$, and *$\lambda$-balanced* if it is both λ-isotype and λ-nice.

Recall that a subgroup K of a group X is said to be *α-pure* if the corresponding short exact sequence is in $\mathrm{Ext}(X/K, K)(\alpha)$. In particular, an α-high subgroup of X is $\alpha+1$-pure. If λ is a limit ordinal, then a λ-pure subgroup is λ-balanced. On several occasions in this paper we could use either notion equally well. The use of λ-balanced subgroups emphasizes the combinatorial nature of the arguments, so we prefer this approach.

The following will be useful in deriving properties of some exact sequences:

LEMMA 2. *Suppose λ is a limit ordinal and we are given a long exact sequence*

$$0 \to W \to X \xrightarrow{\phi} Y \to Z \to 0$$

and for every ordinal $\alpha < \lambda$ this induces an exact sequence

$$0 \to W(\alpha) \to X(\alpha) \to Y(\alpha) \to Z(\alpha).$$

If we denote the image of ϕ by M, then the sequence

$$0 \to W \to X \to M \to 0$$

is λ-balanced, and the sequence

$$0 \to M \to Y \to Z \to 0$$

is λ-isotype.

Proof. If $\alpha < \lambda$ is an ordinal, it is clear that

$$\phi(X(\alpha)) \subseteq M(\alpha) \subseteq M \cap Y(\alpha)$$

and by diagram chasing in

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ X(\alpha) & \longrightarrow & Y(\alpha) & \longrightarrow & Z(\alpha) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\phi} & Y & \longrightarrow & Z \end{array}$$

we can conclude that $M \cap Y(\alpha) \subseteq \phi(X(\alpha))$ and so we have,

$$\phi(X(\alpha)) = M(\alpha) = M \cap Y(\alpha).$$

The second equality immediately implies that M is a λ-isotype subgroup of Y, and the first equality implies that for every ordinal $\alpha < \lambda$, the sequence,

$$0 \to W(\alpha) \to X(\alpha) \to M(\alpha) \to 0$$

is exact, which means the first sequence is λ-balanced.

In order to use the above result we prove the following:

LEMMA 3. *Suppose K and L are pure subgroups of X and Y respectively. Then there are exact sequences*

$$0 \to \mathrm{Tor}(K, L) \to \mathrm{Tor}(X, Y) \xrightarrow{\sigma} \mathrm{Tor}(X/K, Y) \oplus \mathrm{Tor}(X, Y/L) \to \mathrm{Tor}(X/K, Y/L) \to 0$$

and

$$0 \to \mathrm{Tor}(K, L) \to \mathrm{Tor}(X, L) \oplus \mathrm{Tor}(K, Y) \xrightarrow{\tau} \mathrm{Tor}(X, Y) \to \mathrm{Tor}(X/K, Y/L) \to 0.$$

Proof. These are standard exact sequences resulting from diagram chasing in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}(K, L) \arrow[r] \arrow[d] & \mathrm{Tor}(K, Y) \arrow[r] \arrow[d] & \mathrm{Tor}(K, Y/L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tor}(X, L) \arrow[r] \arrow[d] & \mathrm{Tor}(X, Y) \arrow[r] \arrow[d] & \mathrm{Tor}(X, Y/L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tor}(X/K, L) \arrow[r] \arrow[d] & \mathrm{Tor}(X/K, Y) \arrow[r] \arrow[d] & \mathrm{Tor}(X/K, Y/L) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the right exactness of the rows and columns follows from Lemma 1(c).

It should be noted that the maps in Lemma 3 are induced either by the corresponding inclusions and projections or are the diagonal or codiagonal maps associated with them (with a possible change in the sign of one factor to make the sequence exact).

THEOREM 1. *Suppose λ is a limit ordinal and K and L are λ-balanced subgroups of X and Y. Denote the image of σ in Lemma 3 by P. Then*

$$0 \to \operatorname{Tor}(K, L) \to \operatorname{Tor}(X, Y) \to P \to 0 \tag{1}$$

and

$$0 \to P \to \operatorname{Tor}(X/K, Y) \oplus \operatorname{Tor}(X, Y/L) \to \operatorname{Tor}(X/K, Y/L) \to 0 \tag{2}$$

are λ-balanced.

Proof. If $\alpha < \lambda$ is an ordinal then $K(\alpha)$ and $L(\alpha)$ are pure subgroups of $X(\alpha)$ and $Y(\alpha)$, so there is a four term exact sequence

$$\begin{aligned} 0 &\to \operatorname{Tor}(K(\alpha), L(\alpha)) \to \operatorname{Tor}(X(\alpha), Y(\alpha)) \\ &\to \operatorname{Tor}(\{X(\alpha)+K\}/K, Y(\alpha)) \oplus \operatorname{Tor}(X(\alpha), \{Y(\alpha)+L\}/L) \\ &\to \operatorname{Tor}(\{X(\alpha)+K\}/K, \{Y(\alpha)+L\}/L) \to 0. \end{aligned}$$

Since for all $\alpha < \lambda$ we have

$$\{X(\alpha)+K\}/K = (X/K)(\alpha) \quad \text{and} \quad \{Y(\alpha)+L\}/L = (Y/L)(\alpha),$$

we have,

$$\begin{aligned} &\operatorname{Tor}(\{X(\alpha)+K\}/K, Y(\alpha)) = \operatorname{Tor}(X/K, Y)(\alpha) \\ &\operatorname{Tor}(X(\alpha), \{Y(\alpha)+L\}/L) = \operatorname{Tor}(X, Y/L)(\alpha) \\ &\operatorname{Tor}(\{X(\alpha)+K\}/K, \{Y(\alpha)+L\}/L) = \operatorname{Tor}(X/K, Y/L)(\alpha). \end{aligned}$$

So for every $\alpha < \lambda$ there is an exact sequence

$$\begin{aligned} 0 \to \operatorname{Tor}(K, L)(\alpha) \to \operatorname{Tor}(X, Y)(\alpha) \to \operatorname{Tor}(X/K, Y)(\alpha) \oplus \operatorname{Tor}(X, Y/L)(\alpha) \\ \to \operatorname{Tor}(X/K, Y/L)(\alpha) \to 0. \end{aligned}$$

So by Lemma 2 the sequence (1) is λ-balanced and sequence (2) is λ-isotype. Since this last map is also surjective for all $\alpha < \lambda$, the sequence (2) is also λ-balanced.

COROLLARY 2. *Suppose K and L are balanced subgroups of X and Y. Then the sequences in Theorem 1 are both balanced.*

Proof. Clearly K and L are λ-balanced for every limit ordinal λ, hence so are the sequences in Theorem 1.

We say a subgroup K of X is *λ-dense* if for all $\alpha < \lambda$ we have $X = X(\alpha) + K$. Clearly a λ-dense subgroup is λ-nice.

COROLLARY 3. *With the same hypotheses as Theorem* 1, *if* K *and* L *are* λ*-dense then sequence* (2) *is* λ*-dense.*

Proof. In the proof of Theorem 1, if for every $\alpha < \lambda$ we have $\{X(\alpha)+K\}/K = X/Y$ and $\{Y(\alpha)+L\}/L = Y/L$ then the map

$$\mathrm{Tor}(X/K, Y)(\alpha) \oplus \mathrm{Tor}(X, Y/L)(\alpha) \to \mathrm{Tor}(X/K, Y/L)$$

is surjective. This gives that P is λ-dense.

THEOREM 2. *Suppose* λ *is a limit ordinal and* K *and* L *are* λ*-isotype subgroups of* X *and* Y. *Denote the image of* τ *in Lemma* 3 *by* Q (*so* $Q = \mathrm{Tor}(X, L) + \mathrm{Tor}(K, Y)$). *Then,*

$$0 \to \mathrm{Tor}(K, L) \to \mathrm{Tor}(X, L) \oplus \mathrm{Tor}(K, Y) \to Q \to 0 \tag{1}$$

is λ*-balanced and*

$$0 \to Q \to \mathrm{Tor}(X, Y) \to \mathrm{Tor}(X/K, Y/L) \to 0 \tag{2}$$

is λ*-isotype.*

Proof. For every $\alpha < \lambda$, $K(\alpha)$ and $L(\alpha)$ are pure subgroups of $X(\alpha)$ and $Y(\alpha)$. So there is a long exact sequence,

$$\begin{aligned}&0 \to \mathrm{Tor}(K(\alpha), L(\alpha)) \to \mathrm{Tor}(X(\alpha), L(\alpha)) \oplus \mathrm{Tor}(K(\alpha), Y(\alpha))\\ &\quad \to \mathrm{Tor}(X(\alpha), Y(\alpha)) \to \mathrm{Tor}(\{X(\alpha)+K\}/K, \{Y(\alpha)+L\}/L) \to 0.\end{aligned}$$

So the following is exact,

$$\begin{aligned}&0 \to \mathrm{Tor}(K, L)(\alpha) \to \mathrm{Tor}(X, L)(\alpha) \oplus \mathrm{Tor}(K, Y)(\alpha)\\ &\quad \to \mathrm{Tor}(X, Y)(\alpha) \to \mathrm{Tor}(X/K, Y/L)(\alpha)\end{aligned}$$

and so by Lemma 2, we are done.

COROLLARY 4. *With the same hypotheses as Theorem* 2, *if* K *and* L *are* λ*-balanced subgroups of* X *and* Y *then sequence* (2) *is* λ*-balanced.*

Proof. This is clear from examining the proof of Theorem 2.

If λ is a limit ordinal, then we say a subgroup A of a group G is λ*-immediate* if it is λ-isotype and for every $\alpha < \lambda$, the group

$\{G(\alpha)+A\}/A$ is divisible. We then have the following:

COROLLARY 5. *With the same hypotheses as Theorem 2, if K and L are λ-immediate subgroups then sequence* (2) *is λ-immediate.*

Proof. Observe if K and L are λ-immediate subgroups, then for every ordinal α, $\{X(\alpha)+K\}/K$ and $\{Y(\alpha)+L\}/L$ are divisible so,

$$\operatorname{Tor}(\{X(\alpha)+K\}/K, \{Y(\alpha)+L\}/L)$$

is divisible and Q is a λ-immediate subgroup.

We pause now for the following simple lemma.

LEMMA 4. (a) *If K is a subgroup of X then K is a λ-high subgroup of X if and only if $X \to X/K$ maps $X(\lambda)$ isomorphically onto an essential subgroup of X/K.*

(b) *If K and L are essential subgroups of X and Y then* $\operatorname{Tor}(K, L)$ *is an essential subgroup of* $\operatorname{Tor}(X, Y)$.

Proof. In (a), $X(\alpha)$ is mapped injectively iff $K \cap X(\alpha)=0$, and the image is essential iff K is maximal with respect to this property. As for (b), note that $X[p]$ and $Y[p]$ are contained in K and L implies that $\operatorname{Tor}(X, Y)[p]=\operatorname{Tor}(X[p], Y[p])$ is contained in $\operatorname{Tor}(K, L)$.

Due to the importance of λ-high subgroups in the study of the Tor functor, the following is perhaps of interest.

THEOREM 3. *If K and L are λ-high subgroups of X and Y then* $Q=\operatorname{Tor}(X, L)+\operatorname{Tor}(K, Y)$ *is a λ-high subgroup of* $\operatorname{Tor}(X, Y)$.

Proof. Clearly K and L are isotype subgroups, so as in Theorem 2 we identify $\operatorname{Tor}(X, Y)/Q$ with $\operatorname{Tor}(X/K, Y/L)$. Since $X(\lambda)$ and $Y(\lambda)$ map isomorphically onto $\{X(\lambda)+K\}/K$ and $\{Y(\lambda)+L\}/L$,

$$\operatorname{Tor}(X, Y)(\lambda)=\operatorname{Tor}(X(\lambda), Y(\lambda))$$

maps isomorphically onto

$$\operatorname{Tor}(\{X(\lambda)+K\}/K, \{Y(\lambda)+L\}/L).$$

Since $\{X(\lambda)+K\}/K$ and $\{Y(\lambda)+L\}/L$ are essential subgroups of X/K and Y/L, by Lemma 4(b),

$$\operatorname{Tor}(\{X(\lambda)+K\}/K, \{Y(\lambda)+L\}/L)$$

is an essential subgroup of $\operatorname{Tor}(X/K, Y/L)$. So the result follows from Lemma 4(a).

The converse of Theorem 3 is not valid. In fact, if $l(X) = \lambda < l(Y)$, $K = X$ and L is any subgroup of Y, then $\mathrm{Tor}(X, L) + \mathrm{Tor}(K, Y) = \mathrm{Tor}(X, Y)$ is clearly λ-high in $\mathrm{Tor}(X, Y)$.

The following is an elementary result, but because of its generality it is perhaps somewhat surprising.

THEOREM 4. *If K and L are arbitrary subgroups of X and Y then* $\mathrm{Tor}(K, L)$ *is an isotype subgroup of* $\mathrm{Tor}(K, Y) \oplus \mathrm{Tor}(X, L)$.

Proof. Identifying $\mathrm{Tor}(K, L)$ with its image using the diagonal map, the following is easily checked:

$$\begin{aligned}
\mathrm{Tor}(K, L) &\cap \{\mathrm{Tor}(K, Y) \oplus \mathrm{Tor}(X, L)\}(\alpha) \\
&= \mathrm{Tor}(K, L) \cap \{\mathrm{Tor}(K(\alpha), Y(\alpha)) \oplus \mathrm{Tor}(X(\alpha), L(\alpha))\} \\
&= \mathrm{Tor}(K, L) \cap \mathrm{Tor}(K(\alpha), Y(\alpha)) \cap \mathrm{Tor}(X(\alpha), L(\alpha)) \\
&= \mathrm{Tor}(K(\alpha), L(\alpha)) = \mathrm{Tor}(K, L)(\alpha).
\end{aligned}$$

In [**8**] it was shown that if G and H are totally projective groups of length strictly greater than Ω, then $\mathrm{Tor}(G, H)$ is not totally projective. In our next result we observe that if we broaden the category of groups considered we can get a more satisfying result. Recall that an IT group is a group which can be embedded as an isotype subgroup of a totally projective group. We have then:

COROLLARY 6. *If X and Y are* IT *groups, then so is* $\mathrm{Tor}(X, Y)$.

Proof. Suppose X and Y are isotype subgroups of the totally projective groups G and H, and let D and D' denote injective hulls of G and H. Then $\mathrm{Tor}(X, Y)$ is an isotype subgroup of $\mathrm{Tor}(G, H)$, and $\mathrm{Tor}(G, H)$ is an isotype subgroup of

$$\mathrm{Tor}(D, H) \oplus \mathrm{Tor}(G, D') \cong \left(\bigoplus H\right) \oplus \left(\bigoplus G\right)$$

which proves the result.

We conclude this section with one more generalization of the fact that balanced subgroups of X and Y yield balanced subgroups of $\mathrm{Tor}(X, Y)$. We first state the following easy lemma whose proof is left to the reader:

LEMMA 5. *Suppose n is a positive integer, $0 \to A \to B \to C \to 0$ is p^n-pure and H is a p^n-bounded group, then the sequence*

$$0 \to \mathrm{Tor}(A, H) \to \mathrm{Tor}(B, H) \to \mathrm{Tor}(C, H) \to 0$$

splits.

LEMMA 6. *Suppose α is an ordinal, $0 \to A \to B \to C \to 0$ is α-balanced and H is a group with $H(\alpha) = 0$. Then*

$$0 \to \operatorname{Tor}(A, H) \to \operatorname{Tor}(B, H) \to \operatorname{Tor}(C, H) \to 0$$

is balanced.

Proof. Suppose $\alpha = \lambda + n$ where λ is a limit ordinal and n is a non-negative integer. If $\beta < \lambda$, then

$$0 \to A(\beta) \to B(\beta) \to C(\beta) \to 0$$

is pure, so

$$0 \to \operatorname{Tor}(A(\beta), H(\beta)) \to \operatorname{Tor}(B(\beta), H(\beta)) \to \operatorname{Tor}(C(\beta), H(\beta)) \to 0$$

is exact. The sequence

$$0 \to A(\lambda) \to B(\lambda) \to C(\lambda) \to 0$$

is p^n-pure and $H(\lambda)$ is p^n-bounded, so by the last lemma the sequence,

$$0 \to \operatorname{Tor}(A(\lambda), H(\lambda)) \to \operatorname{Tor}(B(\lambda), H(\lambda)) \to \operatorname{Tor}(C(\lambda), H(\lambda)) \to 0$$

splits, which proves the result.

THEOREM 5. *Suppose X and Y are reduced groups containing subgroups K and L, and*

$$\sigma = l(K) \leq \tau$$

where K is τ-balanced, L is σ-balanced, and $L \cap Y(\tau) = 0$. Then $\operatorname{Tor}(K, L)$ *is a balanced subgroup of* $\operatorname{Tor}(X, Y)$.

Proof. Observe that K must in fact be an isotype subgroup of X, and $K \cap X(\tau) = 0$. Let X' and Y' denote τ-high subgroups of X and Y containing K and L. By Theorem 3 of [6], $\operatorname{Tor}(X', Y')$ is a balanced subgroup of $\operatorname{Tor}(X, Y)$. So, since the property of being balanced is transitive, we may assume that $X(\tau) = 0$ and $Y(\tau) = 0$. If we now apply Lemma 6 (with $\alpha = \sigma$) we conclude that $\operatorname{Tor}(K, L)$ is a balanced subgroup of $\operatorname{Tor}(K, Y)$. We now apply Lemma 6 (with $\alpha = \tau$) to get that $\operatorname{Tor}(K, Y)$ is a balanced subgroup of $\operatorname{Tor}(X, Y)$. So we are done by the transitivity of the property of balance.

As a consequence of the last result, observe that if X and Y are reduced and K is a balanced subgroup of X, then for us to know that $\operatorname{Tor}(K, L)$ is a balanced subgroup of $\operatorname{Tor}(X, Y)$, all we need to know is that L is a $l(K)$-balanced subgroup of X.

2. We begin this section by reviewing and developing some facts about C_λ groups.

The following result of Hill [**4**] is pivotal:

THEOREM 6. *If K is an isotype subgroup of a totally projective group and $l(K)$ is countable then K is a d.s.c. group.*

We note the following consequences of Theorem 6.

LEMMA 7. *If $\lambda \leq \Omega$ is a limit ordinal, X is a C_λ group and K is an isotype subgroup of X, then,*

(a) *K is a C_λ group,*

(b) *if $l(K) < \lambda$ then K is a d.s.c. group.*

Proof. If α is an ordinal then $K/K(\alpha)$ can be viewed as an isotype subgroup of $X/X(\alpha)$. The result is therefore clear.

A useful notion in the study of C_λ groups is that of a *λ-high tower* which is defined as an ascending chain of α-high subgroups for $\alpha < \lambda$. The following elementary fact is well known (see, for example, [**6**, Lemma 1]).

LEMMA 8. *If λ is a limit ordinal and $\{K_\alpha\}$ is a λ-high tower in X, then $K = \bigcup_{\alpha<\lambda} K_\alpha$ is a λ-pure subgroup of X and X/K is divisible.*

The following fact is due to Nunke [**12**]:

THEOREM 7. *Suppose α is a countable ordinal and X is a group. If one α-high subgroup of X is a d.s.c. group then all α-high subgroups of X are d.s.c. groups.*

We are now led to the following characterization of C_λ groups.

THEOREM 8. *Suppose $\lambda \leq \Omega$ is a limit ordinal. Then X is a C_λ group if and only if for every $\alpha < \lambda$, X has an α-high subgroup which is a d.s.c. group.*

Proof. Necessity follows immediately from Lemma 7(b). Conversely, if for every $\alpha < \lambda$ there is an α-high subgroup of X which is a d.s.c. group, then using Theorem 7 we can construct a λ-high tower $\{K_\alpha : \alpha < \Omega\}$ of X consisting of d.s.c. groups. To show X is a C_λ group, let $\beta < \lambda$. Define K as follows: if $\lambda < \Omega$ let $K = \bigcup_{\alpha<\lambda} K_\alpha$, and if $\lambda = \Omega$ let $K = K_{\beta+\omega}$. In either case the group K will be a d.s.c. group (in the first case use [**9**, Cor. 1]) which is isotype in X and satisfies $K + X(\beta) = X$. Therefore,

$$X/X(\beta) = \{K + X(\beta)\}/X(\beta) \cong K/\{K \cap X(\beta)\} = K/K(\beta)$$

is a d.s.c. group.

The following is a theorem of Nunke [**12**].

THEOREM 9. *Suppose X and Y are reduced groups with $l(X) < l(Y)$. Then* $\operatorname{Tor}(X, Y)$ *is a d.s.c. group if and only if,*

(a) *X is a d.s.c. group,*

(b) *if α is an ordinal for which the αth Ulm invariant of X is non-zero, then any α-high subgroup of Y is a d.s.c. group.*

COROLLARY 7. *If $\lambda \leq \Omega$ is a limit ordinal, X is a d.s.c. group with $X(\lambda) = 0$ and Y is a C_λ group, then* $\operatorname{Tor}(X, Y)$ *is a d.s.c. group.*

Proof. This follows from the last two results noting that X can be expressed as a direct sum of countable groups of length less than λ.

Observe that a C_λ group of length less than λ is a d.s.c. group. We shall call a C_λ group G *proper* if $l(G) \geq \lambda$.

THEOREM 10. *Let $\lambda \leq \Omega$ be a limit ordinal. If X and Y are C_λ groups then* $\operatorname{Tor}(X, Y)$ *is a C_λ group. Furthermore,* $\operatorname{Tor}(X, Y)$ *is a proper C_λ group if and only if X and Y are proper C_λ groups.*

Proof. Assume first that X and Y are C_λ groups. If D_1 and D_2 are divisible hulls of X and Y respectively, then by Theorem 4, $\operatorname{Tor}(X, Y)$ is an isotype subgroup of $\operatorname{Tor}(X, D_2) \oplus \operatorname{Tor}(D_1, Y)$. This last group is isomorphic to a direct sum of copies of X and Y and so $\operatorname{Tor}(X, Y)$ is a C_λ group by Lemma 7(a).

For the second statement, we observe that X and Y are proper iff $\operatorname{Tor}(X, Y)$ is proper, so we assume now that $\operatorname{Tor}(X, Y)$ is a proper C_λ group. If $\alpha < \lambda$ and K is an α-high subgroup of X, then $\operatorname{Tor}(K, Y)$ is isotype in $\operatorname{Tor}(X, Y)$ and $\operatorname{Tor}(K, Y)(\alpha) = 0$. By Lemma 7(b), $\operatorname{Tor}(K, Y)$ is a d.s.c. group, and by Theorem 9, K is a d.s.c. group. So by Theorem 8, X, and similarly Y, is a C_λ group.

Corollary 8. *Let $\lambda \leq \Omega$ be a limit ordinal, X a proper C_λ group. If Y is a group then* $\mathrm{Tor}(X, Y)$ *is a C_λ group if and only if Y is a C_λ group.*

Proof. We may clearly assume that X and Y are reduced. If $l(Y) < \lambda$ then $\mathrm{Tor}(X, Y)$ is a C_λ group iff it is a d.s.c. group iff Y is a d.s.c. group. If $l(Y) \geq \lambda$ the result follows from Theorem 10.

If λ was a limit ordinal greater than Ω one could define a group G to be a C_λ if for each $\alpha < \lambda$ the quotient $G/G(\alpha)$ is totally projective. As in the case of total projectivity, we now show that Theorem 10 does not extend to limit ordinals λ greater than Ω.

Theorem 11. *If $\lambda > \Omega$ is a limit ordinal then* $\mathrm{Tor}(X, Y)$ *is not a proper C_λ group for any reduced C_λ groups X and Y.*

Proof. To derive a contradiction we assume that X and Y are reduced groups for which $\mathrm{Tor}(X, Y)$ is a proper C_λ group. Replacing X by a direct sum of copies of X in no way affects the fact that $\mathrm{Tor}(X, Y)$ is a proper C_λ, so we may assume that X and similarly Y have the property that each of their non-zero Ulm invariants have cardinality at least $\aleph_1$. Let n be a positive integer such that the $\Omega + n$th Ulm invariant of X is non-zero, and T_1 be a totally projective group of cardinality $\aleph_1$ and length $\Omega + n$ with the property that T_1 has a non-zero Ulm invariant at an ordinal α if and only if the same can be said of X. Clearly the totally projective groups $X/X(\Omega + n)$ and $\{X \oplus T_1\}/\{X \oplus T_1\}(\Omega + n) \cong X/X(\Omega + n) \oplus T_1$ are isomorphic, which implies that X is isomorphic to $X \oplus T_1$, so we can view T_1 as a summand of X. Similarly we can find a totally projective summand T_2 of Y of length $\Omega + m$ for some positive integer m. We may clearly assume $n \leq m$.

Observe that $\mathrm{Tor}(T_1, T_2)$ is isomorphic to a summand of

$$\mathrm{Tor}(X, Y)/\mathrm{Tor}(X, Y)(\Omega + m),$$

which is totally projective. Therefore $\mathrm{Tor}(T_1, T_2)$ must be totally projective and this contradiction proves the result.

Let $\mathscr{T}$ denote the class of C_Ω groups which are the union of some Ω-high tower of subgroups. It is clear that $\mathscr{T}$ is closed with respect to direct sums. By [**6**, Theorem 6], if G is a reduced C_Ω group of length Ω and cardinality $\leq \aleph_1$ then G is in $\mathscr{T}$. Let $\mathscr{T}_s$ denote the class of C_Ω

groups which are summands of groups in $\mathcal{T}$. It is not clear whether a group in $\mathcal{T}_s$ is actually in $\mathcal{T}$.

We have the following consequence of [**6**, Lemma 3].

THEOREM 12. *Let G and H be members of $\mathcal{T}_s$. Then* $\operatorname{Tor}(G, H)$ *is a d.s.c. group.*

Proof. Clearly $\operatorname{Tor}(G, H)$ is a summand of a d.s.c. group so it is a d.s.c. group itself.

We will use extensively the following special case [**6**, Theorem 6]:

COROLLARY 9. *If G and H are C_Ω groups of cardinality at most $\aleph_1$ and $G(\Omega) = H(\Omega) = 0$ then* $\operatorname{Tor}(G, H)$ *is a d.s.c. group.*

We note the following consequence of Theorem 3.

COROLLARY 10. *If G is a member of $\mathcal{T}$ (resp. $\mathcal{T}_s$) and H is a C_Ω group then* $\operatorname{Tor}(G, H)$ *is also a member of $\mathcal{T}$ (resp. $\mathcal{T}_s$).*

Recall that a group is *summable* if its socle is a free valued vector space when its values are those induced by the height function. When λ is a countable limit ordinal a C_λ group of length λ is a d.s.c. group if and only if it is summable. Examples show that this does not generalize to C_Ω groups. We have though,

THEOREM 13. *Suppose G is a reduced group. If G is a summable C_Ω group then G is in $\mathcal{T}$. In fact, G is a summable C_Ω group if and only if G is the union of an Ω-high tower A_α, $\alpha < \Omega$, of α-high subgroups that are d.s.c. groups, and which is continuous in the sense that whenever α is a limit ordinal, we have $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.*

Proof. Suppose G is a summable C_Ω and $G[p] = \bigoplus S_\alpha$, where S_α is a homogeneous space with value α. If α is an ordinal and A_β has been defined for all $\beta < \alpha$, define $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ if α is a limit ordinal, and if α is isolated, let A_α be an α-high subgroup containing $A_{\alpha-1} \oplus S_{\alpha-1}$. In either case we have $A_\alpha[p] = \bigoplus_{\beta<\alpha} S_\beta$ and we are done.

Conversely, if A_α, for $\alpha < \Omega$, is a continuous Ω-high tower of G, let S_α be defined by the equation

$$A_{\alpha+1}[p] = A_\alpha[p] \oplus S_\alpha.$$

It is readily checked that $G[p] = \bigoplus S_\alpha$.

If A is an isotype subgroup of the group G, then the *c-valuation* on G/A is defined by

$$c(x+A) = \sup\{ht(x+a)+1 : a \in A\}$$

for any $x + A \in G/A$. The following is an important recent result of Hill and Megibben [**8**]:

THEOREM 14. *If G is a totally projective group and H and K are isotype subgroups of G, then there is an automorphism of G taking H to K if and only if the c-valuated groups G/H and G/K are isomorphic.*

We make use of this result in the following:

THEOREM 15. *Let G be an* $\Game$T *group with $G(\Omega) = 0$. Then G is in $\mathscr{T}_S$.*

Proof. Suppose G is an isotype subgroup of the totally projective group X. Since G can also be viewed as an isotype subgroup of $X/X(\Omega)$ we may assume X is a d.s.c. group. Let A_α, for $\alpha < \Omega$, denote an Ω-high tower of G, and let $A = \bigcup_{\alpha<\Omega} A_\alpha$. Observe A is also an isotype subgroup of X so clearly A is in $\mathscr{T}$. Note by Lemma 8, G/A is divisible and for all $g+A \in G/A$ we have $c(g+A) = \Omega$. This implies that there is an isomorphism of c-valuated groups $X/A \cong X/G \oplus G/A$. It is also clear that there is an S-group M which is an Ω-pure subgroup of a d.s.c. group H, for which there is an isomorphism of c-valuated groups, $H/M \cong G/A$. Therefore, as c-valuated groups there are isomorphisms,

$$\begin{aligned}(X \oplus H)/(A \oplus H) &\cong X/A \cong X/G \oplus G/A \cong X/G \oplus H/M \\ &\cong (X \oplus H)/(G \oplus M)\end{aligned}$$

and so by Theorem 14 we have $G \oplus M \cong A \oplus H$. Since $A \oplus H$ is clearly in $\mathscr{T}$, we are done.

3. In this section we extend some of the previous results. The techniques employed are similar to those of [**5**].

By a *λ-balanced tower* of a group G we mean an ascending chain of λ-balanced subgroups A_i, which we assume are indexed by some segment $[0, \gamma]$ of the ordinals, such that $A_0 = 0$, $A_\gamma = G$ and for every limit ordinal α we have $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. It is readily checked that if $i \leq j$ then A_i is a λ-balanced subgroup of A_j. If γ is a limit ordinal and $|A_i| < |G|$ for each $i < \gamma$ we will say it is *proper*. If $A_i = G$ for all

$i > 0$ we say the tower is *trivial.* By simply repeating G we may extend a λ-balanced tower to any ordinal we want (though of course it will not remain proper). We state first a sufficient condition for $\mathrm{Tor}(G, H)$ to be a d.s.c. group.

THEOREM 16. *Suppose $\{A_i\}$ and $\{B_i\}$ are λ-balanced towers for the groups G and H, where $G(\lambda) = H(\lambda) = 0$. If for every i*

$$\mathrm{Tor}(A_{i+1}/A_i, B_i) \quad \textit{and} \quad \mathrm{Tor}(A_{i+1}, B_{i+1}/B_i)$$

or

$$\mathrm{Tor}(A_i, B_{i+1}/B_i) \quad \textit{and} \quad \mathrm{Tor}(A_{i+1}/A_i, B_{i+1})$$

are d.s.c. groups then $\mathrm{Tor}(G, H)$ *is a d.s.c. group.*

Proof. We will show that for every i, $\mathrm{Tor}(A_{i+1}, B_{i+1})$ is isomorphic to $\mathrm{Tor}(A_i, B_i) \oplus X_i$ where X_i is a d.s.c. group. This will show that $\mathrm{Tor}(G, H) \cong \bigoplus X_i$ and prove the result.

In the first case there are sequences

$$0 \to \mathrm{Tor}(A_i, B_i) \to \mathrm{Tor}(A_{i+1}, B_i) \to \mathrm{Tor}(A_{i+1}/A_i, B_i) \to 0$$

and

$$0 \to \mathrm{Tor}(A_{i+1}, B_i) \to \mathrm{Tor}(A_{i+1}, B_{i+1}) \to \mathrm{Tor}(A_{i+1}, B_{i+1}/B_i) \to 0.$$

By Lemma 6, these are balanced, and so they must split and

$$\begin{aligned}\mathrm{Tor}(A_{i+1}, B_{i+1}) \cong\ & \mathrm{Tor}(A_i, B_i) \oplus \mathrm{Tor}(A_{i+1}/A_i, B_i) \\ & \oplus \mathrm{Tor}(A_{i+1}, B_{i+1}/B_i).\end{aligned}$$

The second case is handled similarly.

Observe that in Theorem 16, if k is an ordinal, $\mathrm{Tor}(A_k, B_k)$ is a d.s.c. group and the stated conditions are true merely for those $i \geq k$, then the same argument shows that $\mathrm{Tor}(G, H)$ is still a d.s.c. group.

LEMMA 9. *Suppose $\lambda \leq \Omega$ is a limit ordinal, G a group with $G(\lambda) = 0$, A is an isotype subgroup of G and A' is a subgroup of G containing A. Then we can extend A' to an isotype subgroup A'' of G such that*

$$|A''/A| \leq |A'/A|\,|\lambda|.$$

Proof. Let A'/A be denoted by C. For each non-zero $c \in C$, let $\{g_{c,i} \in c\colon i \in I_c\}$ be a collection with $|I_c| \leq |\lambda|$ such that for every

$g + A = c \in A'/A$ there is a $g_{c,i}$ such that $ht_G(g_{c,i}) \geq ht_G(g)$. Let K be a subgroup of G containing each $g_{c,i}$, such that

$$|K| \leq |C|\,|\lambda|$$

and such that for all $g_{c,i}$ we have

$$ht_K(g_{c,i}) = ht_G(g_{c,i}).$$

Let $A_0 = A'$ and $A_1 = A + K$. Then it is easily checked that for all $g \in A_0$, that $ht_{A_1}(g) = ht_G(g)$. If we use the same construction to produce A_{n+1} from A_n, then $A'' = \bigcup_{n<\omega} A_n$ will have the required properties.

The same sort of repeated building of extensions will prove the following:

LEMMA 10. *Suppose $\lambda \leq \Omega$ is a limit ordinal and G is a C_λ group with $G(\lambda) = 0$. For every $\alpha < \lambda$ fix a decomposition*

$$G/G(\alpha) \cong \bigoplus_{j \in I_\alpha} C_{\alpha,j}$$

where each $C_{\alpha,j}$ is countable. Suppose $A \subseteq A'$ are subgroups of G and for each $\alpha < \lambda$ we have

$$\{A + G(\alpha)\}/G(\alpha) = \bigoplus_{j \in S_\alpha} C_{\alpha,j}$$

for some $S_\alpha \subseteq I_\alpha$. Then we can extend A' to a subgroup A'' of G such that

$$|A''/A| \leq |A'/A|\,|\lambda|$$

and for every $\alpha < \lambda$ we have

$$\{A'' + G(\alpha)\}/G(\alpha) = \bigoplus_{j \in T_\alpha} C_{\alpha,j}$$

for some $T_\alpha \subseteq I_\alpha$.

Observe that the groups A and A'' in the last result are actually λ-nice in G since for every $\alpha < \lambda$

$$\frac{G/A}{\{A + G(\alpha)\}/A} \cong \frac{G/G(\alpha)}{\{A + G(\alpha)\}/G(\alpha)}$$

has length at most α, so

$$(G/A)(\alpha) = \{A + G(\alpha)\}/A.$$

LEMMA 11. *Suppose G and H are groups and* $\operatorname{Tor}(G,H) = \bigoplus_{i\in I} C_i$ *with each C_i a countable group.*

(a) *Suppose m is an infinite cardinal and A and B are subgroups of G and H of cardinality at most m. Then there are extensions A' and B' of A and B also of cardinality at most m with* $\operatorname{Tor}(A',B') = \bigoplus_{i\in J} C_i$ *for some $J \subseteq I$.*

(b) *Suppose $|G|$ is infinite and B is a subgroup of H such that* $\operatorname{Tor}(G,B) = \bigoplus_{i\in J} C_i$ *for some $J \subseteq I$. If B' is an extension of B with $|B'/B| \le |G|$ then B' has an extension B'' such that $|B''/B| \le |G|$ and* $\operatorname{Tor}(G,B'') = \bigoplus_{i\in K} C_i$ *for some $K \subseteq I$.*

Proof. As for (a), note that there are subsets J_n of I and a chain of subgroups A_n and B_n starting at A and B also of cardinality at most m such that,

$$\cdots \subseteq \operatorname{Tor}(A_n,B_n) \subseteq \bigoplus_{i\in J_n} C_i \subseteq \operatorname{Tor}(A_{n+1},B_{n+1}) \subseteq \bigoplus_{i\in J_{n+1}} C_i \subseteq \cdots$$

so letting $A' = \bigcup A_n$ and $B' = \bigcup B_n$ does the trick.

As for (b), let X be a subgroup of H such that $|X| \le |G|$ and $B' = B + X$. Let $X \subseteq X'$ with $|X'| \le |G|$ and $\operatorname{Tor}(G,X') = \bigoplus_{i\in L} C_i$. So letting $B'' = B + X'$ does the trick.

We wish now to turn to some necessary conditions for $\operatorname{Tor}(G,H)$ to be a d.s.c. group. We consider first the case where G and H have different cardinalities.

THEOREM 17. *Suppose $\lambda \le \Omega$ is a limit ordinal and G and H are reduced C_λ groups of length λ. If $|\lambda| \le |G| < |H|$ then* $\operatorname{Tor}(G,H)$ *is a d.s.c. group if and only if H has a λ-balanced tower $\{B_i\}$ such that*

$$\operatorname{Tor}(G, B_{i+1}/B_i)$$

is a d.s.c. group for each i. Furthermore, if $\operatorname{Tor}(G,H)$ *is a d.s.c. group we may choose the tower to be proper and so that it satisfies $|B_{i+1}/B_i| \le |G|$ for each i.*

Proof. If we let $\{A_i\}$ denote the trivial λ-balanced tower of G then sufficiency follows from Theorem 16.

As to necessity, if α is the smallest ordinal of cardinality $|H|$, and we index H using the ordinals $\beta < \alpha$, then using a standard "back-and-forth" argument with Lemmas 9, 10 and 11(b), we can construct a λ-balanced tower $\{B_i, i < \alpha\}$ such that for all $i < \alpha$ we have,

(i) $h_i \in B_{i+1}$,

(ii) $\mathrm{Tor}(G, B_i)$ is the direct sum of a subcollection of the terms in a fixed decomposition of $\mathrm{Tor}(G, H)$,

(iii) $|B_{i+1}/B_i| \leq |G|$,

(iv) $|B_i| < |H|$,

which gives the result.

We observe the following consequence of the last result.

THEOREM 18. *If $\lambda \leq \Omega$ is a limit ordinal and G is a C_λ group with $|G| > |\lambda|$ then G has a proper λ-balanced tower, $\{A_i\}$, such that for each i, A_{i+1}/A_i is a C_λ group of cardinality at most $|\lambda|$.*

Proof. If T is a d.s.c. group of length λ and cardinality $|\lambda|$, then $\mathrm{Tor}(T, G)$ is a d.s.c. group, and so the result follows from Theorem 17 and Corollary 8.

The following generalizes Corollary 9.

COROLLARY 11. *Suppose $\lambda \leq \Omega$ is a limit ordinal, and G and H are C_λ groups of cardinality $\aleph_1$ with $G(\lambda) = H(\lambda) = 0$. Then* $\mathrm{Tor}(G, H)$ *is a d.s.c. group.*

Proof. We may assume λ is countable. Let G and H have proper λ-balanced towers, $\{A_i\}$ and $\{B_i\}$, as in Theorem 18. Clearly the four groups mentioned in Theorem 16 are in this case countable, so $\mathrm{Tor}(G, H)$ is a d.s.c. group.

Recall that using the category of balanced exact sequences, we can refer to the *balanced projective dimension* of a group (see [2]). Denote by $\mathscr{C}_1$ the class of reduced C_Ω groups, of length at most Ω, whose balanced projective dimension is at most 1.

We denote by M_Ω the standard S-group which is an Ω-pure subgroup of the "generalized Prüfer group" H_Ω with $H_\Omega/M_\Omega \cong Z_{p^\infty}$.

THEOREM 19. *If G is a C_Ω group with $G(\Omega) = 0$ then* $\mathrm{Tor}(G, M_\Omega)$ *is a d.s.c. group if and only if G is in $\mathscr{C}_1$.*

Proof. By Lemma 6 there is a balanced sequence

$$0 \to \mathrm{Tor}(G, M_\Omega) \to \mathrm{Tor}(G, H_\Omega) \to \mathrm{Tor}(G, Z_{p^\infty})(\cong G) \to 0.$$

Clearly $\mathrm{Tor}(G, H_\Omega)$ is a d.s.c. group and so the result follows.

Since M_Ω is in $\mathcal{T}_s$, by Theorems 12 and 19 we have that any group in $\mathcal{T}_s$, in particular, any isotype subgroup of a d.s.c. group, is in $\mathcal{C}_1$ (a fact which can be shown in other ways, too).

The following shows that under certain circumstances the property that $\operatorname{Tor}(X, Y)$ is a d.s.c. group can be inherited.

THEOREM 20. *Suppose K and L are isotype subgroups of X and Y and* $\operatorname{Tor}(X, Y)$ *is a d.s.c. group. If* $\operatorname{Tor}(X, L)$ *and* $\operatorname{Tor}(K, Y)$ *are d.s.c. groups then* $\operatorname{Tor}(K, L)$ *is also a d.s.c. group.*

Proof. Using the notation of Theorem 2, the d.s.c. group $\operatorname{Tor}(X, L)\oplus \operatorname{Tor}(K, Y)$ contains $\operatorname{Tor}(K, L)$ as a balanced subgroup. The cokernel, Q, is an isotype subgroup of the d.s.c. group $\operatorname{Tor}(X, Y)$. Therefore Q is in $\mathcal{C}_1$, so the balanced projective dimension of Q is at most 1, and we have that $\operatorname{Tor}(K, L)$ is a d.s.c. group.

The following consequence of the last two results presents an interesting property of $\mathcal{C}_1$.

THEOREM 21. *If G is a member of $\mathcal{C}_1$ and A is an isotype subgroup of G, then A is also in $\mathcal{C}_1$.*

Proof. The hypotheses and the last result guarantee that $\operatorname{Tor}(M_\Omega, G)$, $\operatorname{Tor}(H_\Omega, A)$ and $\operatorname{Tor}(H_\Omega, G)$ are d.s.c. groups. So by Theorem 20, $\operatorname{Tor}(M_\Omega, A)$ is a d.s.c. group and we are done.

THEOREM 22. *Suppose H is a C_Ω group with $H(\Omega) = 0$. Then H is in $\mathcal{C}_1$ iff H has a Ω-balanced tower $\{B_i\}$ such that each B_{i+1}/B_i is a C_λ group of cardinality at most $\aleph_1$, with $(B_{i+1}/B_i)(\Omega) = 0$.*

Proof. The result is clear if $|H| \leq \aleph_1$. If $|H| > \aleph_1 = |M_\Omega|$, then by Theorem 17, $\operatorname{Tor}(H, M_\Omega)$ is a d.s.c. group iff we can choose an Ω-balanced tower $\{B_i\}$ such that $\operatorname{Tor}(M_\Omega, B_{i+1}/B_i)$ is a d.s.c. group for each i, where $|B_{i+1}/B_i| \leq \aleph_1$. By Corollary 8, B_{i+1}/B_i is a C_Ω group, and since M_Ω is not a d.s.c. group, we must have $(B_{i+1}/B_i)(\Omega) = 0$. This completes the proof.

In fact, it can easily be seen that the subgroups B_i in the last result are balanced in G.

The following shows that amongst the C_Ω groups, $\operatorname{Tor}(G, H)$ is often a d.s.c. group.

THEOREM 23. *If G and H are in $\mathscr{C}_1$, then* $\operatorname{Tor}(G, H)$ *is a d.s.c. group.*

Proof. Choose Ω-balanced towers $\{A_i\}$ and $\{B_i\}$ as in the last result. If we fix i, then using the Ω-balanced tower $\{B_j\}_{j\leq i}$ of B_i and the trivial Ω-balanced tower for A_{i+1}/A_i then by Theorem 16 and Corollary 9 we can conclude that $\operatorname{Tor}(A_{i+1}/A_i, B_i)$ is a d.s.c. group. Similarly we can conclude that $\operatorname{Tor}(A_{i+1}, B_{i+1}/B_i)$ is a d.s.c. group. So by Theorem 16, $\operatorname{Tor}(G, H)$ is a d.s.c. group.

Observe that the above result is clearly not true if Ω is replaced by a smaller limit ordinal. In fact, if G is any reduced group of countable length then its balanced-projective dimension is at most 1 (this follows from Theorem 6).

We state now a necessary condition for $\operatorname{Tor}(G, H)$ to be a d.s.c. group, which can be applied when G and H have the same cardinality.

THEOREM 24. *Suppose $\lambda \leq \Omega$ is a limit ordinal, G and H are C_λ groups with $G(\lambda) = H(\lambda) = 0$ and $m = |G| = |H| > \aleph_1$. If* $\operatorname{Tor}(G, H)$ *is a d.s.c. group and either λ is countable, or $\lambda = \Omega$ and $m = \aleph_2$, then G and H have proper λ-balanced towers $\{A_i\}$ and $\{B_i\}$ such that* $\operatorname{Tor}(A_i, B_{i+1}/B_i)$ *and* $\operatorname{Tor}(A_{i+1}/A_i, B_i)$ *are d.s.c. groups. If $m = \aleph_2$ then this condition is also sufficient for* $\operatorname{Tor}(G, H)$ *to be a d.s.c. group.*

Proof. Suppose $\operatorname{Tor}(G, H)$ is a d.s.c. group. We can clearly construct proper λ-balanced towers $\{A_i\}$, $\{B_i\}$ such that $\operatorname{Tor}(A_i, B_i)$ is the direct sum of a subcollection of the terms in a fixed decomposition of $\operatorname{Tor}(G, H)$. So $\operatorname{Tor}(A_i, B_i)$ is a summand of $\operatorname{Tor}(A_{i+1}, B_{i+1})$ and hence it is also a summand of $\operatorname{Tor}(A_{i+1}, B_i)$ and $\operatorname{Tor}(A_i, B_{i+1})$. Therefore,

$$\operatorname{Tor}(A_{i+1}, B_i) \cong \operatorname{Tor}(A_i, B_i) \oplus \operatorname{Tor}(A_{i+1}/A_i, B_i)$$

and

$$\operatorname{Tor}(A_i, B_{i+1}) \cong \operatorname{Tor}(A_i, B_i) \oplus \operatorname{Tor}(A_i, B_{i+1}/B_i).$$

If λ is countable, then since $\operatorname{Tor}(A_{i+1}, B_i)$ and $\operatorname{Tor}(A_i, B_{i+1})$ are isotype subgroups of $\operatorname{Tor}(A_{i+1}, B_{i+1})$ they are d.s.c. groups, which gives necessity in this case. If $\lambda = \Omega$ then our cardinality assumption, together with Corollary 11, once again assures that they are d.s.c. groups, so we have proven necessity.

Assume now that G and H have proper λ-balanced towers as above and $m = \aleph_2$. The ordinals $l(A_i)$ form a non-decreasing sequence, which must after some ordinal k be constant. If $l(A_k) < \lambda$ then $l(G) < \lambda$ and G is a d.s.c. group. In this case, then, we clearly have $\operatorname{Tor}(G, H)$

a d.s.c. group, so we may assume $l(A_k) = \lambda$. We may similarly assume $l(B_k) = \lambda$. If λ is countable we make one more assumption on our choice of k. Observe that if $i < j$ and A_j is a d.s.c. group then the same can be said of A_i. So by possibly replacing k with a larger ordinal, we may assume that A_i is either always or never a d.s.c. group for all $i \geq k$.

Our objective is to show that the hypotheses of Theorem 16 are satisfied for the λ-balanced towers $\{A_i\}_{i \geq k}$ and $\{B_i\}_{i \geq k}$. By Corollary 11, $\mathrm{Tor}(A_k, B_k)$ is a d.s.c. group. If $i \geq k$ then since $\mathrm{Tor}(A_i, B_{i+1}/B_i)$ is a d.s.c. group, by Corollary 8, B_{i+1}/B_i is a C_λ group. If $l(B_{i+1}/B_i) \leq \lambda$ then by Corollary 11, $\mathrm{Tor}(A_{i+1}, B_{i+1}/B_i)$ is a d.s.c. group. If $l(B_{i+1}/B_i) > \lambda$ then A_i is a d.s.c. group. If λ is countable then by our choice of k, A_{i+1} will also be a d.s.c. group and hence so will $\mathrm{Tor}(A_{i+1}, B_{i+1}/B_i)$. If $\lambda = \Omega$, then A_i is complete in its Ω-adic topology (see **[10]**), so $l(A_{i+1}/A_i) \leq \Omega$, and since $\mathrm{Tor}(A_{i+1}/A_i, B_i)$ is a d.s.c. group, A_{i+1}/A_i is a C_Ω group. So once again by Corollary 11, $\mathrm{Tor}(A_{i+1}/A_i, B_{i+1})$ is a d.s.c. group. This concludes the proof.

We conclude with the following somewhat interesting result.

Theorem 25. *If $\lambda \leq \Omega$ is a limit ordinal, G, H and K are C_λ groups of cardinality at most $\aleph_2$ and $G(\lambda) = H(\lambda) = K(\lambda) = 0$ then* $\mathrm{Tor}(G, H, K)$ *is a d.s.c. group.*

Proof. Let $\{A_i\}$, $\{B_i\}$ and $\{E_i\}$ for $i \leq \omega_2$ be λ-balanced towers of G, H and K respectively which are trivial if the cardinality of the corresponding group is $< \aleph_2$ and proper otherwise. As in Theorem 18, we may assume A_{i+1}/A_i, B_{i+1}/B_i and E_{i+1}/E_i are C_λ groups for each i. For each i, by Corollary 11 we have that $\mathrm{Tor}(B_i, E_i)$ is a d.s.c. group and by Theorem 1 it is a λ-balanced subgroup of $\mathrm{Tor}(B_{i+1}, E_{i+1})$. If we denote the quotient

$$\mathrm{Tor}(B_{i+1}, E_{i+1})/\mathrm{Tor}(B_i, E_i)$$

by P_i, then as in Theorem 1, P_i is a λ-balanced subgroup of

$$\mathrm{Tor}(B_{i+1}, E_{i+1}/E_i) \oplus \mathrm{Tor}(B_{i+1}/B_i, E_{i+1})$$

so by Theorem 10 and Lemma 7(a), P_i is a C_λ group and $P_i(\lambda) = 0$. So by Corollary 11 we now have that for every i,

$$\mathrm{Tor}(A_{i+1}/A_i, \mathrm{Tor}(B_{i+1}, E_{i+1}))$$

and $\mathrm{Tor}(A_i, P_i)$ are d.s.c. groups and hence by Theorem 16, so is

$$\mathrm{Tor}(G, H, K) \cong \mathrm{Tor}(G, \mathrm{Tor}(H, K)).$$

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